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The domain of null controllability for some nonlinear control systems

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Abstract. In this paper we are concerned with the description of the domain of null controllability for nonlinear control systems. Two cases are considered and a theorem for each case is given with proof.

1. Introduction

Consider the nonlinear control systems

$$x^{(n)} = f(x, \dot{x}, \dots, x^{(n-1)}, u), \tag{1}$$

where $\dot{x} = \frac{dx}{dt}$, and the control function u belongs to the set of measurable functions taking values in some closed bounded subset Ω of the real line.

The aim of this work is to obtain a description of the domain of null controllability for the control system (1) in two cases. The first case is when the function f is independent of $x, \dot{x}, \dots, x^{(n-3)}, x^{(n-1)}$, i.e. when f depends on $x^{(n-2)}$. The second case is when f is independent of $x, \dot{x}, \dots, x^{(n-2)}$, i.e. f depends on $x^{(n-1)}$. Each case is covered by a theorem with its proof.

By the domain of null controllability we mean the set of all initial points which can be steered to the origin by some admissible controls.

Some motivations for choosing this kind of work: Firstly the linear control system

$$\dot{z} = Az + Bu \tag{2}$$

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has received a great attention during the study of control theory and many concepts for it have been studied extensively such as controllability, observability, reachability, the set of reachability and the domain of null controllability when the state and control spaces are finite or infinite dimensional spaces, for example see [1–4] and [6–12]. Secondly the work with nonlinear systems becomes very interesting after overcoming the analytical difficulties and finally this work is more general than the work with linear systems which can be considered as a special case.

2. The first case

In this case we prove the following theorem when f is independent of $x, \dot{x}, \dots, x^{n-3}, x^{n-1}$, i.e. when f depends on x^{n-2}.

Theorem 1. *Consider the control system* (1) *which can be written in the form*

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dots,\ \dot{x}_{n-1} = x_n,\ \dot{x}_n = f(x_{n-1}, u), \tag{3}$$

where $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $\dots$, $x_n = x^{(n-1)}$. *Let*

$$\begin{aligned} \phi(x_{n-1}) &= \inf\{f(x_{n-1}, \omega) : \omega \in \Omega\}, \\ \psi(x_{n-1}) &= \sup\{f(x_{n-1}, \omega) : \omega \in \Omega\}. \end{aligned} \tag{4}$$

Suppose ϕ *and* ψ *are continuous,* ϕ *is negative and* ψ *is positive with* $\phi(0) < 0$, $\psi(0) > 0$ *and* $f(x_{n-1}, \Omega)$ *is connected for all* x_{n-1}. *Then the domain of null controllability, denoted by* D_1, *is*

$$D_1 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : m(x_{n-1}) < x_n < M(x_{n-1})\}, \tag{5}$$

where $m(x_{n-1})$ *and* $M(x_{n-1})$ *are given by*

$$\begin{aligned} m(x_{n-1}) &= -\left\{2\int_{-\infty}^{x_{n-1}} \psi(s)ds\right\}^{\frac{1}{2}}, \\ M(x_{n-1}) &= \left\{-2\int_{x_{n-1}}^{\infty} \phi(s)ds\right\}^{\frac{1}{2}}. \end{aligned} \tag{6}$$

Proof. The system (3) under the conditions described above is locally null controllable and one can find some positive integer δ_1 such that the sphere $S(\mathbf{0}, \delta_1)$ centered at the origin and of radius δ_1 lies in D_1.

Suppose that $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ with $a_{n-1} > 0$ and $-\delta_1 < a_n < 0$. Given the conditions mentioned above in the theorem, we can find a measurable function $\omega(x_{n-1})$ such that $f(x_{n-1}, \omega(x_{n-1}) = 0$.

Using FILIPPOV's lemma [5], we can choose such a function $\omega(x_{n-1}) \in \Omega$ for any x_{n-1}. The solution of (3) with u replaced by ω is given by

$$(7)\qquad \begin{cases} x_1(t) = \dfrac{1}{n!} a_n t^n + \dfrac{1}{(n-1)!} a_{n-1} t^{n-1} + \cdots + a_2 t + a_1, \\ x_2(t) = \dfrac{1}{(n-1)!} a_n t^{n-1} + \dfrac{1}{(n-2)!} a_{n-1} t^{n-2} + \cdots + 2a_3 t + a_2, \\ \quad\vdots \\ x_n(t) = a_n. \end{cases}$$

A similar argument applies to points $(a_1, a_2, \dots, a_n)$ with $a_{n-1} \leq 0$ and $0 < a_n < \delta_1$. Thus we have $D \subseteq D_1$ and

$$\begin{aligned} D = &\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_{n-1} \geq 0,\ -\delta_1 < x_n < 0\} \\ &\cup \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_{n-1} \geq 0,\ 0 < x_n < \delta_1\} \cup S(\mathbf{0}, \delta_1). \end{aligned}$$

Consider the points $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ such that $0 \leq a_n \leq M(a_{n-1})$. Let h be some measurable function for which there is some $B > a_{n-1}$ such that $\int_{a_{n-1}}^{B} h(s)ds + \frac{a_n^2}{2} = 0$ and the smallest such B also satisfies $h(B) < 0$.

Now we want to show that the solution of

$$(8)\qquad \dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dots,\ \dot{x}_n = h(x_{n-1})$$

with the initial conditions $x_1(0) = a_1$, $x_2(0) = a_2, \dots, x_n(0) = a_n$ reaches the set D. From the last two equations in (8), we get the differential equation

$$x_n \frac{dx_n}{dx_{n-1}} = h(x_{n-1}),$$

which has the solution

$$(9)\qquad \frac{1}{2}(x_n^2 - a_n^2) = \int_{a_{n-1}}^{x_{n-1}} h(s)ds.$$

Since $\int_{a_{n-1}}^{B} h(s)ds = -\frac{a_n^2}{2}$, while for $x_{n-1} \in [a_{n-1}, B]$, $\int_{a_{n-1}}^{B} h(s)ds + \frac{a_n^2}{2} > 0$, the solution of (8) is invariant under reflection through the x_{n-1}-axis and passes through the point $(0, 0, \ldots, B, 0)$. Since $a_n < M(a_{n-1})$, we get

$$a_n^2 < -2 \int_{a_{n-1}}^{\infty} \phi(s)ds, \tag{10}$$

and so we can find some division K of the interval $[a_{n-1}, N]$ such that the upper associated with K of the function ϕ is equal to $-\frac{a_n^2}{2}$. It is also no real restriction to suppose that in the final interval of K, $\sup \phi(x_{n-1}) < 0$. Now we can construct the required function $h(x_{n-1})$. It will be a piecewise constant function defined on $[a_{n-1}, N]$ which takes the values

$$\sup\{\phi(x_{n-1}) : x_{n-1} \in I\}$$

on every interval I of K. Then, by the choice the upper sum, the associated with K of the function ϕ is the same as $\int_{a_{n-1}}^{N} h(s)ds$ and so $\int_{a_{n-1}}^{N} h(s)ds + \frac{a_n^2}{2} = 0$. It is clear that N is the smallest point ($> a_{n-1}$) for which this holds.

Again, using Filippov's lemma with the conditions stated in the theorem, we can find a function $\omega(x_{n-1})$ taking values in Ω such that

$$f(x_{n-1}, \omega(x_{n-1})) = h(x_{n-1}).$$

The function $u(t) = \omega(x_{n-1}(t))$ is measurable and takes values in Ω. Hence by the above discussion and comments this control steers the points $(a_1, a_2, \ldots, a_n)$ into the set D.

Conversely, suppose that $(a_1, a_2, \ldots, a_n)$ and $a_n \geq M(a_{n-1})$, then we can find some ν such that

$$\int_{a_{n-1}}^{\infty} \phi(s)ds + \frac{a_n^2}{2} - \nu = 0. \tag{11}$$

Now let u be any admissible control and consider the solutions of (8) which lie in the upper halfspace. This means that $x_n > 0$ and x_{n-1} is strictly monotonic. Thus there is always a unique x_{n-1} which allows us to regard u as a function of x_{n-1}.

Let us compare the solutions of the two systems

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \ldots,\ \dot{x}_{n-1} = x_n,\ \dot{x}_n = f(x_{n-1}, u(x_{n-1})), \tag{12}$$

and

$$\dot{\alpha}_1 = \alpha_2,\, \dot{\alpha}_2 = \alpha_3,\, \dots,\, \dot{\alpha}_{n-1} = \alpha_n,\, \dot{\alpha}_n = \phi(\alpha_{n-1}), \tag{13}$$

where these systems (12) and (13) have the same initial conditions $(a_1, a_2, \dots, a_n)$. From the last two equations in (12), we get

$$\frac{d(x_n^2/2)}{dx_{n-1}} = f(x_{n-1}, u(x_{n-1})), \tag{14}$$

and from the last two equations in (13), we get

$$\frac{d(\alpha_n^2/2)}{d\alpha_{n-1}} = \phi(\alpha_{n-1}). \tag{15}$$

We always have $f(x_{n-1}, u(x_{n-1})) \geq \phi(\alpha_{n-1})$ and so by comparison we obtain

$$\alpha_n^2(\alpha_{n-1}) \leq x_n^2(x_{n-1}).$$

Since we are in the upper halfspace, it follows that

$$\alpha_n(\alpha_{n-1}) \leq x_n(x_{n-1}).$$

However, the solution of (15) is described by the curve

$$\int_{\alpha_{n-1}}^{\infty} \phi(s)ds + \alpha_n - \nu = 0.$$

Hence the slolution of (12) with the initial conditions $(a_1, a_2, \dots, a_n)$ does not enter the set $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 < x_n < M(x_{n-1})\}$. This argument covers the upper halfspace. A similar argument covers the lower halfspace. If $(0, 0, \dots, x_{n-1}, 0) \in \mathbb{R}^n$ and there is some ω in Ω such that $f(x_{n-1}, \omega) = 0$, then the constant control $u(t) = \omega$ will leave the x_{n-1}-axis and if t is small enough it will stay in the open set described. On the other hand, if $f(x_{n-1}, \Omega) = 0$ then the point $(0, 0, \dots, x_{n-1}, 0)$ will remain unaltered by any admissible control and so will not belong to D_1.

3. The second case

In this case we state and prove the following theorem where f is independent of $x, \dot{x}, \dots, x^{(n-2)}$ i.e. f depends on $x^{(n-1)}$.

Theorem 2. *Consider the control system* (1) *which can be written in the form*

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \ldots,\ \dot{x}_n = f(x_n, u), \tag{16}$$

where $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $\ldots$, $x_n = x^{(n-1)}$. *Let*

$$\xi(x_n) = \inf\{f(x_n, \omega) : \omega \in \Omega\}, \quad \eta(x_n) = \sup\{f(x_n, \omega) : \omega \in \Omega\}. \tag{17}$$

In this case the domain of null controllability, denoted by D_2, *is*

$$D_2 = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : q < x_n < Q\}, \tag{18}$$

where q *is the largest negative root of* $\eta = 0$, Q *is the smallest positive root of* $\xi = 0$ *and* i) ξ *and* η *are continuous,* ii) 0 *lies in the interior of* $f(0, \Omega)$.

Proof. Suppose that $S(\mathbf{0}, \delta_2)$ is an open sphere centered at the origin and of radious δ_2 which lies in D_2. Now f is independent of $x_1, x_2, \ldots, x_{n-1}$. Thus we can translate these by any real numbers $a_1, a_2, \ldots, a_{n-1}$ without altering either the conditions in the hypothesis or the nature of the equation. In this way we can see that all the points in the sphere $S((a_1, a_2, \ldots, a_{n-1}), \delta_2)$ may be steered to $(a_1, a_2, \ldots, a_{n-1}, 0)$ by any admissible control. It is easy to show that the set $\{(x_1, x_2, \ldots, x_{n-1}) : |x_n| < \delta_2\}$ is a subset of D_2. Let $(a_1, a_2, \ldots, a_{n-1}, b)$ be any point in $\mathbb{R}^n$ such that $0 < b < Q$. The theorem will be proved if we show that $(a_1, a_2, \ldots, a_{n-1}, b) \in D_2$.

Consider the equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \ldots,\ \dot{x}_n = \xi(x_n) \tag{19}$$

with the initial conditions $x_1(0) = a_1$, $x_2(0) = a_2$, $\ldots$, $x_n(0) = b$. If we are in the upper halfspace, we conclude that $\dot{x}_{n-1} = x_n > 0$ and so x_{n-1} is monotonically increasing and for $0 < x_n < Q$, $x_n = \xi(x_n)$ it is negative, so $x_n(t)$ is monotonically decreasing. We can see that $\xi(x_n) \leq r \leq 0$ for $x_n \in [0, b]$ where $r = \sup\{\xi(x_n) : x_n \in [0, b]\}$. Now, from the last two equations in (19), we get

$$\frac{dx_n}{dx_{n-1}} = \frac{\xi(x_n)}{x_n},$$

i.e.

$$\frac{d(x_n^2/2)}{dx_{n-1}} = \xi(x_n), \tag{20}$$

by comparing the solution of this equation with the solution of the equation

$$\frac{d(x_n^2/2)}{dx_{n-1}} = r, \tag{21}$$

which is given in the form

$$\frac{1}{2}x_n^2 = rx_{n-1} + \mu_1,$$

where μ_1 is a constant. By using the initial conditions, we have

$$x_n^2 - b^2 = 2r(x_{n-1} - a_{n-1}). \tag{22}$$

Also, from the $(n-2)$-th and the n-th equations in (19), we have

$$\frac{dx_n}{dx_{n-2}} = \frac{\xi(x_n)}{x_{n-1}}. \tag{23}$$

Substituting from (22) into (23), we get

$$\frac{dx_n}{dx_{n-2}} = \frac{2r\xi(x_n)}{x_n^2 - b + 2ra_{n-1}}, \tag{24}$$

by comparing the solution of the equation (24) with the solution of

$$\frac{dx_n}{dx_{n-2}} = \frac{2r\xi(x_n)}{x_n^2 - b + 2ra_{n-1}},$$

which is given in the form

$$x_n^3 - 3(b - 2ra_{n-1})x_n = 6r^2x_{n-2} + 3\mu_2,$$

where μ_2 is a constant. Again by using the initial conditions, we have

$$x_n^3 - 3(b - 2ra_{n-1})x_n = 6r^2x_{n-2} + b^3 - 3(b - 2ra_{n-1})b - 2r^2a_{n-2}. \tag{25}$$

Now $\xi(x_n) \le r$, $0 \le x_n \le b$ and for $t \ge 0$, the solutions of the differential equations discussed above must lie in the upper halfspace under the branch of the curves of the intersection of the surfaces obtained above. From this and the comments on the monotonicity of $x_{n-1}(t)$ and $x_n(t)$ we conclude that the above solutions must enter the set $\{(x_1, x_2, \dots, x_n) : |x_n| < \delta_2\}$.

Again using Filippov's lemma, we can find a measurable function $\omega : \mathbb{R} \to \Omega$ taking values in Ω such that $f(x_n, \omega(x_n)) = \xi(x_n)$. The composition $u(t) = \omega(x_n(t))$ is also measurable, i.e. we have found an admissible control which steers $(a_1, a_2, \ldots, a_{n-1}, b)$ into the set $\{(x_1, x_2, \ldots, x_n) : |x_n| < \delta_2\}$ from where it may be steered to the origin.

Conversely, suppose that the point $(a_1, a_2, \ldots, a_{n-1}, b) \in \mathbb{R}^n$ is such that $b \geq Q$. Then

$$\xi(Q) = \inf\{(Q, \omega) : \omega \in \Omega\} = 0 \tag{26}$$

which means that no solution of (16) can go below $x_n = Q$.

This argument covers the upper halfspace $x_n > 0$ in $\mathbb{R}^n$. We can reason in the same way for the lower halfspace $x_n < 0$ to make the proof complete.

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