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Operators obeying a-Weyl's theorem

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Abstract. This article treatises several problems relevant to a-Weyl's theorem for bounded operators on Banach spaces. There are presented sufficient conditions for an operator T, such that a-Weyl's theorem holds for T. If a-Weyl's theorem holds for an a-isoloid operator T, and F is a finite rank operator commuting with T, then a-Weyl's theorem holds for $T+F$. The algebraic view point for a-Weyl's theorem is considered in the sense of the spectral mapping theorem for a special part of the spectrum. If T^* is a quasihyponormal operator on a Hilbert space, f is a regular function in a neighbourhood of the spectrum of T and f is not constant on the connected components of its domain, we prove that a-Weyl's theorem holds for $f(T)$. The article also contains some related results.

1. Introduction

In this article we only consider bounded operators on a complex infinite-dimensional Banach space X. We use I to denote the identity operator on X, and $K(X)$ to denote the ideal of all compact operators on X. For an arbitrary operator T on X, $\mathcal{N}(T)$ denotes its kernel and $\mathcal{R}(T)$ denotes its image. We set $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim X/\mathcal{R}(T)$. Also, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ denote the sets of Fredholm and semi-Fredholm operators on X respectively. For a semi-Fredholm operator T we define the index $i(T) = \alpha(T) - \beta(T)$. Let us consider two classes of operators: $\Phi_0(T) = \{T \in \Phi(X) : i(T) = 0\}$ is the set of Weyl operators on X, and $\Phi_+^-(X) = \{T \in \Phi_+(X) : i(T) \le 0\}$ which is introduced in [11]. It is well-known that the sets $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_0(X)$ and $\Phi_+^-(X)$ form multiplicative semigroups in $B(X)$.

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For a subset V of an arbitrary topological space, $\overline{V}$ denotes the closure of V. Let $\mathbb{C}$ denotes the complex plane. If S is a subset of $\mathbb{C}$, then $\operatorname{iso} S$ denotes the set of all isolated points of S and $\operatorname{acc} S$ denotes the set of all points of accumulation of S.

We use $\sigma(T)$ and $\sigma_a(T)$ to denote the spectrum and the approximate point spectrum of T respectively. The Weyl spectrum of T is

$$\sigma_w(T) = \bigcap_{K \in K(X)} \sigma(T+K) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_0(X)\}.$$

The essential approximate point spectrum of T is (see [11]):

$$\sigma_{ea}(T) = \bigcap_{K \in K(X)} \sigma_a(T+K) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+^-(X)\}.$$

Recall that all of these spectra are compact non-empty subsets of $\mathbb{C}$. Also, we use the following notation: $\pi_{00}(T) = \{\lambda \in \mathbb{C} : \lambda \in \operatorname{iso} \sigma(T)$ and $0 < \alpha(T - \lambda I) < \infty\}$, $\pi_{a0}(T) = \{\lambda \in \mathbb{C} : \lambda \in \operatorname{iso} \sigma_a(T)$ and $0 < \alpha(T - \lambda I) < \infty\}$. The set $\pi_{00}(T)$ (respectively $\pi_{a0}(T)$), consists of all isolated eigenvalues of $\sigma(T)$ ($\sigma_a(T)$) of finite geometric multiplicity.

A complex function f belongs to the set $\operatorname{Hol}(T)$, if f is regular in a neighbourhood of $\sigma(T)$ and f is not constant on the connected components of its domain of definition.

The following terminology may be found in [8], [9], [10], [11] and [13]. We say that Weyl's theorem holds for T provided that $\sigma_w(T) = \sigma(T) \backslash \pi_{00}(T)$, and a-Weyl's theorem holds for T provided that $\sigma_{ea}(T) = \sigma_a(T) \backslash \pi_{a0}(T)$. If T obeys a-Weyl's theorem, then it obeys Weyl's theorem and the converse is not true [13].

The paper is organized as follows.

In Section 2 we set $\Omega(T) = \{\lambda \in \mathbb{C} : \mathcal{R}(T - \lambda I)$ is closed$\}$ and find sufficient conditions for an operator T such that the next equality holds:

$$(\sigma_a(T) \backslash \sigma_{ea}(T)) \cap \Omega(T) = \pi_{a0}(T) \cap \Omega(T).$$

In Section 3 we prove a perturbation theorem for a-Weyl's theorem. Namely, if a-Weyl's theorem holds for an a-isoloid operator T, and if F is a finite rank operator commuting with T, then we prove that a-Weyl's theorem holds for $T + F$.

In Section 4 we consider primitive Banach algebras and the R-spectrum $\sigma_R(t)$ in the sense of KORDULA and MÜLER [7]. We prove the spectral mapping theorem for the set $\sigma_R(t)\backslash\pi_R(t)$, where $\pi_R(t)$ consists of all isolated points of $\sigma_R(t)$ which are eigenvalues of t of finite geometric multiplicity.

In Section 5 we consider quasihyponormal operators on a Hilbert space. If T^* is a quasihyponormal operator, we prove that a-Weyl' theorem holds for $f(T)$, provided that f is a regular function in a neighbourhood of $\sigma(T)$ and f is not constant on the connected components of its domain.

2. Sufficient conditions for a-Weyl's theorem

We begin with the following useful statement.

Lemma 2.1. *If $\lambda \in \pi_{a0}(T)$ and $\mathcal{R}(T-\lambda)$ is closed, then*

$$\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T).$$

PROOF. If $\lambda \in \pi_{a0}(T)$, then $\lambda \in \operatorname{iso}\sigma_a(T)$ and $0 < \alpha(T-\lambda I) < \infty$. Since $\mathcal{R}(T-\lambda I)$ is closed, we get that $T-\lambda I \in \Phi_+(X)$. Also, there exists a number $\epsilon > 0$, such that for all $\mu \in \mathbb{C}$, if $0 < |\lambda-\mu| < \epsilon$ then $\alpha(T-\mu I) = 0$ and $\mathcal{R}(T-\mu I)$ is closed, so $T-\mu I \in \Phi_+^-(X)$. By the continuity of the index, we get that $T-\lambda I \in \Phi_+^-(X)$ and $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. $\square$

The next important result will be also used in the proof of our main theorem.

Theorem 2.2. *If $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$, then for all complex numbers $\lambda \neq 0$ we have: $\lambda \in \sigma_a(T)$ if and only if $\lambda \in \sigma_a(T_1)$, where T_1 is the restriction of T to its invariant subspace $\mathcal{R}(T)$.*

PROOF. Since $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$, then it follows from [6, Satz 72.4 and Satz 101.2] that 0 is the pole of the resolvent of T of order equal to 1 or $0 \notin \sigma(T)$. Furthermore, if P is the spectral projection corresponding to $\{0\}$, then

$$\mathcal{N}(T) = \mathcal{R}(P) \quad \text{and} \quad \mathcal{R}(T) = \mathcal{N}(P),$$

hence $\mathcal{R}(T)$ is closed.

If I_0 and I_1 are the identity operators on $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively, we can write $T-\lambda I = (-\lambda I_0)\oplus(T_1-\lambda I_1)$ with respect to the decomposition

$X = \mathcal{N}(T) \oplus \mathcal{R}(T)$. We see that $\mathcal{N}(T - \lambda I) = \mathcal{N}(T_1 - \lambda I_1)$, so $T - \lambda I$ is one-to-one if and only if $T_1 - \lambda I_1$ is one-to-one. Also, $\mathcal{R}(T - \lambda I) = \mathcal{N}(T) \oplus \mathcal{R}(T_1 - \lambda I_1)$. We shall prove that $\mathcal{R}(T - \lambda I)$ is closed if and only if $\mathcal{R}(T_1 - \lambda I_1)$ is closed.

Suppose that $\mathcal{R}(T_1 - \lambda I_1)$ is closed and $x \in \overline{\mathcal{R}(T - \lambda I)}$. Then there exists a sequence (x_n), $x_n \in X$, such that $\lim(T - \lambda I)x_n = x$. Now, $x = u + v$, $x_n = u_n + v_n$, where $u, u_n \in \mathcal{N}(T)$ and $v, v_n \in \mathcal{R}(T)$. Let P be a bounded projection of X onto $\mathcal{N}(T)$, such that $\mathcal{N}(P) = \mathcal{R}(T)$. We get

$$u = Px = P \lim(T - \lambda I)x_n = -\lambda \lim u_n.$$

Now,

$$\begin{aligned} v = x - u &= \lim(T - \lambda I)x_n + \lim(\lambda u_n) \\ &= \lim(T - \lambda I)v_n = \lim(T_1 - \lambda I_1)v_n. \end{aligned}$$

It follows that there exists a vector $z \in \mathcal{R}(T)$, such that $(T - \lambda I)z = (T_1 - \lambda I_1)z = v$. Now, $(T - \lambda I)\left(-\frac{1}{\lambda}u \oplus z\right) = x$ and we get that $\mathcal{R}(T - \lambda I)$ is closed.

Suppose that $\mathcal{R}(T - \lambda I)$ is closed and $x \in \overline{\mathcal{R}(T_1 - \lambda I_1)} \subset \mathcal{R}(T)$. Then there exists a sequence (x_n) in $\mathcal{R}(T)$, such that $\lim(T_1 - \lambda I_1)x_n = x$. It follows that $\lim(T - \lambda I)x_n = x$, so there exists a vector $z \in X$ such that $(T - \lambda I)z = x = 0 \oplus x$. We can find $u \in \mathcal{N}(T)$ and $v \in \mathcal{R}(T)$ such that $z = u + v$. Now, $0 \oplus x = (T - \lambda I)z = -\lambda u \oplus (T_1 - \lambda I_1)v$. Consequently, $0 = -\lambda u$ and $(T_1 - \lambda I_1)v = x$.

The previous consideration shows that $\lambda \in \sigma_a(T)$ if and only if $\lambda \in \sigma_a(T_1)$. □

We say that an operator T is regular (or g-invertible), provided that there exists an operator S, such that $T = TST$. It is well-known that T is regular if and only if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are complemented subspaces of X. An operator T is simply polar, provided that $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$ [5]. If T is simply polar, as in the proof of Theorem 2.2 it follows that $\mathcal{R}(T)$ is closed. Obviously, if T is simply polar, then T is regular.

In the proof of the main Theorem 2.3 we shall also use the essential Browder approximate point spectrum of T (see [12]), defined as follows:

$$\sigma_{ab}(T) = \bigcap_{\substack{K \in K(X) \\ TK = KT}} \sigma_a(T + K) = \sigma_{ea}(T) \cup \operatorname{acc} \sigma_a(T). \tag{1}$$

Recall that $\sigma_{ab}(T)$ is non-empty compact subset of $\mathbb{C}$ for all bounded operators T on X [12].

Theorem 2.3. *Suppose that T is simply polar and suppose that for an arbitrary finite dimensional T-invariant subspace M of $\mathcal{R}(T)$ there exists a closed T-invariant subspace N of $\mathcal{R}(T)$, such that $M \oplus N = \mathcal{R}(T)$. Then*

$$\sigma_a(T) \backslash \sigma_{ea}(T) \subset \pi_{a0}(T).$$

Proof. Suppose that $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. We consider two cases:

Case I. If $\lambda = 0$, then $T \in \Phi_+^-(X)$ and $0 < \alpha(T) < \infty$. We have to prove that 0 is an isolated point of $\sigma_a(T)$. By (1) it is enough to prove that $0 \notin \sigma_{ab}(T)$. Let P be a continuous projection of X onto $\mathcal{N}(T)$, such that $\mathcal{N}(P) = \mathcal{R}(T)$. It follows that $P \in K(X)$ and we shall prove that $TP = PT$ and $0 \notin \sigma_a(T + P)$. Let $x = u + v$, where $u \in \mathcal{N}(T)$ and $v \in \mathcal{R}(T)$. Then $TPx = TPu = 0 = PTx$, so P and T mutually commute. If $(T + P)x = 0$, then $u = Pu = -Tv$, where $u \in \mathcal{N}(T)$ and $-Tv \in \mathcal{R}(T)$, so we get that $u = 0$ and $v = 0$. Since $T \in \Phi_+^-(X)$, it follows that $T+P \in \Phi_+^-(X)$, so $\mathcal{R}(T+P)$ is closed. Consequently, $0 \notin \sigma_a(T+P)$, $0 \notin \sigma_{ab}(T)$ and $0 \in \operatorname{iso}\sigma_a(T)$. It follows that $0 \in \pi_{a0}(T)$.

Case II. Now, suppose that $\lambda \neq 0$. We get that $T = 0 \oplus T_1$ with respect to the decomposition $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$, where T_1 is the restriction of T to $\mathcal{R}(T)$. Since $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$, we get that $T - \lambda I \in \Phi_+^-(X)$, so $\mathcal{R}(T - \lambda I)$ is closed and $0 < \alpha(T - \lambda I) < \infty$. By Theorem 2.2 and its proof we get that $\lambda \in \sigma_a(T_1)$, $\mathcal{R}(T_1 - \lambda I_1)$ is closed, $0 < \alpha(T_1 - \lambda I_1) < \infty$ and $i(T_1 - \lambda I_1) = i(T - \lambda I) \leq 0$, so $T_1 - \lambda I_1 \in \Phi_+^-(\mathcal{R}(T))$. There is a neighbourhood $U(\lambda)$ of λ, such that $0 \notin U(\lambda)$. So for all $\mu \in U(\lambda)$, using Theorem 2.2 we get that $\mu \in \sigma_a(T)$ if and only if $\mu \in \sigma_a(T_1)$, so $\lambda \in \operatorname{acc}\sigma_a(T)$ if and only if $\lambda \in \operatorname{acc}\sigma_a(T_1)$. To prove that $\lambda \in \pi_{a0}(T)$, it is enough to prove that $\lambda \in \operatorname{iso}\sigma_a(T)$, or $\lambda \in \operatorname{iso}\sigma_a(T_1)$. To prove that $\lambda \in \operatorname{iso}\sigma_a(T_1)$, it is enough to prove that $\lambda \notin \sigma_{ab}(T_1)$.

We shall use the similar method as the one in the Case I. Since $\mathcal{N}(T_1 - \lambda I_1)$ is the finite dimensional eigenspace of T_1, there exists a closed T_1-invariant subspace M, such that $\mathcal{R}(T) = \mathcal{N}(T_1 - \lambda I_1) \oplus M$. Let Q be the continuous projection of $\mathcal{R}(T)$ onto $\mathcal{N}(T_1 - \lambda I_1)$, such that $\mathcal{N}(Q) = M$. It is obvious that $Q \in K(\mathcal{R}(T))$. We have to prove that $QT_1 = T_1Q$ and $\lambda \notin \sigma_a(T_1 + Q)$.

Suppose that $x = u + v$, such that $u \in \mathcal{N}(T_1 - \lambda I_1)$ and $v \in M$. Then

$$QT_1x = QT_1u + QT_1v = QT_1u = -\lambda u = T_1Qu + T_1Qv = T_1Qx.$$

The second equality follows from $T_1 v \in M = \mathcal{N}(Q)$, and the fourth equality follows from $v \in M = \mathcal{N}(Q)$. Since $T_1 - \lambda I_1 \in \Phi_+^-(\mathcal{R}(T))$, it is well-known that $T_1 + Q - \lambda I_1 \in \Phi_+^-(\mathcal{R}(T))$, so $\mathcal{R}(T_1 + Q - \lambda I_1)$ must be closed in $\mathcal{R}(T)$. We only have to prove that $T_1 - \lambda I_1 + Q$ is one-to-one. Suppose that $x = u + v$, $u \in \mathcal{N}(T_1 - \lambda I_1)$, $v \in M$ and $(T_1 - \lambda I_1 + Q)x = 0$. Since $(T_1 - \lambda I_1)u = 0$, $Qu = u$ and $Qv = 0$, we get that $u = -(T_1 - \lambda I_1)v$. Since $u \in \mathcal{N}(T_1 - \lambda I_1)$ and $-(T_1 - \lambda I_1)v \in M$, we get that $u = 0$, so $v \in M \cap \mathcal{N}(T_1 - \lambda I_1)$, $v = 0$ and $x = 0$. Consequently $\lambda \notin \sigma_a(T_1 + Q)$, $\lambda \notin \sigma_{ab}(T_1)$, so $\lambda \in \operatorname{iso} \sigma_a(T_1)$, $\lambda \in \operatorname{iso} \sigma_a(T)$ and $\lambda \in \pi_{a0}(T)$. $\square$

Now, following [8], we introduce the next notation: $\Omega(T) = \{\lambda \in \mathbb{C} : \mathcal{R}(T - \lambda I) \text{ is closed}\}$. Note that $T - \lambda I$ is assumed to be regular in [8], which is much stronger condition in Banach spaces. The next corollary follows from Lemma 2.1 and Theorem 2.3.

Corollary 2.4. *Suppose that the conditions from Theorem 2.3 are valid for T. Then*

$$(\sigma_a(T) \backslash \sigma_{ea}(T)) \cap \Omega(T) = \pi_{a0}(T) \cap \Omega(T).$$

3. Perturbations by a commuting finite rank operator

If a-Weyl's theorem holds for T, T is a-isoloid and F is a finite rank operator commuting with T, then we prove that a-Weyl's theorem holds for $T + F$. This problem for Weyl's theorem is proposed in [10] and solved in [9]. It is convenient to give the proof of the next statement.

Lemma 3.1. *If $\alpha(T) = n$ and $\dim \mathcal{R}(F) = m$, then*

$$\alpha(T + F) \leq n + m,$$

where m and n are nonnegative integers.

Proof. We have that $X = \mathcal{N}(T) \oplus M$ for a closed subspace M of X. Notice that the restriction $T|_M$ is one-to-one. Let $W = \{v \in M : Tv \in \mathcal{R}(F)\}$. Since $T|_M$ is one-to-one, we get that $\dim W \leq m$, and $\dim(\mathcal{N}(T) \oplus W) \leq n + m$. Now, suppose that $x \in \mathcal{N}(T + F)$. Then $x = u + v$, where $u \in \mathcal{N}(T)$, $v \in M$ and

$$0 = (T + F)(u + v) = Tv + Fx.$$

It follows that $Tv = -Fx \in \mathcal{R}(F)$ and $v \in W$. We get that if $x \in \mathcal{N}(T + F)$, then $x \in \mathcal{N}(T) \oplus W$, so $\alpha(T + F) \leq n + m$. $\square$

The next result is very useful.

Theorem 3.2. *If F is an arbitrary finite rank operator on X, such that $FT = TF$, then for all $\mu \in \mathbb{C}$:*

$$\mu \in \operatorname{acc}\sigma_a(T) \quad \textit{if and only if } \mu \in \operatorname{acc}\sigma_a(T+F).$$

Proof. Firstly, we prove that if T is one-to-one and $TF = FT$, then $\mathcal{R}(F) \subset \mathcal{R}(T)$. Since F is a finite rank operator, there exist two systems: a system of linearly independent vectors $(y_i)_{i=1}^n$, and a system of non-zero bounded linear functionals $(g_i)_{i=1}^n$ on X, such that for all $x \in X$: $F(x) = \sum_{i=1}^n g_i(x)y_i$. Now, we get that $TFx = \sum_{i=1}^n g_i(x)Ty_i$ and $FTx = \sum_{i=1}^n g_i(Tx)y_i$. Since T is one-to-one, we get that $Ty_1, \dots, Ty_n$ are linearly independent, so we get that $\{\sum_{i=1}^n g_i(x)Ty_i : x \in X\} = \operatorname{span}\{y_1, \dots, y_n\} = \operatorname{span}\{Ty_1, \dots, Ty_n\}$, and $\mathcal{R}(F) \subset \mathcal{R}(T)$. Similarly, if $T - \lambda I$ is one-to-one for a number $\lambda \in \mathbb{C}$, then $\mathcal{R}(F) \subset \mathcal{R}(T - \lambda I)$.

Now, suppose that $\mu \notin \operatorname{acc}\sigma_a(T)$. There exists a number $\epsilon > 0$, such that for all $\lambda \in \mathbb{C}$, if $0 < |\lambda - \mu| < \epsilon$ then $\alpha(T - \lambda I) = 0$ and $\mathcal{R}(T - \lambda I)$ is closed. Also, there exists a bounded operator $T_1 : \mathcal{R}(T - \lambda I) \to X$, such that $(\lambda I - T)T_1 = I_{\mathcal{R}(T-\lambda I)}$ and $T_1(\lambda I - T) = I_X$. Notice that $\mathcal{R}(F)$ is a finite dimensional subspace of a Banach space $\mathcal{R}(T - \lambda I)$, so we may find a closed subspace M, such that $\mathcal{R}(F) \oplus M = \mathcal{R}(T - \lambda I)$.

Suppose that $\lambda \in \sigma_a(T + F)$. Then there exists a sequence $(x_n)_n$, $x_n \in X$ and $\|x_n\| = 1$ for all $n \geq 1$, such that $\lim(T + F - \lambda I)x_n = 0$. We can assume that $\lim Fx_n = x \in \mathcal{R}(F)$. Now,

$$0 = \lim T_1(T + F - \lambda I)x_n = \lim(x_n + T_1Fx_n).$$

Since the limit $\lim T_1Fx_n = T_1x$ exists, we get $\lim x_n = -T_1x$. Since $\|x_n\| = 1$, it follows that $x \neq 0$. We verify that $x = \lim Fx_n = -FT_1x \in \mathcal{R}(F)$. Also, $(T - \lambda I)x = -(T - \lambda I)FT_1x = -Fx$ and $(T + F - \lambda I)x = 0$. We get that if $\lambda \in \sigma_a(T+F)$, then λ is an eigenvalue of $T+F$. It is known that eigenvectors corresponding to distinct eigenvalues of $T+F$ are linearly independent. But, we get that all such eigenvectors are contained in the finite dimensional subspace $\mathcal{R}(F)$. It follows that $\sigma_a(T + F)$ may contain only finitely many points λ, such that $0 < |\lambda - \mu| < \epsilon$. We get that $\mu \notin \operatorname{acc}\sigma_a(T + F)$. The opposite implication is analogous. □

Definition 3.3. We say that an operator T is a-isoloid, if all isolated points of $\sigma_a(T)$ are eigenvalues of T.

Recall that an operator T is isoloid, provided that all isolated points of $\sigma(T)$ are eigenvalues of T. Now, $\sigma_a(T)$ contains all isolated points of $\sigma(T)$, so if T is a-isoloid then it is also isoloid.

The next theorem is the main result in this section.

Theorem 3.4. *Suppose that F is an arbitrary finite rank operator and $TF = FT$. If T is a-isoloid and a-Weyl's theorem holds for T, then a-Weyl's theorem holds for $T + F$.*

Proof. It is enough to prove that $0 \in \sigma_a(T+F)\backslash\sigma_{ea}(T+F)$ if and only if $0 \in \pi_{a0}(T+F)$. Firstly we prove the implication $\Longrightarrow$. Now, if $0 \in \sigma_a(T+F)\backslash\sigma_{ea}(T+F)$, then $T+F \in \Phi_+^-(X)$ and $0 < \alpha(T+F) < \infty$. We need to prove that $0 \in \operatorname{iso}\sigma_a(T+F)$. It follows that $T \in \Phi_+^-(X)$, so $0 \notin \sigma_{ea}(T)$. It is possible that $0 \notin \sigma_a(T)$. In this case we get $0 \notin \operatorname{acc}\sigma_a(T)$ and by Theorem 3.2 it follows that $0 \notin \operatorname{acc}\sigma_a(T+F)$, so $0 \in \pi_{a0}(T+F)$. The second possibility is that $0 \in \sigma_a(T)$. Since a-Weyl's theorem holds for T, we get that $0 \notin \operatorname{acc}\sigma_a(T)$ and again $0 \in \pi_{a0}(T+F)$.

To prove the opposite implication $\Longleftarrow$, suppose that $0 \in \pi_{a0}(T+F)$. Then $0 \in \operatorname{iso}\sigma_a(T+F)$ and $0 < \alpha(T+F) < \infty$. By Theorem 3.2 we get $0 \notin \operatorname{acc}\sigma_a(T)$ and by Lemma 3.1 it follows that $0 \leq \alpha(T) < \infty$. Again, we distinguish two cases. Firstly, if $0 \notin \sigma_a(T)$, then $T \in \Phi_+^-(X)$ and $T+F \in \Phi_+^-(X)$, so $0 \in \sigma_a(T+F)\backslash\sigma_{ea}(T+F)$. On the other hand, if $0 \in \sigma_a(T)$ then $0 \in \operatorname{iso}\sigma_a(T)$. Since T is a-isoloid, we get that $0 < \alpha(T) < \infty$ and $0 \notin \sigma_{ea}(T)$. Now, we have $T \in \Phi_+^-(X)$, $T+F \in \Phi_+^-(X)$ and $0 \in \sigma_a(T+F)\backslash\sigma_{ea}(T+F)$. □

4. Spectral mapping theorems in Banach algebras

In this section we shall prove a spectral mapping theorem for a part of the spectrum, which is relevant to Weyl's theorems. Also, we shall use the partial case of Theorem 4.4 to prove an important result in Section 5 (Theorem 5.6). Let $\mathcal{A}$ be a primitive Banach algebra with the identity 1, and let $\operatorname{Min}(\mathcal{A})$ be the set of all minimal idempotents in $\mathcal{A}$. If $t \in \mathcal{A}$ and $e \in \operatorname{Min}(\mathcal{A})$ then $t^\wedge$ denotes the element in $B(\mathcal{A}e)$ defined as $t^\wedge(ae) = tae$ for all $ae \in \mathcal{A}e$. Notice that we use $B(X)$ to denote the set of all bounded operators on the Banach space X. The mapping $t \to t^\wedge$ is called the left regular representation of the primitive Banach algebra $\mathcal{A}$ on the Banach space $\mathcal{A}e$. It is well-known that the rank, nullity and defect of $t^\wedge$ do not depend on the choice of $e \in \mathrm{Min}(\mathcal{A})$ (see [1]), so we write $\alpha(t) = \alpha(t^\wedge)$. Notice that $(ts)^\wedge = t^\wedge s^\wedge$ and $(t-\lambda)^\wedge = t^\wedge - \lambda I$, where I denotes the identity operator on $\mathcal{A}e$. Let $\mathcal{A}^{-1}$ denote the set of all invertible elements in $\mathcal{A}$.

We restate a result which enable us to define the point spectrum of an element in a primitive Banach algebra $\mathcal{A}$.

Lemma 4.1 ([1, Example F.2.2]). *Let X be a Banach space and $T \in B(X)$. Then*

$$\sigma_p(T) = \sigma_p(T^\wedge).$$

Definition 4.2. If $t \in \mathcal{A}$ and $e \in \mathrm{Min}(\mathcal{A})$, then the point spectrum of t is defined by $\sigma_p(t) = \sigma_p(t^\wedge)$, where $t^\wedge \in B(\mathcal{A}e)$.

By [1], Definition 4.2 does not depend on the choice of $e \in \mathcal{A}$ and by Lemma 4.1 it coincides with the usual definition of the point spectrum of a bounded operator on a Banach space. We say that the set $\sigma_p(t)$ consists of eigenvalues of t.

Let $R \neq \emptyset$ be a regularity of $\mathcal{A}$ (see [7]), i.e. R satisfies the following conditions:

(a) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R \iff a^n \in R$;

(b) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1$, then $ab \in R \iff a \in R$ and $b \in R$.

The R-spectrum of $t \in \mathcal{A}$ is defined as follows:

$$\sigma_R(t) = \{\lambda \in \mathbb{C} : t - \lambda \notin R\}.$$

It is well known that if $a \in \mathcal{A}^{-1}$ and $ab = ba$, then $ab \in R$ if and only if $b \in R$, and $\mathcal{A}^{-1} \subset R$. Also, the spectral mapping theorem $f(\sigma_R(t)) = \sigma_R(f(t))$ holds for all $t \in \mathcal{A}$ and $f \in \mathrm{Hol}(t)$. We shall always assume that R is an open regularity of $\mathcal{A}$, so $\sigma_R(t)$ is (possibly empty) compact subset of $\sigma(t)$ [7].

Consider the set $\pi_R(t) = \{\lambda \in \mathbb{C} : \lambda \in \mathrm{iso}\,\sigma_R(t) \text{ and } 0 < \alpha(t-\lambda) < \infty\}$. The set $\pi_R(t)$ consists of all isolated points of $\sigma_R(t)$ which are eigenvalues of t of finite geometric multiplicity. Now, we introduce a general definition inspired by Definition 3.3.

Definition 4.3. We say that $t \in \mathcal{A}$ is R-isoloid, provided that $\operatorname{iso}\sigma_R(t) \subset \sigma_p(t)$, i.e. all isolated points of $\sigma_R(t)$ are eigenvalues of t.

We shall prove the spectral mapping theorem for the set $\sigma_R(t)\backslash\pi_R(t)$. If T is a bounded operator on a Banach space, the analogous problem for the set $\sigma(T)\backslash\pi_{00}(T)$ and polynomials is considered in [10].

Theorem 4.4. *Let R be an open regularity of $\mathcal{A}$, such that $\sigma_R(t) \neq \emptyset$ for all $t \in \mathcal{A}$. If $t \in \mathcal{A}$ is R-isoloid and $f \in \operatorname{Hol}(t)$ is arbitrary, then*

$$\sigma_R(f(t))\backslash\pi_R(f(t)) = f(\sigma_R(t)\backslash\pi_R(t)).$$

Proof. To prove the inclusion $\subset$, let us take $\lambda \in \sigma_R(f(t))\backslash\pi_R(f(t)) \subset f(\sigma_R(t))$ and distinguish three cases.

Case I. If λ is a limit point of $\sigma_R(f(t))$, then λ is also a limit point of $f(\sigma_R(t))$, so there is a sequence (μ_n) in $\sigma_R(t)$, such that $f(\mu_n) \to \lambda$. Now, $\sigma_R(t)$ is compact, so we can take that $\mu_n \to \mu \in \sigma_R(t)$. We get that $\lambda = f(\mu) \in f(\sigma_R(t)\backslash\pi_R(t))$.

Case II. Now, let λ be an isolated point of t, but $\alpha(t-\lambda) = 0$. We have that

$$f(t) - \lambda = (t-\mu_1)\cdots(t-\mu_n)g(t), \tag{2}$$

where $\mu_1, \dots, \mu_n \in \sigma(t)$, elements on the right side of (2) mutually commute and $g(t)$ is invertible. Since $\lambda \in f(\sigma_R(t))$, we know that some μ_{i_0} belongs to $\sigma_R(t)$. Since λ is not an eigenvalue of $f(t)^\wedge$, it follows that non of $\mu_1, \dots, \mu_n$ can be an eigenvalue of $t^\wedge$. Therefore $\lambda = f(\mu_{i_0}) \in f(\sigma_R(t)\backslash\pi_R(t))$.

Case III. Let λ be an isolated eigenvalue of $f(t)$ of infinite geometric multiplicity. Notice that (2) also holds. Since λ is an eigenvalue of $f(t)^\wedge$ of infinite multiplicity, there exists an μ_{i_0}, such that μ_{i_0} is an eigenvalue of $t^\wedge$ of infinite multiplicity. We get that $\lambda = f(\mu_{i_0}) \in f(\sigma_R(t)\backslash\pi_R(t))$.

To prove the inclusion $\supset$, let us take $\lambda \in f(\sigma_R(t)\backslash\pi_R(t)) \subset \sigma_R(f(t))$. Suppose that $\lambda \in \pi_R(f(t))$. Then λ is isolated in $\sigma_R(f(t))$ and (2) holds. If some μ_i is in $\sigma_R(t)$, then μ_i is isolated in $\sigma_R(t)$ and it must be an eigenvalue of t, since t is R-isoloid. Now, λ is an eigenvalue of $f(t)$ of finite multiplicity, so all $\mu_i \in \sigma_R(t)$ are eigenvalues of t of finite multiplicities. We get that all $\mu_i \in \sigma_R(t)$ are also in $\pi_R(t)$. This is in contradiction with the assumption $\lambda \in f(\sigma_R(t)\backslash\pi_R(t))$. □

Remark 4.4. Notice that we can prove the inclusion $\subset$ in Theorem 4.3 assuming that R is an open subset of $\mathcal{A}$, which satisfies the following:

(c) $\mathcal{A}^{-1} \subset R$;

(d) if $a, b \in R$ and $ab = ba$, then $ab \in R$.

Namely, if R satisfies (c) and (d), then the inclusion $\sigma_R(f(t)) \subset f(\sigma_R(t))$ holds for all $t \in \mathcal{A}$ and $f \in \mathrm{Hol}(t)$. In this partial case t need not to be R-isoloid.

In the rest of this section we shall consider a generalization of the Browder spectrum. This part of the paper is not the main object of our investigation, but Theorem 4.7 plays an important role in the Fredholm theory. Till the end of this section we can assume that $\mathcal{A}$ is an arbitrary complex Banach algebra with the identity 1.

Let J be any closed two-sided ideal of $\mathcal{A}$. If $t \in \mathcal{A}$ and $\lambda \in \mathrm{iso}\,\sigma(t)$, let $p = p(\lambda, t)$ denote the spectral idempotent of t, corresponding to λ. Define the set of all isolated points of finite algebraic multiplicity (with respect to J) as:

$$\pi_0(t) = \{\lambda \in \mathbb{C} : \lambda \in \mathrm{iso}\,\sigma(t) \text{ and } p(\lambda, t) \in J\}.$$

We shall prove the spectral mapping theorem for the set $\sigma(t)\backslash\pi_0(t)$.

If T is a bounded operator on a Banach space X, $\mathcal{A} = B(X)$ is the Banach algebra of all bounded operators on X, and $J = K(X)$ is the ideal of all compact operators on X, then $\sigma(T)\backslash\pi_0(T)$ is the Browder spectrum of T. Recall that a projection is a compact operator if and only if it is a finite rank operator. So, we may call the set $\sigma(t)\backslash\pi_0(t)$ the Browder spectrum of an element t in a Banach algebra $\mathcal{A}$. Other generalizations of the Browder spectrum may be found in [1], [5] and [7].

We shall need the following result of Dunford and Schwartz [3, Theorem 19, p. 574] (interpreted for elements of an arbitrary Banach algebra).

Theorem 4.5. *Let $t \in \mathcal{A}$, f is a regular function in neighbourhood of $\sigma(t)$, and let κ be a spectral set of $\sigma(f(t))$. Then $\sigma(t) \cap f^{-1}(\kappa)$ is a spectral set of $\sigma(t)$ and*

$$p(\kappa, f(t)) = p(f^{-1}(\kappa), t).$$

Also, we shall use the next statement.

Lemma 4.6. *Let $\mathcal{A}$ be an algebra and let J be a two-sided ideal of $\mathcal{A}$. If a, b are idempotents in $\mathcal{A}$, such that $a+b \in J$ and $ab = ba$, then $a, b \in J$.*

Proof. Since $ab = ba$, we get $(a+b)^2 = a + 2ab + b \in J$ and $ab \in J$. Now $a(a+b) = a + ab \in J$ and $a \in J$. □

We prove the spectral mapping theorem for the Browder spectrum.

Theorem 4.7. *If $a \in \mathcal{A}$ and $f \in \mathrm{Hol}(a)$, then*

$$\sigma(f(a)) \backslash \pi_0(f(a)) = f(\sigma(a) \backslash \pi_0(a)).$$

Proof. Let $\lambda \in \sigma(f(a)) \backslash \pi_0(f(a)) \subset f(\sigma(a))$. We distinguish two cases.

Case I. Suppose that λ is not an isolated point of $\sigma(f(a))$. Then there exists a sequence (μ_n), $\mu_n \in \sigma(a)$, such that $f(\mu_n) \to \lambda$ and $\mu_n \to \mu_0$. Now $\lambda = f(\mu_0) \in f(\sigma(a) \backslash \pi_0(a))$.

Case II. Suppose that λ is an isolated point of $\sigma(f(a))$, but $p(\lambda, f(a)) \notin J$. We have

$$f(a) - \lambda = (a - \mu_1) \cdots (a - \mu_n) g(a), \tag{3}$$

where $g(a)$ is invertible and all μ_i are isolated points of $\sigma(a)$. By Theorem 4.5, it follows that

$$p(\lambda, f(a)) = p(\{\mu_1, \dots, \mu_n\}, a) = p(\mu_1, a) + \cdots + p(\mu_n, a). \tag{4}$$

If μ is not a point of accumulation of $\sigma(a)$, then it is well-known that $p(\mu, a) = 0$ if and only if $\mu \notin \sigma(a)$. If all idempotents on the right side of (4) are in J, then $p(\lambda, f(a)) \in J$ also. So there exists an $\mu_i \in \sigma(a)$, such that $p(\mu_i, a) \notin J$ and $\lambda = f(\mu_i) \in f(\sigma(a) \backslash \pi_0(a))$.

We prove the opposite inclusion. Let $\lambda \in f(\sigma(a) \backslash \pi_0(a)) \subset \sigma(f(a))$. Suppose that $\lambda \in \pi_0(f(a))$. Then λ is isolated in $\sigma(f(a))$ and we get again (3) and (4). It is well-known that idempotents on the right side of (4) are mutually orthogonal. Since $p(\lambda, f(a)) \in J$, by Lemma 4.6 we get $p(\mu_i, a) \in J$ for all i. So if $\lambda = f(\mu)$ and $\mu \in \sigma(a)$, then $\mu \in \pi_0(a)$. This is in contradiction with the assumption $\lambda \in f(\sigma(a) \backslash \pi_0(a))$. □

Remark 4.8. Notice that the spectral mapping theorem for the Browder spectrum holds for bounded operators on Banach spaces. Constructions in [1], [5] and [7] also imply the spectral mapping theorem for the Browder spectrum.

5. Quasihyponormal operators

Throughout this section H denotes a complex infinite-dimensional Hilbert space and T is always a bounded operator on H. We say that an operator T on a Hilbert space H is hyponormal, provided that $\|T^*x\| \le \|Tx\|$ holds for all $x \in H$. T is quasihyponormal, provided that $\|T^*Tx\| \le \|T^2x\|$ for all $x \in H$. Obviously, if T is hyponormal, then it is quasihyponormal. If T^* is a quasihyponormal operator, we prove that a-Weyl's theorem holds for $f(T)$, provided that f is a regular function in a neighbourhood of $\sigma(T)$ and f is not constant on the connected components of its domain. This problem for Weyl's theorem and hyponormal operators is partially proposed in [10] and solved in [9]. A general solution for Weyl's theorem may be found in [15].

The next lemma is proved in the Erovenko's paper [4].

Lemma 5.1. *Let T be a quasihyponormal operator on H. If $\lambda \in \mathbb{C}\backslash\{0\}$, then $\alpha(T-\lambda I) \le \alpha(T-\lambda I)^*$. If $\alpha(T) < \infty$ or $\beta(T) < \infty$, then $\alpha(T) \le \alpha(T^*)$.*

If f is an arbitrary regular function in a neighbourhood of $\sigma(T)$, then it is well-known that $\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T))$ [12]. This inclusion may be proper even if f is a polynomial [11]. For the class of quasihyponormal operators we have the more precise result.

We use the notation $\Phi_+(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(X)\}$. Recall the notation [14]

$$\mathcal{S}_+(X) = \{T \in B(X) :\ i(\lambda I - T) \le 0 \quad \text{for all } \lambda \in \Phi_+(T), \\ \text{or } i(\lambda I - T) \ge 0 \quad \text{for all } \lambda \in \Phi_+(T)\}.$$

Theorem 5.2. *If T^* is a quasihyponormal operator, $f \in \mathrm{Hol}(T)$, then*

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)).$$

Proof. If T^* is quasihyponormal, $\lambda \in \mathbb{C}$ and $\lambda I - T \in \Phi_+(X)$, then $i(\lambda I - T) \ge 0$, by Lemma 5.1. From [14, Theorem 2] it follows that

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$$

for all $f \in \mathrm{Hol}(T)$, since $T \in \mathcal{S}_+(X)$. □

Also, we shall use the next result (see [2]).

Theorem 5.3. *If T^* is quasihyponormal, then a-Weyl's theorem holds for T.*

The next theorem is a generalization of the Oberai's theorem [10].

Theorem 5.4. *Let T be a-isoloid and let T obey a-Weyl's theorem. If $f \in \mathrm{Hol}(T)$, then $f(T)$ obeys a-Weyl's theorem if and only if $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$.*

Proof. We shall use Theorem 4.4 assuming that the regularity R is the set of all operators on H, which are one-to-one with closed range. Now the proof follows from

$$f(\sigma_a(T)\backslash\pi_{a0}(T)) = \sigma_a(f(T))\backslash\pi_{a0}(f(T)). \qquad \square$$

Now, the next statement is expected.

Theorem 5.5. *If T^* is quasihyponormal, then T is a-isoloid.*

Proof. Suppose that $\lambda \in \mathrm{iso}\,\sigma_a(T)$. Then there exists a number $\epsilon > 0$, such that for all $\mu \in \mathbb{C}$, if $0 < |\lambda - \mu| < \epsilon$ then $\alpha(T - \mu I) = 0$ and $\mathcal{R}(T - \mu I)$ is closed. By Lemma 5.1 it follows that $\alpha(T - \mu I)^* = 0$, so $T - \mu I$ and $(T - \mu I)^*$ are invertible. It follows that $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$. It is well-known that the quasihyponormal operator T^* is isoloid, i.e. all isolated points of $\sigma(T^*)$ are eigenvalues of T^* (see the comment in [4, Теорема 5). We get that $0 < \alpha(T - \lambda I)^* \leq \alpha(T - \lambda I)$, so λ is an eigenvalue of T. $\square$

We are able to prove the following general result.

Theorem 5.6. *Let $T \in B(X)$ be a-isoloid and let T obey a-Weyl's theorem. Then the following assertions are equivalent:*

(1) *$T \in \mathcal{S}_+(X)$;*

(2) *for each $f \in \mathrm{Hol}(T)$ a-Weyl's theorem holds for $f(T)$;*

(3) *for each non-constant polynomial p a-Weyl's theorem holds for $p(T)$.*

Proof. (1) $\Longrightarrow$ (2) Let $T \in \mathcal{S}_+(X)$. In [14, Theorem 2] it is proved that $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for all $f \in \mathrm{Hol}(T)$. From Theorem 4.4 we get $\sigma_a(f(T)) \setminus \pi_{a0}(f(T)) = f(\sigma_a(T) \setminus \pi_{a0}(T))$ for all $f \in \mathrm{Hol}(T)$. Since a-Weyl's theorem holds for T, it follows that a-Weyl's theorem holds for $f(T)$ for all $f \in \mathrm{Hol}(T)$.

(2) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (1) Suppose that $T \notin \mathcal{S}_+(X)$. In [14, Theorem 2] SCHMOEGER constructed a non-constant polynomial p, such that $\sigma_{ea}(p(T)) \neq p(\sigma_{ea}(T))$. From Theorem 4.4 we still have $\sigma_a(p(T)) \setminus \pi_{a0}(p(T)) = p(\sigma_a(T) \setminus \pi_{a0}(T))$. Since a-Weyl's theorem holds for T, it follows that a-Weyl's theorem does not hold for $p(T)$. □

The analogous result for Weyl's theorem is considered in [15, Theorem 1].

Now, the next statement is very general and considered in [9], [10], [15].

Corollary 5.7. *If T^* is a quasihyponormal operator, f is a regular function in a neighbourhood of $\sigma(T)$ and f is not constant on the connected components of its domain, then a-Weyl's theorem holds for $f(T)$.*

PROOF. This proof follows from Theorems 5.2–5.6. □

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