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On the completeness of certain sequences

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Abstract. For $A = \{a_1 \le \dots a_n \le \dots\} \subset N$ we define $P(A) = \{\sum \varepsilon_i a_i, \varepsilon_i = 0$ or 1, $\sum \varepsilon_i$ is finite$\}$. A is said to be subcomplete if $P(A)$ contains an infinite arithmetic progression.

We prove: for every infinite sequence $\{b_m\}$ of integers and for every Graham's sequence $\{[2^n\alpha]\}$ the sequence $\{b_m[2^n\alpha]\}$ is subcomplete.

1. Introduction

Let $A = \{a_1 \le a_2 \le \dots \le a_n \le \dots\}$ be a sequence of positive integers. Define

$$P(A) = \left\{ \sum_{i=1}^{\infty} \varepsilon_i a_i : \varepsilon_i \in \{0,1\}; \sum_{i=1}^{\infty} \varepsilon_i < \infty \right\}.$$

Alternatively, we may consider A to be a multiset, and define $P(A)$ to be the set of all sums of the elements of finite submultisets of A.

A is called complete if every sufficiently large integer belongs to $P(A)$. We call A subcomplete if there is an integer $d \ne 0$ such that $P(A)$ contains an infinite arithmetic progression with difference d. For example the powers of 2 form a complete sequence. Let $1 \le a_1 < a_2 < \dots < a_k < \dots$ be a sequence of integers. Let $A(n) = \sum_{a_i \le n} 1$. Clearly we have $|\{\sum_{i=1}^k \varepsilon_i a_i : \varepsilon_i \in \{0,1\}\}| \le 2^k$. Thus if $\limsup_{n\to\infty}(n - 2^{A(n)}) = \infty$ then the sequence

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$A = \{1 \le a_1 < \dots < a_k < \dots\}$ cannot be complete. This implies that if A is complete then

$$A(n) \ge (1 + o(1)) \log_2 n.$$

Denote the set of natural numbers by $\mathbb{N}$, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathbb{R}^+$ be the set of positive real numbers. Given a set A of positive integers, we denote by $A + A$ the set of pairwise sums, $A + A = \{a + a' : a, a' \in A\}$. Let $dA = \{da : a \in A\}$.

Let $g(n) = A(n)/\log n$. Many authors investigated sequences for which $g(n)$ tends to infinity very "slowly" as n tends to infinity. For example P. Erdős conjectured and J. Birch proved in 1959 (see [1]) that if $(p, q) = 1$, then the sequence formed by the values $p^k q^m$ is complete, where k and m are natural numbers. Clearly these sets have $\Theta(\log^2 n)$ elements less than n and so $g(n) < C(p, q) \log n$.

A few years later J.W. Cassels proved in [2] the following theorem: Assume that

$$\lim_{n\to\infty} (A(2n) - A(n))/\log\log n = \infty$$

and for every real θ; $0 < \theta < 1$,

$$\sum_{k=1}^{\infty} \|a_k\theta\| = \infty.$$

Then A is complete, where $\|x\| = \min_{-\infty<n<\infty}\{|x - n|\}; n \in \mathbb{N}$.

It is easy to see that the theorem of Cassels contains Birch's result.

In 1969 R.L. Graham asked [3] (see also [4]): for which pairs of positive reals (α, β) is the sequence $A_{\alpha\beta} := \{[2^n\alpha], [2^m\beta] : n, m \in \mathbb{N}\}$ complete? The first author investigated the completeness of this sequence in [5] and [6]. Clearly, here $A_{\alpha\beta}(n) < c(\alpha, \beta) \log n$ and so $g(n) = o(1)$.

The aim of this note is to investigate an Erdős-Birch-Graham-type sequence A for which the conditions of Cassels' theorem are not valid.

Definition. Let $B = \{b_1 \le b_2 \le \dots\}$ be an infinite sequence of integers. Let

$$A(\alpha, B) := \{b_m[2^n\alpha] : a, m \in \mathbb{N}; \alpha \in \mathbb{R}\}.$$

We prove the following

Theorem. *For every real number $\alpha > 0$ and every infinite sequence B of positive integers, $A(\alpha, B)$ is subcomplete.*

We mention here that if $B(n) = O(\log\log n)$ then we cannot use Cassels' theorem, i.e. our theorem is not covered by Cassels' argument. On the other hand it is easy to construct a sequence B so that the second condition of Cassels' theorem is not valid for the sequence $A(\alpha, B)$ either (see [7]). Indeed, first we are going to count the number of elements of $A(\alpha, B)$ between $2^N, 2^{N+1}$, where N is large and a fixed number. So let us assume that

$$2^N \le [2^n\alpha]b_k < 2^{N+1}$$

or equivalently

$$\frac{2^N}{[2^n\alpha]} \le b_k < \frac{2^{N+1}}{[2^n\alpha]}. \tag{1}$$

Since $[x] > x/2$ if $x \ge 2$, for $\alpha \ge 2^{1-n}$ we get

$$2^{N-n}\alpha^{-1} \le b_k \le 2^{N+2-n}\alpha^{-1}. \tag{2}$$

This yields

$$\left|\left[2^N, 2^{N+1}\right] \cap A(\alpha, B)\right| \le \left|\cup_{n\ge 0}\left\{\left(2^{N-n}\alpha^{-1}, 2^{N+2-n}\alpha^{-1}\right) \cap B\right\}\right|$$
$$\le 2B\left(2^{N+2}\alpha^{-1}\right) = O\left(\log\log 2^N\right).$$

So we conclude that the first condition of Cassels' theorem is not valid for the sequence $A(\alpha, B)$ with $B(n) = O(\log\log n)$.

2. Lemmas

Let $a_n = [2^n\alpha]$ and let $\alpha = \sum_{n=-k}^{\infty} \varepsilon_n(\alpha)2^{-n}$ where $\varepsilon_n(\alpha) = 0$ or 1, and assume that $\varepsilon_n(\alpha) = 0$ holds infinitely often.

We need some lemmas:

Lemma 1. *For every n*

$$a_{n+1} = 2a_n + \varepsilon_{n+1}(\alpha).$$

Indeed

$$2^n\alpha = \varepsilon_{-k}(\alpha)\varepsilon_{-k+1}(\alpha)\ldots\varepsilon_n(\alpha)\cdot\varepsilon_{n+1}(\alpha)\ldots$$

and

$$2^{n+1}\alpha = \varepsilon_{-k}(\alpha)\varepsilon_{-k+1}(\alpha)\ldots\varepsilon_n(\alpha)\varepsilon_{n+1}(\alpha)\cdot\ldots$$

(in base 2).

So $a_{n+1} = \varepsilon_{-k}(\alpha)\dots\varepsilon_{n+1}(\alpha) = 2a_n + \varepsilon_{n+1}(\alpha)$ *(in base 2).*

Lemma 2. *Let* $\alpha = \sum_{n=-k}^{\infty} \varepsilon_n(\alpha)2^{-n}$. *Assume that* $\varepsilon_n(\alpha) = 1$ *holds infinitely often. Let* $H = \{h_1 < h_2 < \dots\}$ *be an infinite sequence of integers. Then* $P(\{a_n : n \in \mathbb{N}\} \cup H)$ *contains arbitrarily large intervals.*

This lemma is essentially Theorem 3 in [6].

Lemma 3. *Let* $X = \{x_1 < x_2 < \dots\}$ *be an infinite sequence of integers. Assume that there exists a positive integer* $T \geq x_1$ *such that for every* n

$$x_{n+1} \leq x_1 + x_2 + \dots + x_n + T. \tag{3}$$

Let $P(X) = \{y_1 < y_2 < \dots\}$. *Then* $P(X)$ *has bounded gaps, in fact*

$$y_{n+1} - y_n \leq T$$

for $n = 1, 2, \dots$.

Proof of Lemma 3. Clearly, we only have to show that for every n the longest gap of $P(\{x_1, \dots, x_n\})$ in the interval $[1, s_n]$, where $s_n = x_1 + \dots + x_n$, is at most T. We prove it by induction on n. For $n = 1$ this is the condition for x_1. Let now $n > 1$. By the inductive hypothesis the longest gap in $[1, s_{n-1}]$ is at most T, so we are going to seek the longest gap in $[s_{n-1}, s_n]$. Since $x_n < s_{n-1} + T$, and by

$$P(\{x_1, \dots, x_n\}) = \{0, x_n\} + P(\{x_1, \dots, x_{n-1}\}),$$

the longest gap in $[1, s_n]$ is at most $\max\{T, \text{the longest gap in} [1, s_{n-1}]\} = T$. □

Lemma 3 is proved.

Corollary. *Let* $X = \{x_1 < x_2 < \dots\}$ *be an infinite sequence of integers. Assume that there exists a positive integer* N *such that for every* $n > N$

$$x_{n+1} \leq x_1 + \dots + x_n.$$

Then $P(X)$ *has bounded gaps.*

Lemma 4. *Let* $x_1, x_2, \dots, x_r$ *be any* r *integers. Then there is a* $y > 0$, $y \in P(\{x_1, \dots, x_r\})$ *for which* $r | y$.

Lemma 4 is well-known.

3. Proof of the Theorem

Let us assume that there exists an integer N, such that for $n > N$ $\varepsilon_n(\alpha) = 0$. By Lemma 1 we get

$$P(\{b_1 a_N, b_1\cdot 2a_N, b_1\cdot 4a_N, \dots\}) = \left\{b_1 a_N \sum_{i=1}^{\infty} \delta_i 2^i : \delta_i \in \{0,1\}, \sum \delta_i < \infty\right\}$$
$$= \{b_1 a_N \cdot n : n \in \mathbb{N}\} \subset P(A(\alpha, B)),$$

so $A(\alpha, B)$ is subcomplete in this case.

Now let us assume $|\{n : \varepsilon(\alpha) = 1\}| = \infty$. Let $r = b_1$ and let us consider the integers $b_2, b_3, \dots, b_{r+1}$. By Lemma 4 we can select integers $b_{i_1}, b_{i_2}, \dots, b_{i_t}$, $1 \le t \le r$ for which

$$r | b_{i_1} + b_{i_2} + \dots + b_{i_t}. \tag{4}$$

Let

$$h_n = \frac{b_{i_1} + b_{i_2} + \dots + b_{i_t}}{r} \cdot a_n,$$

$H = \{h_1 < h_2 < \dots\}$.

Then by Lemma 2 $P(\{a_n\} \cup H)$ contains arbitrarily large intervals. By (4)

$$P(\{b_1 a_n\} \cup \{(b_{i_1} + b_{i_2} + \dots, b_{i_t}) \cdot a_n\}) = b_1 \cdot P(\{a_n\} \cup H), \tag{5}$$

thus for every K there is an integer z such that

$$\{z, z + b_1, z + 2b_1, \dots, z + Kb_1\} \subset b_1 \cdot P(\{a_n\} \cup H) \subset P(A(\alpha, B)). \tag{6}$$

Now let us consider the elements $b_{r+2}, b_{r+3}, \dots, b_{2r+1}$ and $b_{2r+2}, b_{2r+3}, \dots, b_{3r+1}$. By Lemma 4 we can choose elements

$$\{b_{j_1}, b_{j_2}, \dots, b_{j_s}\} \subseteq \{b_{r+2}, b_{r+3}, \dots, b_{2r+1}\}$$

and

$$\{b_{k_1}, b_{k_2}, \dots, b_{k_y}\} \subseteq \{b_{2r+2}, b_{2r+3}, \dots, b_{3r+1}\}$$

for which

$$r | U := \{b_{j_1} + b_{j_2} + \dots + b_{j_s}\} \tag{7}$$

and

$$r|V := \{b_{k_1} + b_{k_2} + \dots, b_{k_y}\}. \tag{8}$$

Clearly $P(\{Ua_n\} \cup \{Va_m\}) \subset P(A(\alpha, B))$.

Lemma 5. *$P(\{Ua_n\} \cup \{Va_m\})$ has bounded gaps.*

Proof of Lemma 5. Let $U' = U/r$, $V' = V/r$. Let $\{U'a_n\} \cup \{V'a_m\} = X = \{x_1 \le x_2 \le \dots \le x_n \le \dots\}$. We are going to show that $P(X)$ has bounded gaps. This implies that $rP(X)$ has also bounded gaps as we wanted. By Lemma 3 we only have to show that there exists N such that for $n > N$

$$x_{n+1} \le x_n + x_{n-1} + \dots + x_1. \tag{9}$$

For brevity let $A = V'\alpha \log_2 \frac{U'}{V'} + 2U' + 2$. Since $\lim_{z\to\infty} 2^z/z = \infty$ we can choose a fixed integer z for which

$$2^z > \min\{z + A/(U'\alpha), V'/(U'\alpha)\}. \tag{10}$$

Let us define s by

$$a_s \le \frac{U'}{V'} a_z \le a_{s+1} \tag{11}$$

and let $x_{n+1} = U'a_z$. By (10) an easy calculation shows that $2^{s-1}\alpha > 1$. By (11), for $n > N$ we have

$$\{U'a_1, U'a_2, \dots, U'a_{z-1}, V'a_1, \dots, V'a_s\} \subset \{x_1, \dots, x_n\}. \tag{12}$$

We claim that

$$U'a_z \le U'a_1 + \dots + U'a_{z-1} + V'a_1 + \dots + V'a_s, \tag{13}$$

which by (12) implies (9). Suppose the claim is false and so

$$\begin{aligned} U'2^z\alpha &\ge U'a_z \ge U'(a_1 + \dots + a_{z-1}) + V'(a_1 + \dots + a_s) \\ &> U'\alpha(1 + 2 + \dots + 2^{z-1} - z) + V'\alpha(1 + 2 + \dots + 2^s - s) \\ &= U'\alpha(2^z - 1 - z) + V'\alpha(2^{s+1} - 1 - s), \end{aligned} \tag{14}$$

hence by (11) and (14)

$$
\begin{aligned}
U'\alpha(z+1) &\geq V'\alpha(2^{s+1}-s-1)\\
&= V'2^{s+1}\alpha - V'\alpha(s+1)\\
&\geq V'a_{s+1} - V'\alpha(s+1)\\
&\geq U'a_z - V'\alpha(s+1).
\end{aligned} \tag{15}
$$

We claim

$$(s+1) \leq \log_2\left(\frac{U'}{V'}\right) + z + 2. \tag{16}$$

Indeed, by (10) and (11) we have

$$2^{s-1}\alpha \leq 2^s\alpha - 1 \leq a_s \leq \frac{U'}{V'}a_z \leq \frac{U'}{V'}2^z\alpha,$$

which implies (16). So by (15)

$$U'a_z - V'\alpha(s+1) \geq U'(2^z\alpha - 1) - V'\Big[\log_2\frac{U'}{V'} + z + 2\Big],$$

which yields $2^z \leq z + A/(U'\alpha)$. This contradicts (10) which proves the lemma.

Let $P(\{Ua_n\} \cup \{Va_m\}) = \{y_1 < y_2 < \dots y_n < \dots\}$. Lemma 5 yields that there is a positive integer L for which

$$y_{n+1} - y_n \leq L; n \in \mathbb{N} \tag{17}$$

and

$$r|y_n. \tag{18}$$

Now by (6) there is an integer z for which

$$\{z, z+b_1, z+2b_1, \dots, z+Lb_1\} \subset b_1P(\{a_n\} \cup H). \tag{19}$$

Our last step is to find an infinite arithmetic progression in $P(A(\alpha, B))$.

We claim $\{(y_1+z)+kb_1\}_{k=1}^{\infty} \subset P(A(\alpha, B))$. Let us consider the element $(y_1+z)+mb_1$ and let us define n by

$$y_n \leq y_1 + z + mb_1 < y_{n+1}. \tag{20}$$

By (7), (8) and (20) we conclude that

$$m + \frac{y_1 - y_n}{b_1} \in \mathbb{N}$$

and

$$0 \le m + \frac{y_1 - y_n}{b_1} \le L.$$

By (19) we have $z + b_1\Big(m + \frac{y_1 - y_n}{b_1}\Big) \in b_1 P(\{a_n\} \cup H)$. This implies

$$(y_1 + z) + mb_1 = y_n + z + b_1\Big(m + \frac{y_1 - y_n}{b_1}\Big) \in P(A(\alpha, B)),$$

as we wanted. $A(\alpha, B)$ is subcomplete and this esthablishes our theorem.

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