On Hilbert's 16th problem

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Abstract. The aim of this paper is to study D. Hilbert's 16th problem for a Liénard type system of differential equations, in connection with some earlier results obtained by T. R. Blows and N. G. Lloyd. The method is based on the properties of the successive function.

1. Introduction

In 1900 at the Congress of Mathematicians held in Paris David Hilbert introduced his famous list of 23 problems. The 16th problem is still unsolved. This is the following: let S be the set of autonomous systems of differential equations of second order

$$(1) \qquad \begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y), \end{cases}$$

where P and Q are polynomials such that $P(0,0) = Q(0,0) = 0$. When $n \geq 2$ take $S_n \subset S$ having the property $\deg(P) \leq n$, $\deg(Q) \leq n$. Define the map $\pi : S \to \{0, 1, 2, \ldots, +\infty\}$ as follows: for $s \in S$, let $\pi(s)$ be the number of the limit cycles of the system s. Then we take

$$\mathcal{H}_n = \sup\{\pi(s) \mid s \in S_n\}.$$

The 16th problem requires to provide an *estimation* for $\mathcal{H}_n$ depending on n and to provide the *pictures* of all possible limit cycles. Both parts of

this problem (the estimation and the pictures) turn out to be very difficult. Short introductions to the estimation aspect may be found in [7] and [4].

Here we are interested in giving an estimation of the number of limit cycles of small amplitude for a particular system of differential equations, from where an answer is obtained to the 16th problem for this system. The results of this paper have been proved using the method of BLOWS and LLOYD from [4] and introduced in [12]. In this paper we use the method of successive function to get the desired results.

A *quadratic dynamical system* is a system of the form (1) such that $\deg(P) \leq 2$ and $\deg(Q) \leq 2$, and at least one of them has degree equal to 2. A *critical point* [14, p. 19], [9, VII.1] is a *fine focus* of (1) if it is a center for the corresponding linearized system. The *order* of a fine focus is defined as follows: Suppose that the origin is a fine focus. It is known ([14, pp. 119–120]) that there exists a function V defined on a neighborhood of O such that $\dot{V}$, its derivative with respect to the system (1), is of the form $\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \cdots$. The coefficients η_{2k} are called *focal values* and they depend on the coefficients of the polynomials P and Q. Then the origin is a *fine focus of order k* if

$$\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0 \quad \text{and} \quad \eta_{2k+2} \neq 0.$$

N. BAUTIN was the first who got a result in this field, [3]. He used the properties of successive function to prove that the origin is a fine focus of order 3 (or that it has a cyclicity of order 3). More precisely we have

Theorem 1.1. *There exist quadratic dynamical systems such that around a focus or a center there are 3 limit cycles. There is no quadratic dynamical system such that around a focus or a center there are 4 limit cycles.*

It follows immediately that

Theorem 1.2. $\mathcal{H}_2 \geq 3$.

The above theorem has been improved by SHI SONGLING [19].

Theorem 1.3. $\mathcal{H}_2 \geq 4$.

This result has been discussed in several papers, e.g. [20], [5], [1], [23].

The first step towards solving Hilbert's 16th problem in the case of cubic systems was performed by SIBIRSKII [18]. He proved that any system of the form

(2)
$$\begin{cases} \dot{x} = \lambda x - y + \sum_{j+l=3} a_{j,l} x^j y^l \\ \dot{y} = x + \lambda y + \sum_{j+l=3} b_{j,l} x^j y^l \end{cases}$$

has around the origin at most 5 limit cycles and there exist such systems with 5 limit cycles around the origin. Thus $\mathcal{H}_3 \geq 5$. In [1] there were studied cubic systems. Since the computations for such a system rapidly become very difficult to handle (η_8 has more than 600 terms), a particular cubic system was studied, namely

(3)
$$\begin{cases} \dot{x} = \lambda x + y + cy^2 + hxy^2 + ky^3 \\ \dot{y} = -x + \lambda y + d(x^2 - y^2) + lx^3 - hx^2 y + nxy^2. \end{cases}$$

One of the results proved in [11], Theorem 2.5, says that $\mathcal{H}_3 \geq 6$. This conclusion has been proved more recently, [16], for the following system:

$$\dot{x} = x, \quad \dot{y} = -x + \lambda y + Ax^2 + Cy^2 + Kx^3 + 3Lx^2 y + Mxy^2 + Ny^3.$$

Recent results on cubic systems may be found in [10], [6], [17], [15], [21]. To the author's knowledge, the newest result for this case, is contained in [22] and it says that $\mathcal{H}_3 \geq 11$.

The method used in [3] by BAUTIN (which is one of bifurcation) was used by several authors, among other things, to determine the number of small amplitude limit cycles of the Liénard equation [4], [13].

In what follows, by the method described in [3], [8], [18], we study the number of small amplitude limit cycles of the system

(4)
$$\begin{cases} \dot{x} = F_1(x) + F_3(y) \\ \dot{y} = F_2(x) + F_4(y). \end{cases}$$

We suppose that

(I_1) $F_1(x) = \sum_{i=1}^{N} a_i x^i$, where $a_1 = \lambda$;

(I_2) $F_2(x) = \sum_{i=1}^{\infty} b_i x^i$, an odd series converging in a neighborhood of the origin;

(I_3) $F_3(y) = \sum_{i=1}^{\infty} c_i y^i$, a convergent series in a neighborhood of the origin;

(I_4) $F_4(y) = \sum_{i=1}^{M} d_i y^i$, an odd polynomial;

(I_5) $a_{2i-1} d_{2i-1} \geq 0$, $i \in \mathbb{N}$.

Moreover, we suppose that in a neighborhood of the origin (4) has the canonical form [2, p. 141], that is

(I_6) $d_1 = \lambda$ and $b_1 = -c_1 \neq 0$;

(I_7) there exists a neighborhood of the origin such that the origin is the only singular point in it for any small perturbations of the coefficients of the polynomials F_1 and F_4.

We note that hypothesis (I_7) will be used only in the proof of Theorem 2.5.

The method consists in starting with a system such that the origin is a center for the linearized system; then, by successive perturbations of the coefficients of the functions involved, to bifurcate out limit cycles. The limit cycles are said to be of *small amplitude* since they can be arranged to lie in a prescribed neighborhood of the origin.

We may suppose that $b_1 = 1$.

At least two definitions of the first return map (Poincaré map) are known. The first one was given by Poincaré [14, pp. 119–120]. Hereafter we will use the definition of the first return map given in [8].

Let us consider the following system of differential equations

$$(5) \qquad \begin{cases} \dot{x} = \; ax + by + P_2(x, y) + P_3(x, y) + \cdots, \\ \dot{y} = -bx + ay + Q_2(x, y) + Q_3(x, y) + \cdots, \end{cases}$$

defined in a neighborhood of the origin, where P_n and Q_n are homogeneous polynomials in x and y, $n \geq 2$. We suppose that $b \neq 0$ and in this way the origin is a center or a focus of the system (5) (Theorem 3.2, [9, p. 215]). Then there exists a neighborhood U of the origin on which we may define the first return map, [14, pp. 119–120]. If we perform a transformation to polar coordinates by $x = r \cos\theta$, $y = r \sin\theta$ ([2, p. 168]) then the system (5) becomes

$$(6) \qquad \begin{cases} \dot{r} = ar + \sum_{k=2}^{\infty} r^k [P_k(\cos\theta, \sin\theta) \cos\theta + Q_k(\cos\theta, \sin\theta) \sin\theta], \\ \dot{\theta} = b \; + \sum_{k=2}^{\infty} r^k [P_k(\cos\theta, \sin\theta) \sin\theta - Q_k(\cos\theta, \sin\theta) \cos\theta]. \end{cases}$$

Consider the Liapunov function in polar coordinates

$$V = \frac{r^2}{2} + r^3(V_{3,0}\cos^3\theta + \cdots + V_{0,3}\sin^3\theta) + r^4(\ldots) + \cdots,$$

with its derivative with respect to the system (6) having the form

$$\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \cdots.$$

Let $\rho = \sqrt{2V}$. For r small enough but strictly positive, $V > 0$ uniformly in respect to θ, thus ρ is well-defined. Moreover, for small but positive values of r, ρ behaves similarly to r, that is $\rho/r \to 1$ if $r \to 0$. Now the first return map is defined as follows:

$$\delta(r_0) = \rho(r_1, 0) - \rho(r_0, 0) = \int_0^T \dot{\rho}(r(t), \theta(t))dt,$$

where T is the time necessary for a solution of (6) which starts at the point $(r_0, 0) = (r_0, \theta(0))$ to meet again the positive horizontal semiaxis in $(r_1, 0) = (r_1, \theta(T))$. Then

$$\delta(r_0) = \int_0^T \left(\frac{d\rho}{dt}\right)dt = \int_0^{2\pi} \left(\frac{d\rho}{d\theta}\right)d\theta = \int_0^{2\pi} \frac{d\rho}{dt}\frac{1}{\frac{d\theta}{dt}}d\theta = \int_0^{2\pi} \dot{V}\frac{1}{\sqrt{2V}}\frac{1}{\dot{\theta}}d\theta.$$

Taking $r = r_0$ and expanding it in a series with respect to r we have

$$\delta(r) = \int_0^{2\pi} (\eta_2 r^2 + \eta_4 r^4 + \ldots)\frac{1}{r}[1 + r(\ldots) + \ldots]\frac{1}{b}[1 + r(\ldots) + \ldots]d\theta.$$

Denote $q_0 = 1$ and $q_n = q_n(\theta)$ for $n \geq 1$. For r small enough we may write

$$\delta(r) = \frac{1}{b}\int_0^{2\pi} (\eta_2 r + \eta_4 r^3 + \ldots)\left(\sum_{n=0}^{\infty} q_n r^n\right)d\theta$$

$$= \frac{1}{b}\sum_{n=1}^{\infty} r^n \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \eta_{2k}\int_0^{2\pi} q_{n-(2k-1)}(\theta)\,d\theta.$$

We have

$$\delta(0) = 0,$$

$$\delta'(r) = \frac{1}{b}\eta_2\int_0^{2\pi} q_0 d\theta + r(\ldots) \implies \delta'(0) = \frac{2\pi}{b}\eta_2.$$

If $\delta'(0) = 0$ then $\eta_2 = 0$ and

$$\delta''(r) = \frac{1}{b}2!\eta_2 \int_0^{2\pi} q_1(\theta)\, d\theta + r(\dots) \implies \delta''(0) = 0$$

$$\delta'''(r) = \frac{1}{b}3! \left[\eta_2 \int_0^{2\pi} q_2(\theta)d\theta + \eta_4 \int_0^{2\pi} q_0 d\theta \right] + r(\dots)$$

$$\implies \delta'''(0) = \frac{2\pi}{b}3!\eta_4.$$

Generally we have

$$(7) \qquad \begin{cases} \delta(0) = \delta'(0) = \cdots = \delta^{(2n)}(0) = 0, \\ \delta^{(2n+1)}(0) = \frac{2\pi}{b}(2n+1)!\,\eta_{2n+2}, \quad n = 1, 2, \dots \end{cases}$$

2. Results

We *seek* a Liapunov function $V = V(x, y)$ having the form

$$V(x, y) = V_2 + V_3 + V_4 + V_5 + \cdots,$$

where the V_k are homogeneous polynomials in x and y, that is

$$(8) \qquad V_2 = \frac{1}{2}(x^2 + y^2) \quad \text{and} \quad V_k = \sum_{i=0}^{k} V_{k-i,i} x^{k-i} y^i, \ k \geq 3,$$

such that $\dot{V}$, the derivative of V with respect to the system (4), has the form

$$\dot{V} = \eta_2(x^2 + y^2) + \eta_4(x^2 + y^2)^2 + \cdots.$$

Hence the unknowns are the coefficients $V_{i,j}$ and η_{2k}. The coefficients $V_{i,j}$ whose first subscript is an odd number are said to be *odd coefficients*. The coefficients $\eta_2, \eta_4, \dots$ are integer functions depending on a_i, b_j, c_k, d_m and they are said to be *focal values*. Another definition may be found in [8].

Denote

$$V_{k,x} = \frac{\partial V_k}{\partial x}, \qquad V_{k,y} = \frac{\partial V_k}{\partial y}, \qquad k \geq 2.$$

Then the derivative of V with respect to the system (4) is

$$\dot{V} = \frac{\partial V}{\partial x}(F_1(x) + F_3(y)) + \frac{\partial V}{\partial y}(F_2(x) + F_4(y))$$

$$= \left(\sum_{k=2}^{\infty} V_{k,x}\right)\left(\sum_{i=1}^{N} a_i x^i + \sum_{i=1}^{\infty} c_i y^i\right) + \left(\sum_{k=2}^{\infty} V_{k,y}\right)\left(\sum_{i=1}^{\infty} b_i x^i + \sum_{i=1}^{M} d_i y^i\right)$$

$$= \sum_{k=2}^{\infty} D_k,$$

where D_k is the degree k term

$$(9)\quad D_k = \sum_{i=1}^{k}[a_i x^i V_{k+1-i,x} + b_i x^i V_{k+1-i,y} + c_i y^i V_{k+1-i,x} + d_i y^i V_{k+1-i,y}].$$

Let us take $p, i, j \in \mathbb{N}$, $i \le j \le p$ and $\lambda \in \mathbb{R}$. Consider the following square matrix

$$(10)\qquad\qquad\qquad M(p,i,j;\lambda)$$

$$= \begin{pmatrix} p\lambda & i & 0 & 0 & \ldots & 0 & 0 & 0 \\ -(p+1-i) & p\lambda & i+1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -(p-i) & p\lambda & i+2 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -(p-i-1) & p\lambda & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & p\lambda & j-1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & -(p+2-j) & p\lambda & j \\ 0 & 0 & 0 & 0 & \ldots & 0 & -(p+1-j) & p\lambda \end{pmatrix}$$

of order $j - i + 2$, that is $a_{m,m} = p\lambda$, $a_{m,m+1} = i + m - 1$ and $a_{m+1,m} = -(p + 2 - i - m)$ for $m \in \{1, \ldots, j - i + 2\}$, all the other elements being equal to zero, and let $\Delta(p, i, j; \lambda)$ be the determinant of $M(p, i, j; \lambda)$.

Proposition 2.1. *Under the above-mentioned conditions we have that*

(i) *all coefficients of λ in the expansion of the determinant $\Delta(p, i, j; \lambda)$ are non-negative;*

(ii) $\Delta(p, i, j; \lambda) = (-1)^{j-i}\Delta(p, p - j + 1, p - i + 1; -\lambda)$;

(iii) *for fixed $k \in \mathbb{N}$, the map $\mathbb{R} \ni \lambda \mapsto \Delta(2k + 1, 1, 2k + 1; \lambda)$ is an even polynomial;*

(iv)

$$\Delta(p, i, j; 0) = \left(j - i + 1 - 2\left[\frac{j - i + 1}{2}\right]\right)\prod_{m=0}^{\left[\frac{j-i}{2}\right]}(p+1-i-2m)(i+2m),$$

where [ ] means the integer part;

(v) $\Delta(p, 1, 2k + 1; 0) = (2k + 1)!! \prod_{m=0}^{k}(p - 2m)$;

(vi) $\Delta(p, 1, 2k + 1; \lambda) \geq (2k + 1)!! \prod_{m=0}^{k}(p - 2m)$.

PROOF. (i) This property is obvious if we expand the determinant (10) in terms of the first column, then in terms of the second column, etc. In other words, we compute

$$\Delta(p, i, i; \lambda) = p^2\lambda^2 + i(p + 1 - i),$$
$$\Delta(p, i, i + 1; \lambda) = p^3\lambda^3 + p(2pi - 2i^2 + p)\lambda,$$

and then we take into account the following recurrence relation valid for any $j \geq i + 2$:

$$\Delta(p, i, j; \lambda) = p\lambda\Delta(p, i + 1, j; \lambda) + (p - i + 1)i\Delta(p, i + 2, j; \lambda).$$

(ii) Interchange the first row with the $(j - i + 2)^{\text{nd}}$, the second row with the $(j - i + 1)^{\text{st}}$, ..., interchange the first column with the $(j - i + 2)^{\text{nd}}$, the second column with the $(j - i + 1)^{\text{st}}$, ..., and at the end we see that each row admits (-1) as factor.

(iii) To prove that $\mathbb{R} \ni \lambda \mapsto \Delta(2k+1, 1, 2k+1; \lambda)$ is an even polynomial it is sufficient to show that

$$\Delta(2k + 1, 1, 2k + 1; \lambda) = \Delta(2k + 1, 1, 2k + 1; -\lambda), \quad \text{for all } \lambda \in \mathbb{R}.$$

But the last equality results from (ii) by $p = 2k + 1$, $i = 1$, $j = 2k + 1$.

(iv) According to the computations there results from (i) that

$$\Delta(p, i, i; 0) = i(p + 1 - i), \quad \Delta(p, i, i + 1; 0) = 0,$$

and for $j \geq i + 2$ we have the recurrence relation

$$\Delta(p, i, j; 0) = (p - i + 1)i\Delta(p, i + 2, j; 0),$$

from where the desired equality follows.

(v) Can be obtained from (iv).

(vi) Follows from (i), (iii) and (v).   □

Let $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$. Denote by $M_{1,4k}(\lambda)$ the square matrix $(a_{p,q})_{1 \leq p,q \leq 4k+1}$ of order $4k + 1$ built up so that

$\quad$ – the first column contains the following elements: $a_{2j,1} = 0$, $j \in \{1, 2, \ldots, 2k\}$, $a_{2j-1,1} = \binom{2k}{j-1}$, $j \in \{1, 2, \ldots, 2k + 1\}$;

$\quad$ – the columns $2, 3, \ldots, 2k + 1$ contain from the rows $1, 2, \ldots, 2k$ the elements of $M(4k, 1, 2k - 1; \lambda)$;

$\quad$ – the columns $2k + 2, \ldots, 4k + 1$ contain from the rows $1, \ldots, 2k$ the 0 only;

$\quad$ – the row $(2k + 1)^{\text{st}}$ contains 0 only, but $a_{2k+1,2k+1} = -(2k + 1)$, $a_{2k+1,2k+2} = 2k + 1$;

$\quad$ – the columns $2, \ldots, 2k + 1$ contain from the rows $2k + 2, \ldots, 4k + 1$ the 0 only;

$\quad$ – the columns $2k+2, \ldots, 4k+1$ contain from the rows $2k+2, \ldots, 4k+1$ the elements of $M(4k, 2k + 2, 4k; \lambda)$.

$\quad$ Let $\Delta_{1,4k}(\lambda)$ be the determinant of $M_{1,4k}(\lambda)$.

Proposition 2.2. *Let $\Delta_{1,4k}(\lambda)$ be defined under the above-mentioned conditions. Then*

(i) *the function $\mathbb{R} \ni \lambda \mapsto \Delta_{1,4k}(\lambda)$ is an even polynomial whose coefficients are all non-negative;*

(ii)

$$\Delta_{1,4k}(0) = \left[\frac{(4k)!!}{(2k)!!}\right]^2 \sum_{i=0}^{2k} \binom{2k}{i}(4k - 1 - 2i)!! \, (2i - 1)!! > 0,$$

$$\text{with } (-1)!! = 1;$$

(iii) *$\Delta_{1,4k}(\lambda) \geq \Delta_{1,4k}(0) > 0$, for any $\lambda \in \mathbb{R}$.*

$\quad$ PROOF. (i) If we expand the determinant $\Delta_{1,4k}(\lambda)$ in terms of the first column we get

$$\Delta_{1,4k}(\lambda) = \sum_{i=0}^{k} \binom{2k}{i} \Delta(4k, 1, 2i - 1; \lambda)[-(4k - 2i)] \ldots [-(2k + 1)]$$

$$\times \Delta(4k, 2k + 2, 4k; \lambda)$$

$$+ \sum_{i=k+1}^{2k} \binom{2k}{i} \Delta(4k, 1, 2k - 1; \lambda)(2k + 1) \ldots (2i) \Delta(4k, 2i + 2, 4k; \lambda)$$

$$= \Delta(4k, 1, 2k - 1; \lambda)\left[\sum_{i=0}^{k} \binom{2k}{i} \Delta(4k, 1, 2i - 1; \lambda)\frac{(4k - 2i)!}{(2k)!}\right.$$

$$+ \left.\sum_{i=k+1}^{2k} \binom{2k}{i} \Delta(4k, 1, 4k - 2i - 1; \lambda)\frac{(2i)!}{(2k)!}\right],$$

where, by definition, $\Delta(4k, 1, -1; \lambda) = \Delta(4k, 4k + 2, 4k; \lambda) = 1$. The conclusion results immediately if we take into account (i) and (iii) from Proposition 2.1.

(ii) We expand the determinant $\Delta_{1,4k}(\lambda)$ in terms of the columns $2, 4, \ldots, 2k$, then in terms of the rows $2k + 2$, $2k + 4, \ldots, 4k$, and finally in terms of the first column.

(iii) Results from (i) and (ii).   □

Let $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$. Denote by $M_{2,4k+2}(\lambda)$ the square matrix $(a_{p,q})_{1 \leq p, q \leq 4(k+1)}$ of order $4(k + 1)$ defined by

– the first column contains $a_{2j,1} = 0$, $j \in \{1, 2, \ldots, k + 1\}$, $a_{2j-1,1} = \binom{2k+1}{j-1}$, $j \in \{1, 2, \ldots, k + 1\}$, $a_{2j-1,1} = 0$, $j \in \{k + 2, \ldots, 2k + 1\}$, $a_{2j,1} = \binom{2k+1}{j}$, $j \in \{k + 2, \ldots, 2k + 1\}$,

– the columns $2, 3, \ldots, 2k + 2$ contain from the rows $1, 2, \ldots, 2k + 1$ the elements of $M(4k + 2, 1, 2k; \lambda)$;

– the column $(2k + 3)^{\mathrm{rd}}$ contains from the rows $1, \ldots, 2k + 1$ the 0, except $a_{2k+1,2k+3} = 2k + 1$;

– the row $(2k+2)^{\mathrm{nd}}$ has 0 only, but $a_{2k+2,2k+2} = -(2k+2)$, $a_{2k+2,2k+3} = (2k + 1)\lambda$;

– the row $(2k+3)^{\mathrm{nd}}$ has 0 only, but $a_{2k+3,2k+4} = 2k + 2$, $a_{2k+3,2k+3} = (2k + 1)\lambda$;

– the columns $2, \ldots, 2k + 3$ contain from the rows $2k + 4, \ldots, 4(k + 1)$ the 0 only, but $a_{2k+4,2k+3} = -(2k + 1)$;

– the columns $2k + 4, \ldots, 4(k + 1)$ contain from the rows $2k + 4, \ldots, 4(k + 1)$ the elements in $M(4k + 2, 2k + 3, 4k + 2; \lambda)$.

Denote by $\Delta_{2,4k+2}(\lambda)$ the determinant of the square matrix $M_{2,4k+2}(\lambda)$.

Proposition 2.3. *Let* $\Delta_{2,4k+2}(\lambda)$ *be defined under the above-mentioned conditions. Then*

(i) *the function* $\mathbb{R} \ni \lambda \mapsto \Delta_{2,4k+2}(\lambda)$ *is an even polynomial whose coefficients are all non-negative;*

(ii)

$$\Delta_{2,4k+2}(0) = -\left[\frac{(4k + 2)!!}{(2k)!!}\right]^2 \sum_{i=0}^{2k+1} \binom{2k + 1}{i}(4k + 1 - 2i)!!(2i - 1)!! < 0,$$

$$\text{with } (-1)!! = 1;$$

(iii) $\Delta_{2,4k+2}(\lambda) \leq \Delta_{2,4k+2}(0) < 0$, *for all $\lambda \in \mathbb{R}$.*

PROOF. (i) First let us show that $\Delta_{2,4k+2}(\lambda) = \Delta_{2,4k+2}(-\lambda)$. We start with $\Delta_{2,4k+2}(-\lambda)$ and we effect the following transformations

– interchange the column i^{th} with the column $(4k + 6 - i)^{\text{th}}$, $i \in \{2, 3, \ldots, 2k + 2\}$. This being an odd number of changes of columns the determinant $\Delta_{2,4k+2}(-\lambda)$ changes its sign;

– interchange the row i^{th} with the row $(4k + 5 - i)^{\text{th}}$, $i \in \{1, 2, \ldots, 2k + 1\}$. This being an even number of changes of rows the determinant $\Delta_{2,4k+2}(-\lambda)$ keeps its sign;

– in the columns $2, 3, \ldots, 4k + 4$ we have (-1) as factor. Thus we get $\Delta_{2,4k+2}(\lambda)$.

Expanding the determinant $\Delta_{2,4k+2}(\lambda)$ in terms of the elements of the first column we obtain

$$\Delta_{2,4k+2}(\lambda) = \sum_{i=0}^{k} \binom{2k+1}{i} \Delta(4k+2, 1, i; \lambda)[-(4k - 2i + 2)] \ldots$$

$$\ldots [-(2k+2)](2k+1)$$

$$\times [\lambda \Delta(4k+2, 2k+3, 4k+2; \lambda) + (2k+2)\Delta(4k+2, 2k+3, 4k+2; \lambda)]$$

$$- \sum_{i=k+1}^{2k+1} \binom{2k+1}{i}(2k+1)\Big[\lambda \Delta(4k+2, 1, 2k; \lambda)$$

$$+ (2k+2)\Delta(4k+2, 1, 2k-1; \lambda)\Big](2k+2) \ldots$$

$$\ldots (2i)\Delta(4k+2, 2i+2, 4k+2; \lambda).$$

Taking into account (ii) from Proposition 2.1 we obtain that all the coefficients of λ at even powers are negative.

(ii) We expand $\Delta_{2,4k+2}(0)$, first in terms of the elements of the columns $4k + 4, 4k + 2, \ldots, 2k + 4$, then in terms of the elements of the rows $2k + 2, 2k, \ldots, 2$, and finally in terms of the elements of the first column.

(iii) Follows from (i) and (ii).   □

Let $k \in \mathbb{N}$, $a, d, \lambda \in \mathbb{R}$. $M'_{1,4k}(a, d; \lambda)$ is the square matrix $(a_{p,q})_{1 \leq p, q \leq 4k+1}$ of order $4k + 1$ such that

– $a_{1,1} = a$, $a_{4k+1,1} = d$, $a_{i,1} = 0$, $i \in \{2, 3, \ldots, 4k\}$;

– the ith column of this matrix coincides with the ith column of the matrix $M_{1,4k}$, $i \in \{2, 3, \ldots, 4k + 1\}$;

Let $\Delta'_{1,4k}(a, d; \lambda)$ be the determinant of the matrix $M'_{1,4k}(a, d; \lambda)$.

Proposition 2.4. *Under the above-mentioned conditions there holds the equality*

$$\Delta'_{1,4k}(a, d; \lambda) = (a + d)\Delta(4k, 1, 2k - 1; \lambda) \prod_{m=0}^{2k-1} (4k - m).$$

PROOF. Expanding this determinant in terms of the elements of the first column we find that

$$\Delta'_{1,4k}(a, d; \lambda) = a \prod_{m=0}^{2k-1} [-(4k - m)]\Delta(4k, 4k + 2, 4k; \lambda)$$

$$+ d\Delta(4k, 1, 2k - 1; \lambda) \prod_{m=0}^{2k-1} (4k - m),$$

and then we apply (ii) from Proposition 2.1. □

Let $k \in \mathbb{N}$, $a, d, \lambda \in \mathbb{R}$. Denote by $M'_{2,4k+2}(a, d; \lambda)$ the square matrix $(a_{p,q})_{1 \leq p,q \leq 4(k+1)}$ of order $4(k + 1)$ defined by

– $a_{1,1} = a$, $a_{4(k+1),1} = d$, $a_{i,1} = 0$, $i \in \{2, 3, \ldots, 4k + 3\}$;

– the ith column of this matrix coincides with the ith column of the matrix $M_{2,4k}$, $i \in \{2, 3, \ldots, 4(k + 1)\}$.

Denote by $\Delta'_{2,4k+2}(a, d; \lambda)$ the determinant of the square matrix $M'_{2,4k+2}(a, d; \lambda)$.

Proposition 2.5. *Under the above-mentioned conditions there holds the equality*

$$\Delta'_{2,4k}(a, d; \lambda) = -(a + d)(2k + 1) \prod_{m=0}^{2k} (4k + 2 - m)$$

$$\times [\lambda \Delta(4k + 2, 1, 2k; \lambda) + (2k + 2)\Delta(4k + 2, 1, 2k - 1; \lambda)].$$

PROOF. Expand $\Delta'_{2,4k+2}(a, d; \lambda)$ in terms of the elements of the first column and then use (ii) from Proposition 2.1

$$
\begin{aligned}
\Delta'_{2,4k}(a, d; \lambda) = a \prod_{m=0}^{2k} [-(4k+2-m)][(2k+1)\Delta(4k+2, 2k+3, 4k+2; \lambda) \\
+ (2k+1)(2k+2)\Delta(4k+2, 2k+4, 4k+2; \lambda)] \\
- d \prod_{m=0}^{2k} (4k+2-m)[(2k+1)\Delta(4k+2, 2k+3, 4k+2; \lambda) \\
+ (2k+1)(2k+2)\Delta(4k+2, 2k+4, 4k+2; \lambda)]. \qquad \square
\end{aligned}
$$

Let $k \in \mathbb{N}$ and $\lambda, q_1, q_2, \ldots$ be real numbers. We denote by $M''_{1,4k}$ the square matrix $(a_{p,q})_{1 \leq p,q \leq 4k+1}$ of order $4k+1$ defined by

– the first column contains $a_{m,1} = 0$, $m \in \{1, 3, 5, \ldots, 4k+1\}$, $a_{m,1} = q_m$, $m \in \{2, 4, \ldots, 4k\}$;

– the mth column coincides with the mth column of the matrix $M_{1,4k}(\lambda)$, $m \in \{2, 3, \ldots, 4k+1\}$.

We denote by $\Delta''_{1,4k}$ the determinant of the square matrix $M''_{1,4k}$.

Proposition 2.6. *Under the above-mentioned conditions $\Delta''_{1,4k}(\lambda)$ may be written as a product of λ by a polynomial in λ.*

PROOF. It is sufficient to show that $\Delta''_{1,4k}(0) = 0$. We expand the determinant $\Delta''_{1,4k}(0)$ in terms of the first row, the third row, $\ldots$, the $(4k-1)^{\text{th}}$ row. In this way we get a determinant whose last row is zero, hence $\Delta''_{1,4k}(0) = 0$. $\qquad \square$

Let $i \in \{3, 5, \ldots, 2k+1, 2k+2, 2k+4, \ldots, 4k\}$ be arbitrary, but fixed. We denote by $\Delta'''_{1,4k}(\lambda)$ the determinant of the square matrix $(a_{p,q})_{1 \leq p,q \leq 4k+1}$ of order $4k+1$ obtained so that the ith column is the first column of the matrix $M''_{1,4k}(\lambda)$, all the others coincide with the columns of the matrix $M_{1,4k}(\lambda)$.

Proposition 2.7. *$\Delta'''_{1,4k}(\lambda)$ may be written as a product of λ by a polynomial in λ.*

PROOF. It is sufficient to show that $\Delta'''_{1,4k}(0) = 0$. We expand the determinant $\Delta'''_{1,4k}(0)$ in terms of the 2$^{\text{nd}}$, the 4$^{\text{th}}$, $\ldots$, the $(2k)^{\text{th}}$, the

$(2k+3)^{\text{th}}$, the $(2k+5)^{\text{th}}, \ldots$, the $(4k+1)^{\text{st}}$ column and then we get a zero column (the ith column). □

Let $k \in \mathbb{N}$ and $\lambda, q_1, q_2, \ldots$ be real numbers. Let $\Delta''_{2,4k+2}(\lambda)$ be the determinant of the square matrix $(a_{p,q})_{1 \leq p,q \leq 4(k+1)}$ of order $4(k+1)$ denoted by $M''_{2,4k+2}(\lambda)$ and obtained so that
  – the first column contains $a_{2j,1} = q_j$, $a_{2j-1,1} = 0$, $j \in \{1, 2, \ldots, k+1\}$, $a_{2m-1,1} = q_m$, $a_{2m,1} = 0$, $m \in \{k+2, k+3, \ldots, 2k+1\}$;
  – the mth column coincides with the mth column of the matrix $M_{2,4k+2}(\lambda)$, $m \in \{2, 3, \ldots, 4k+4\}$.

Proposition 2.8. *Under the above conditions, $\Delta''_{2,4k+2}(\lambda)$ may be written as a product of λ by a polynomial in λ.*

PROOF. It is sufficient to show that $\Delta''_{2,4k+2}(0) = 0$. We expand the determinant $\Delta''_{2,4k+2}(0)$ in terms of the first, the $3^{\text{rd}}, \ldots$, the $(2k+1)^{\text{st}}$ lines, then in terms of the $(2k+4)^{\text{th}}$, the $(2k+6)^{\text{th}}, \ldots$, the $(4k+2)^{\text{th}}$ lines. We get a determinant whose last line is zero, hence $\Delta''_{2,4k+2}(0) = 0$. □

Let be $i \in \{3, 5, \ldots, 4k+3\}$ arbitrary but fixed. We denote by $\Delta'''_{2,4k+2}(\lambda)$ the determinant of the square matrix of order $4k+4$ which has in its ith column the first column of $M''_{2,4k+2}(\lambda)$, all the other columns being equal to the corresponding columns of $M_{2,4k+2}(\lambda)$.

Proposition 2.9. *Let $\Delta'''_{2,4k+2}(\lambda)$ be defined by the above-mentioned conditions. Then $\Delta'''_{2,4k+2}(\lambda)$ may be written as a product of λ by a polynomial in λ.*

PROOF. It is sufficient to show that $\Delta'''_{2,4k+2}(0) = 0$. We expand $\Delta'''_{2,4k+2}(0)$ in terms of the second, the $4^{\text{th}}, \ldots$, the $(2k+2)^{\text{th}}$, the $(4k+4)^{\text{th}}$, the $(4k+2)^{\text{th}}, \ldots$, the $(2k+4)^{\text{th}}$ lines and we obtain a zero column (the ith). □

Let be $i, j \in \mathbb{N}$, $i \leq j$ and $j - i$ an even number. Let $m \in \{1, 3, \ldots, j-i+1\}$ be arbitrary but fixed. Consider, also a sequence $(q_k)_{k \geq 1}$ of real numbers. Denote by $M'(p, i, j; \lambda)$ the square matrix of order $j - i + 2$ obtained so that
  – the mth column contains $a_{2k-1,m} = q_k$, $a_{2k,m} = 0$, $k \in \{1, 2, \ldots, \frac{j-i+2}{2}\}$;
  – all the other columns being equal to the corresponding columns of $M(p, i, j; \lambda)$.

Proposition 2.10. *Denote by* $\Delta'(p, i, j; \lambda)$ *the determinant of* $M'(p, i, j; \lambda)$. *Then* $\Delta'(p, i, j; \lambda)$ *may be written as a product of* λ *by a polynomial in* λ.

PROOF. It is enough to show that $\Delta'(p, i, j; 0) = 0$. We expand successively $\Delta'(p, i, j; 0)$ in terms of the elements of the $(j - i + 2)^{\text{nd}}$, the $(j - i)^{\text{th}}, \ldots$, the second column and we get a zero column, namely the mth column. $\qquad\square$

Lemma 2.1 (The general form of the focal values). *Under the* (I_1)–(I_6) *hypotheses the following statements are valid:*
(i) $\eta_2 = \lambda$;
(ii) *the odd coefficients in* V_3 *admit* λ *as a factor, more precisely*

$$V_{3,0} = -3\lambda[(7 + 9\lambda^2)a_2 + 2c_2]/\Delta_3(\lambda),$$

$$V_{2,1} = -9[(1 + 3\lambda^2)a_2 - 2\lambda^2 c_2]/\Delta_3(\lambda),$$

$$V_{1,2} = -9\lambda[2a_2 + (1 + 3\lambda^2)c_2]/\Delta_3(\lambda),$$

$$V_{0,3} = -3[2a_2 + (1 + 3\lambda^2)c_2]/\Delta_3(\lambda),$$

where $\Delta_3(\lambda) = \Delta(3, 1, 3; \lambda)$;
(iii)

$$\eta_4 = [\Delta'_{1,4}(a_3, d_3; \lambda) + \lambda P_4]/\Delta_4(\lambda)$$

$$V_{3,1} = [a_3 P_{3,1,a_3} + d_3 P_{3,1,d_3} + \lambda P_{3,1,\lambda}]/\Delta_4(\lambda)$$

$$V_{1,3} = [a_3 P_{1,3,a_3} + d_3 P_{1,3,d_3} + \lambda P_{1,3,\lambda}]/\Delta_4(\lambda),$$

where $\Delta_4(\lambda) = \Delta_{1,4}(\lambda)$, $P_{3,1,a_3}$, $P_{3,1,d_3}$, $P_{1,3,a_3}$, $P_{1,3,d_3}$ *are polynomials in* λ, *and* P_4, $P_{3,1,\lambda}$ *and* $P_{1,3,\lambda}$ *are polynomials in* λ, a_2, c_2, b_3 *and* c_3;
(iv)

$$V_{5,0} = [a_3 P_{5,0,a_3} + d_3 P_{5,0,d_3} + \lambda P_{5,0,\lambda}]/\Delta_5(\lambda),$$

$$V_{3,2} = [a_3 P_{3,2,a_3} + d_3 P_{3,2,d_3} + \lambda P_{3,2,\lambda}]/\Delta_5(\lambda),$$

$$V_{1,4} = [a_3 P_{1,4,a_3} + d_3 P_{1,4,d_3} + \lambda P_{1,4,\lambda}]/\Delta_5(\lambda),$$

where $\Delta_5(\lambda) = \Delta(5, 1, 5; \lambda)$, *and* $P_{5,0,a_3}$, $P_{5,0,d_3}$, $P_{3,2,a_3}$, $P_{3,2,d_3}$, $P_{1,4,a_3}$, $P_{1,4,d_3}$, $P_{5,0,\lambda}$, $P_{3,2,\lambda}$ *and* $P_{1,4,\lambda}$ *are polynomials in* λ, a_2, a_4, c_2, c_4, b_3 *and* c_3;

(v)

$$\eta_6 = [\Delta'_{2,6}(a_5, d_5; \lambda) + a_3 P_{6,a_3} + d_3 P_{6,d_3} + \lambda P_6]/\Delta_6(\lambda),$$

$$V_{5,1} = [a_5 P_{5,1,a_5} + d_5 P_{5,1,d_5} + a_3 P_{5,1,a_3} + d_3 P_{5,1,d_3} + \lambda P_{5,1,\lambda}]/\Delta_6(\lambda)$$

$$V_{3,2} = [a_5 P_{3,2,a_5} + d_5 P_{3,2,d_5} + a_3 P_{3,2,a_3} + d_3 P_{3,2,d_3} + \lambda P_{3,2,\lambda}]/\Delta_6(\lambda)$$

$$V_{1,5} = [a_5 P_{1,5,a_5} + d_5 P_{1,5,d_5} + a_3 P_{1,5,a_3} + d_3 P_{1,5,d_3} + \lambda P_{1,5,\lambda}]/\Delta_6(\lambda),$$

where $\Delta_6(\lambda) = \Delta_{2,6}(\lambda)$, P_6, $P_{5,1,\lambda}$, $P_{3,2,\lambda}$, $P_{1,5,\lambda}$, P_{6,a_3}, P_{6,d_3}, $P_{5,1,a_3}$, $P_{5,1,d_3}$, $P_{3,2,a_3}$, $P_{3,2,d_3}$, $P_{1,5,a_3}$, $P_{1,5,d_3}$ are polynomials in λ, a_2, a_4, c_2, c_4, b_3 and c_3, and $P_{5,1,a_5}$, $P_{5,1,d_5}$, $P_{3,2,a_5}$, $P_{3,2,d_5}$, $P_{1,5,a_5}$, $P_{1,5,d_5}$, are polynomials in λ;

(vi) Suppose that for an $n \in \mathbb{N}$, $n \geq 6$,
– all odd coefficients in V_m, $3 \leq m \leq n$, may be written as fractions; each denominator is equal to $-\Delta_m(\lambda)$, where

$$\Delta_m(\lambda) = \begin{cases} \Delta(m, 1, m; \lambda), & \text{for odd } m \\ \Delta_{1,m}(\lambda), & \text{for } m \equiv 0 \ (\mathrm{mod}\ 4) \\ \Delta_{2,m}(\lambda), & \text{otherwise;} \end{cases}$$

each numerator is a sum of products, each product being that of an $a_{2\nu-1}$ or $d_{2\nu-1}$, $1 \leq 2\nu-1 < m$ and of a polynomial in λ, a_2, $c_2 \ldots, a_\mu$, b_μ, c_μ, d_μ, $\mu < m$;
– if $m \equiv 0 \ (\mathrm{mod}\ 4)$, $4 \leq m \leq n$, then η_m is a fraction whose denominator is $-\Delta_{1,m}(\lambda)$ and the numerator is a sum of products, each product being that of an $a_{2\nu-1}$ or $d_{2\nu-1}$, $1 \leq 2\nu - 1 < m$ and of a polynomial in λ, a_2, c_2, $\ldots, a_\mu$, b_μ, c_μ, d_μ, $\mu < m$;
– if $m \equiv 0 \ (\mathrm{mod}\ 2)$ and $m \neq 0 \ (\mathrm{mod}\ 4)$, $4 \leq m \leq n$, then η_m is a fraction whose denominator is $-\Delta_{2,m}(\lambda)$ and the numerator is a sum of products, each product being that of an $a_{2\nu-1}$ or $d_{2\nu-1}$, $1 \leq 2\nu - 1 < m$ and of a polynomial in λ, a_2, c_2, $\ldots, a_\mu$, b_μ, c_μ, d_μ, $\mu < m$;
then
– all odd coefficients in V_m, $3 \leq m \leq n + 1$, may be written as fractions; each denominator is equal to $-\Delta_m(\lambda)$, where

$$\Delta_m(\lambda) = \begin{cases} \Delta(m, 1, m; \lambda), & \text{for odd } m \\ \Delta_{1,m}(\lambda), & \text{for } m \equiv 0 \ (\mathrm{mod}\ 4) \\ \Delta_{2,m}(\lambda), & \text{otherwise;} \end{cases}$$

each numerator is a sum of products, each product being that of an $a_{2\nu-1}$ or $d_{2\nu-1}$, $1 \le 2\nu - 1 < m$ and of a polynomial in λ, a_2, c_2, $\ldots$, a_μ, b_μ, c_μ, d_μ, $\mu < m$;

– if $m \equiv 0 \pmod 4$, $4 \le m \le n+1$, then η_m is a fraction whose denominator is $-\Delta_{1,m}(\lambda)$ and the numerator is a sum of products, each product being that of an $a_{2\nu-1}$ or $d_{2\nu-1}$, $1 \le 2\nu - 1 < m$ and of a polynomial in λ, a_2, c_2, $\ldots$, a_μ, b_μ, c_μ, d_μ, $\mu < m$;

– if $m \equiv 0 \pmod 2$, and $m \not\equiv 0 \pmod 4$, $4 \le m \le n+1$, then η_m is a fraction whose denominator is $-\Delta_{2,m}(\lambda)$ and the numerator is a sum of products, each product being that of an $a_{2\nu-1}$ or $d_{2\nu-1}$, $1 \le 2\nu - 1 < m$ and of a polynomial in λ, a_2, c_2, $\ldots$, a_μ, b_μ, c_μ, d_μ, $\mu < m$.

PROOF. (i) From (9) we have $D_2 = a_1 x^2 + c_1 xy + b_1 xy + d_1 y^2$. By ($I_1$) and ($I_6$) $D_2 = \lambda(x^2 + y^2)$, hence $\eta_2 = \lambda$.

(ii) We consider D_3 from the derivative of the Liapunov function V with respect to the system (4). Taking into account that $b_2 = d_2 = 0$, from (I_2) and (I_4) there results the following system of linear equations for the coefficients of the terms in V_3

$$
\begin{aligned}
V_{2,1} &= -a_2 \\
-3V_{3,0} + 2V_{1,2} &= 0 \\
-2V_{2,1} + 3V_{0,3} &= -c_2 \\
-V_{1,2} &= 0.
\end{aligned}
$$

The determinant of this system is equal to $\Delta(3,1,3;\lambda)$ which, based on (vi) from Proposition 2.1, is strictly positive. Hence the system has a unique solution.

(iii) We consider D_4 from the derivative of the Liapunov function V with respect to the system (4) and we identify it with $\eta_4(x^2 + y^2)^2$. There follows a linear system having 5 equations in 6 unknowns:

$$
\begin{aligned}
-\eta_4 + 4\lambda V_{4,0} + V_{3,1} &= -2a_3 V_{2,0} - 3a_2 V_{3,0} - b_2 V_{2,1} \\
-4V_{4,0} + 4\lambda V_{3,1} + 2V_{2,2} &= -2a_2 V_{2,1} - 2b_2 V_{1,2} - 2b_3 V_{2,0} \\
-2\eta_4 - 3V_{3,1} + 4\lambda V_{2,2} + 3V_{1,3} &= -a_2 V_{1,2} - 3b_2 V_{0,3} - 3c_2 V_{3,0} \\
&\quad - d_2 V_{2,1} \\
-2V_{2,2} + 4\lambda V_{1,3} + 4V_{0,4} &= -2c_2 V_{2,1} - 2c_3 V_{2,0} - 2d_2 V_{1,2} \\
-\eta_4 - V_{1,3} + 4\lambda V_{0,4} &= -c_2 V_{1,2} - 3d_2 V_{0,3} - 2d_3 V_{2,0}.
\end{aligned}
$$

Since we look for *a solution* of this system, we may take $V_{2,2} = 0$. Then the remaining system has 5 equations in 5 unknowns and it has a unique solution, its determinant being equal to $\Delta_4(\lambda) = -\Delta_{1,4}(\lambda) \leq -\Delta_{1,4}(0) < 0$, cf. (iii) in Proposition 2.2.

Taking into account the hypotheses (I_2) and (I_4) and the fact that (by (ii)) $V_{3,0}$ and $V_{1,2}$ admit λ as a factor, the right-hand side can be written as

$$
-\begin{pmatrix} a_3 \\ 0 \\ 0 \\ 0 \\ d_3 \end{pmatrix} - \lambda \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} - \begin{pmatrix} 0 \\ -b_3 - 2a_2 V_{2,1} \\ 0 \\ -c_3 - 2c_2 V_{2,1} \\ 0 \end{pmatrix},
$$

hence the last matrix contains all the terms which do not have λ as a factor.

We apply Propositions 2.4, 2.6 and 2.7 with $k = 1$ and we get that

$$
\eta_4 = [\Delta'_{1,4}(a_3, d_3; \lambda) + \lambda P_4]/\Delta_{1,4}(\lambda)
$$

$$
V_{3,1} = [a_3 P_{3,1,a_3} + d_3 P_{3,1,d_3} + \lambda P_{3,1}]/\Delta_{1,4}(\lambda)
$$

$$
V_{1,3} = [a_3 P_{1,3,a_3} + d_3 P_{1,3,d_3} + \lambda P_{1,3}]/\Delta_{1,4}(\lambda),
$$

where $P_{3,1,a_3}$, $P_{3,1,d_3}$, $P_{1,3,a_3}$ and $P_{1,3,d_3}$ are polynomials in λ. P_4, $P_{1,3}$ and $P_{3,1}$ are polynomials in λ, a_2, c_2, b_3 and c_3.

(iv) We consider the term D_5 in the derivative of the Liapunov V function with respect to the system (4). Identifying this term with 0 and using (I_2) and (I_4) we get the following system of linear equations:

$$
\begin{aligned}
5\lambda V_{5,0} + V_{4,1} &= -4a_2 V_{4,0} - 3a_3 V_{3,0} - a_4 \\
&\quad - b_3 V_{2,1} \\
-5V_{5,0} + 5\lambda V_{4,1} + 2V_{3,2} &= -3a_2 V_{3,1} - 2a_3 V_{2,1} - 2b_3 V_{1,2} \\
-4V_{4,1} + 5\lambda V_{3,2} + 3V_{2,3} &= -2a_2 V_{2,2} - a_3 V_{1,2} - 3b_3 V_{0,3} \\
&\quad - 4c_2 V_{4,0} \\
-3V_{3,2} + 5\lambda V_{2,3} + 4V_{1,4} &= -a_2 V_{1,3} - 3c_2 V_{3,1} - 3c_3 V_{3,0} \\
&\quad - d_3 V_{2,1} \\
-2V_{2,3} + 5\lambda V_{1,4} + 5V_{0,5} &= -2c_2 V_{2,2} - 2c_3 V_{2,1} - c_4 \\
&\quad - 2d_3 V_{1,2} \\
-V_{1,4} + 5\lambda V_{0,5} &= -c_2 V_{1,3} - c_3 V_{1,2} - 3d_3 V_{0,3}.
\end{aligned}
$$

The determinant of this system is equal to $\Delta_{1,5}(\lambda) = \Delta(5, 1, 5; \lambda) \geq \Delta(5, 1, 5; 0) > 0$, hence it admits a unique solution. We remark that the

right-hand sides of the second, the fourth, and the sixth equation can be written as linear combinations of products of a_3, d_3, respectively λ and as entire functions in λ, a_2, c_2, a_3, a_4, b_3, c_3, c_4 and d_3. Consequently, from the sixth equation we have that $V_{1,4}$ can be written as a sum of two terms. The first one admits λ as factor, the second one admits as factor d_3. From the fourth equation it follows that $V_{3,2}$ can be written as a sum of three terms. The first term admits as a factor λ, the second term admits as a factor a_3, while the third term d_3. Analogously we infer that $V_{5,0}$ is a sum of three terms. The first term has λ as a factor, the second a_3, while the third d_3. Thus the odd coefficients of V_5 are fractions whose denominators are equal to $\Delta_{1,5}(\lambda)$, and the numerators are sums of products. Each product is that of λ or a_3 or d_3 and of a polynomial in λ, a_2, c_2, a_3, a_4, b_3, c_3, c_4 and d_3. We may use Proposition 2.10 to get the same result.

(v) We take the term D_6 from the derivative of the Liapunov function V with respect to the system (4) and we identify it with $\eta_6(x^2 + y^2)^3$. Then we get a system of linear equations with left-hand sides

$$
\begin{aligned}
-\eta_6 \quad &+6\lambda V_{6,0} \quad +V_{5,1} \\
&-6V_{6,0} \quad +6\lambda V_{5,1} \quad +2V_{4,2} \\
-3\eta_6 \quad\quad &\qquad -5V_{5,1} \quad +6\lambda V_{4,2} \quad +3V_{3,3} \\
&\qquad\qquad\qquad -4V_{4,2} \quad +6\lambda V_{3,3} \quad +4V_{2,4} \\
-3\eta_6 \quad\quad &\qquad\qquad\qquad\qquad\qquad -3V_{3,3} \quad +6\lambda V_{2,4} \quad +5V_{1,5} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2V_{2,4} \quad +6\lambda V_{1,5} \quad +6V_{0,6} \\
-\eta_6 \quad\quad &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -V_{1,5} \quad +6\lambda V_{0,6},
\end{aligned}
$$

and right-hand sides

$$
-\sum_{j=2}^{5}(7-j)a_j V_{7-j,0} - 1\sum_{j=2}^{4} b_j V_{6-j,1}
$$

$$
-\sum_{j=2}^{4}(6-j)a_j V_{6-j,1} - 2\sum_{j=2}^{5} b_j V_{5-j,2}
$$

$$
-\sum_{j=2}^{4}(5-j)a_j V_{5-j,2} - 3\sum_{j=2}^{4} b_j V_{4-j,3} \quad -5\sum_{j=2}^{2} c_j V_{5,2-j} - \sum_{j=2}^{2}(3-j)d_j V_{4,3-j}
$$

$$
-\sum_{j=2}^{3}(4-j)a_j V_{4-j,3} - 4\sum_{j=2}^{3} b_j V_{3-j,4} \quad -4\sum_{j=2}^{3} c_j V_{4,3-j} - \sum_{j=2}^{3}(4-j)d_j V_{3,4-j}
$$

$$-\sum_{j=2}^{2}(3-j)a_j V_{3-j,4} - 5\sum_{j=2}^{2} b_j V_{2-j,5} - 3\sum_{j=2}^{4} c_j V_{3,4-j} - \sum_{j=2}^{4}(5-j)d_j V_{2,5-j}$$

$$-2\sum_{j=2}^{5} c_j V_{2,5-j} - \sum_{j=2}^{4}(6-j)d_j V_{1,6-j}$$

$$-\sum_{j=2}^{4} c_j V_{1,6-j} \ -\sum_{j=2}^{5}(7-j)d_j V_{0,7-j}.$$

Since we have 8 unknowns and only 7 equations we decompose the fourth equation into two equations preserving some symmetry properties. This may be done so:

$$-4V_{4,2} + 3\lambda V_{3,3} \qquad = -\sum_{j=2}^{3}(4-j)a_j V_{4-j,3} - 4\sum_{j=2}^{3} b_j V_{3-j,4}$$

$$3\lambda V_{3,3} + 4V_{2,4} = -4\sum_{j=2}^{3} c_j V_{4,3-j} - \sum_{j=2}^{3}(4-j)d_j V_{3,4-j}$$

Thus we get a new system of linear equations whose determinant is equal to $-\Delta_{2,6}(\lambda) > 0$ (by (iii) Proposition 2.3), hence it has a unique solution. Its right-hand side can be written as

$$-\begin{pmatrix} a_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d_3 \end{pmatrix} - \lambda \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{pmatrix} - a_3 \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{pmatrix} - d_3 \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{pmatrix} - \begin{pmatrix} 0 \\ \ldots \\ 0 \\ \ldots \\ \ldots \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

since
     – the right-hand side of the first equation has two sums. In the case of the first sum, if j is even, then the first subscript of $V_{7-j,0}$ is odd; if j is odd, then the subscript of a_j is odd, too. In the case of the second sum if j is even, then $b_j = 0$ (from (I_2)), and if j is odd, then the first subscript of $V_{6-j,1}$ is odd.

– the right-hand side of the third equation has 4 sums. The first and the second sum can be discussed as we have already done earlier. The first subscript of $V_{5,2-j}$ from the third sum is odd. The last sum is zero, since $d_2 = 0$ from (I_4).

– the right-hand side of the sixth equation has 4 sums. The first three sums can be discussed as already done earlier. At the last sum if j is even then $d_j = 0$, and for $j = 3$ it has d_3 as a factor.

– the right-hand side of the 8th equation has two sums. The first sum contains coefficients V having the first subscript odd. In the second sum if j is even, then $d_j = 0$ by (I_4), and if j is odd, then d_3 and d_5 are factors.

Applying Propositions 2.5, 2.8 and 2.9 we get the form of η_6 and of the odd coefficients of V_6.

(vi) Suppose that $n + 1$ is odd. We consider the term D_{n+1} from the derivative of the Liapunov function V with respect to the system (4) and we identify this term with 0. There results a system of linear equations. Let us denote $\bar{n} = n + 1$. Then the left-hand side of this system is

$$
\begin{array}{llll}
\bar{n}\lambda V_{n+1,0} & +V_{n,1} & & \\
-\bar{n}V_{n+1,0} & +\bar{n}\lambda V_{n,1} & +2V_{n-1,2} & \\
& -nV_{n,1} & +\bar{n}\lambda V_{n-1,2} & \\
\end{array}
$$

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \quad \vdots \\
\end{array}
$$

$$
\begin{array}{lll}
-3V_{3,n-2} & +\bar{n}\lambda V_{2,n-1} & +nV_{1,n} \\
& -2V_{2,n-1} & +\bar{n}\lambda V_{1,n} & +\bar{n}V_{0,n+1} \\
& & -V_{1,n} & +\bar{n}\lambda V_{0,n+1},
\end{array}
$$

and the right-hand side is

$$
-\sum_{j=2}^{n}(n + 2 - j)a_j V_{n+2-j,0} - 1\sum_{j=2}^{n-1} b_j V_{n+1-j,1}
$$

$$
-\sum_{j=2}^{n-1}(n + 1 - j)a_j V_{n+1-j,1} - 2\sum_{j=2}^{n} b_j V_{n-j,2}
$$

$$
-\sum_{j=2}^{n-1}(n - j)a_j V_{n-j,2} - 3\sum_{j=2}^{n-1} b_j V_{n-1-j,3} - n\sum_{j=2}^{2} c_j V_{n,2-j} - \sum_{j=2}^{2}(3 - j)d_j V_{n-1,3-j}
$$

$$
-\sum_{j=2}^{n-2}(n - 1 - j)a_j V_{n-1-j,3} - 4\sum_{j=2}^{n-2} b_j V_{n-2-j,4}
$$

$$
-(n - 1)\sum_{j=2}^{3} c_j V_{n-1,3-j} - \sum_{j=2}^{3}(4 - j)d_j V_{n-2,4-j}
$$

$$-\sum_{j=2}^{n-3}(n-2-j)a_j V_{n-2-j,4} - 5\sum_{j=2}^{n-3} b_j V_{n-3-j,5}$$

$$-(n-2)\sum_{j=2}^{4} c_j V_{n-2,4-j} - \sum_{j=2}^{4}(5-j)d_j V_{n-3,5-j}$$

$$\vdots$$

$$-\sum_{j=2}^{n-k}(n-k+1-j)a_j V_{n-k+1-j,k+1} - (k+2)\sum_{j=2}^{n-k} b_j V_{n-k-j,k+2}$$

$$-(n-k+1)\sum_{j=2}^{k+1} c_j V_{n-k+1,k+1-j} - \sum_{j=2}^{k+1}(k+2-j)d_j V_{n-k,k+2-j}$$

$$\vdots$$

$$-\sum_{j=2}^{2}(3-j)a_j V_{3-j,n-1} - n\sum_{j=2}^{2} b_j V_{2-j,n} - 3\sum_{j=2}^{n-1} c_j V_{3,n-1-j} - \sum_{j=2}^{n-1}(n-j)d_j V_{2,n-j}$$

$$-2\sum_{j=2}^{n} c_j V_{2,n-j} - \sum_{j=2}^{n-1}(n+1-j)d_j V_{1,n+1-j}$$

$$-\sum_{j=2}^{n-1} c_j V_{1,n+1-j} - \sum_{j=2}^{n}(n+2-j)d_j V_{0,n+2-j}.$$

The linear system has $n+2$ equations with $n+2$ unknowns and it has a unique solution, since its determinant is equal to $-\Delta(n+1,1,n+1;\lambda)\neq 0$, for all $\lambda\in\mathbb{R}$ ((vi), Proposition 2.1).

We see that the right-hand sides can be written as

$$-\lambda\begin{pmatrix}\cdots\\\cdots\\\cdots\\\cdots\\\cdots\\\vdots\\\cdots\\\cdots\end{pmatrix} - a_3\begin{pmatrix}\cdots\\\cdots\\\cdots\\\cdots\\\cdots\\\vdots\\\cdots\\\cdots\end{pmatrix} - d_3\begin{pmatrix}\cdots\\\cdots\\\cdots\\\cdots\\\cdots\\\vdots\\\cdots\\\cdots\end{pmatrix} - \cdots$$

$$\ldots - a_{n-1} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \\ \cdots \end{pmatrix} - d_{n-1} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \\ \cdots \end{pmatrix} - \begin{pmatrix} \cdots \\ 0 \\ \cdots \\ 0 \\ \cdots \\ \vdots \\ \cdots \\ 0 \end{pmatrix}$$

since the right-hand side of the $(2p)^{\text{th}}$ equation is

$$- \sum_{j=2}^{n-2p+2} (n - 2p + 3 - j) a_j V_{n-2p+3-j,2p-1} - 2p \sum_{j=2}^{n-2p+2} b_j V_{n-2p+2-j,2p}$$

$$- (n - 2p + 3) \sum_{j=2}^{2p-1} c_j V_{n-2p+3,2p-1-j} - \sum_{j=2}^{2p-1} (2p - j) d_j V_{n-2p+2,2p-j}.$$

In the first sum if j is even, then $V_{n-2p+3-j,2p-1}$, by hypothesis, can be written as a sum of products, and each product has as a factor $a_{2\nu-1}$ or $d_{2\nu-1}$. In the second sum if j is odd then $V_{n-2p+2-j,2p}$ has the same representation. In the third sum $V_{n-2p+3,2p-1-j}$ admits a representation of the same type depending on $a_{2\nu-1}$ and $d_{2\nu-1}$. In the last sum we have, in fact, only d_j with odd subscript. Taking into account the Proposition 2.10 there results that all the odd coefficients in V_{n+1} can be written as sums of products, each product having a factor $a_{2\nu-1}$ or $d_{2\nu-1}$, $2\nu - 1 < n+1$.

We suppose that $n + 1 \equiv 0 \pmod 4$. Consider the term D_{n+1} in the derivative of the Liapunov function V with respect to the system (4) and identify it with $\eta_{4k}(x^2 + y^2)^{2k}$, where $n+1 = 4k$. There results a system of $4k+1$ equations with $4k+2$ unknowns, namely $\eta_{4k}, V_{4k,0}, \ldots, V_{0,4k}$. Since we seek *a solution*, we take $V_{2k,2k} = 0$. Then there remains a quadratic system of linear equations, having determinant equal to $-\Delta_{1,4k}(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$ ((iii), Proposition 2.2).

Using the above-mentioned remarks the right-hand side can be written as

$$
-\begin{pmatrix} a_{4k-1} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ d_{4k-1} \end{pmatrix} - \lambda \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \\ \cdots \end{pmatrix} - a_3 \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \\ \cdots \end{pmatrix} - d_3 \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \vdots \\ \cdots \\ \cdots \end{pmatrix} - \cdots - \begin{pmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ \vdots \\ \cdots \\ 0 \end{pmatrix}.
$$

By Propositions 2.4, 2.6 and 2.7 we get that η_{4k}, $V_{4k-1,1}, V_{4k-3,3} \dots, V_{1,4k-1}$ admit the following representations:

$$
(11)\ \ \eta_{4k}=\frac{\Delta'_{1,4k}(a_{4k-1},d_{4k-1};\lambda)+\sum_{i=1}^{2k-2}\left(a_{2i+1}P_{4k,a_{2i+1}}+d_{2i+1}P_{4k,d_{2i+1}}\right)+\lambda P_{4k}}{\Delta_{1,4k}(\lambda)},
$$

$$
V_{4k-p,p} = \left[\sum_{i=1}^{2k-1}\left(a_{2i+1}P_{4k-p,p,a_{2i+1}} + d_{2i+1}P_{4k-p,p,d_{2i+1}}\right) + \lambda P_{4k-p,p}\right]/\Delta_{1,4k}(\lambda),
$$

$$
p \in \{1,3,\dots,4k-1\}
$$

where $P_{4k-p,p,a_{4k-1}}$, $P_{4k-p,p,d_{4k-1}}$, for all $p \in \{1,3,\dots,4k-1\}$ are polynomials in λ. P_{4k}, $P_{4k,a_{2i+1}}$, $P_{4k,d_{2i+1}}$, $i \in \{1,2,3,\dots,2k-2\}$, $P_{4k-p,p,a_{2j+1}}$, $P_{4k-p,p,d_{2j+1}}$, $j \in \{1,2,\dots,2k-2\}$, $p \in \{1,3,\dots,4k-1\}$ are polynomials in a_ν, b_ν, c_ν and d_ν where $\nu < 4k$.

Finally, we suppose that $n+1 \equiv 0 \pmod 2$ and $n+1 \not\equiv 0 \pmod 4$. We consider the term D_{n+1} in the derivative of the Liapunov function V with respect to the system (4) and we identify it with $\eta_{4k+2}(x^2+y^2)^{2k+1}$, where $n+1 = 4k+2$. There results a system of linear equations having $4k+3$ equations and $4k+4$ unknowns, namely η_{4k+2}, $V_{4k+2,0}, \dots, V_{0,4k+2}$. We decompose the $(2k+2)^{\mathrm{nd}}$ equation into two equations. The decomposition is done so that the left-hand sides of these equations have the form

$$
-(2k+2)V_{2k+2,2k}+(2k+1)\lambda V_{2k+1,2k+1}
$$
$$
(2k+1)\lambda V_{2k+1,2k+1} + (2k+2)V_{2k+2,2k},
$$

while the right-hand side of the first equation contains only the terms in which the coefficients of the sums are a_j and b_j; the second contains all the other terms, namely those terms in which the coefficients of the sums

are c_j and d_j). This new system of linear equations is quadratic and it has a unique solution since its determinant is different from zero. ((iii) Proposition 2.3).

Using Propositions 2.5, 2.8 and 2.9 it results that η_{4k+2} and the odd coefficients in V_{4k+2} have the following representations:

$$
\begin{aligned}
\eta_{4k+2} = \Bigg[& \Delta'_{2,4k+2}(a_{4k+1}, d_{4k+1}; \lambda) \\
(12) \qquad & + \sum_{i=1}^{2k-1} \big(a_{2i+1} P_{4k+2, a_{2i+1}} + d_{2i+1} P_{4k+2, d_{2i+1}} \big) \\
& + \lambda P_{4k+2} \Bigg] / \Delta_{2,4k+2}(\lambda),
\end{aligned}
$$

$$
\begin{aligned}
V_{4k+2-p,p} = \Bigg[& \sum_{i=1}^{2k} \big(a_{2i+1} P_{4k+2-p,p,a_{2i+1}} + d_{2i+1} P_{4k+2-p,p,d_{2i+1}} \big) \\
& + \lambda P_{4k+2-p,p} \Bigg] / \Delta_{2,4k+2}(\lambda), \quad p \in \{1, 3, \ldots, 4k+1\}. \qquad \square
\end{aligned}
$$

Remark. It is easy to see that the Liénard system in [4] is a particular case of the system 4, among other things $F_4(\cdot) = 0$. Specializing $V_{3,1}$ on the lines of BLOWS and LLOYD in [4] we see that their result $(V_{3,1} = \frac{-1}{8} a_3)$ does not agree with ours.

Let $m = \max\{N, M\}$ and $\alpha = \left[\dfrac{m-1}{2} \right]$, the integer part of $\dfrac{m-1}{2}$.

Theorem 2.4 (The canonical forms of the focal values). *Consider the system* (4) *and suppose that the assumptions* (I$_1$)–(I$_6$) *are satisfied. Then, for a convenient λ, the focal values can be written as*

$$
(13) \qquad
\begin{cases}
\eta_2 = \tilde{\eta}_2 = \lambda, \\
\eta_{2k} = \tilde{\eta}_{2k} + \displaystyle\sum_{j=1}^{k-2} \big(a_{2j+1} R_{2k, a_{2j+1}} + d_{2j+1} R_{2k, d_{2j+1}} \big) + \lambda(\ldots), \; k \geq 2,
\end{cases}
$$

where

$$
(14) \qquad \tilde{\eta}_{2k} = \frac{(2k-1)!!}{\sum_{j=0}^{k} \binom{k}{j} (2k-1-2j)!!(2j-1)!!} (a_{2k-1} + d_{2k-1})
$$

$$
\text{with } (-1)!! = 1,
$$

and $R_{2k,a_{2j+1}}$ as well as $R_{2k,d_{2j+1}}$ are polynomials depending on the coefficients of the system (4) such that if a_ν or d_ν appears in these polynomials, then $\nu \geq 2j + 1$.

PROOF. Let

$$\Delta_{2k}(\lambda) = \begin{cases} \Delta_{1,2k}(\lambda), & k \text{ even} \\ \Delta_{2,2k}(\lambda), & k \text{ odd} \end{cases} \quad \text{and}$$

$$\Delta'_{2k}(a,d;\lambda) = \begin{cases} \Delta'_{1,2k}(a,d;\lambda), & k \text{ even} \\ \Delta'_{2,2k}(a,d;\lambda), & k \text{ odd}. \end{cases}$$

By Lemma 2.1, from (11) and (12) we infer

$$\begin{cases} \eta_2 = \lambda \\ \eta_{2k} = [\Delta'_{2k}(a_{2k-1},d_{2k-1};\lambda) + \sum_{j=1}^{k-2} \left(a_{2j+1}P_{2k,a_{2j+1}} + d_{2j+1}P_{2k,d_{2j+1}} \right) \\ \qquad + \lambda P_{2k}]/\Delta_{2k}(\lambda), \; 2 \leq k. \end{cases}$$

We rearrange the terms in $\sum_{j=1}^{k-2} \left(a_{2j+1}P_{2k,a_{2j+1}} + d_{2j+1}P_{2k,d_{2j+1}} \right)$ in order to get

$$\sum_{j=1}^{k-2} \left(a_{2j+1}Q_{2k,a_{2j+1}} + d_{2j+1}Q_{2k,d_{2j+1}} \right)$$

with the property that if the polynomials $Q_{2k,a_{2j+1}}$ and $Q_{2k,d_{2j+1}}$ contain a_ν or d_ν with ν odd, then $\nu \geq 2j + 1$. Taking into account the Propositions 2.2 and 2.3 we can write that $\Delta_{2k}(\lambda) = \Delta_{2k}(0) + \lambda^2(\ldots)$ and for a convenient λ we have

$$\eta_{2k} = \frac{\Delta'_{2k}(a_{2k-1},d_{2k-1};0) + \sum\limits_{j=1}^{k-2} \left(a_{2j+1}Q_{2k,a_{2j+1}} + d_{2j+1}Q_{2k,d_{2j+1}} \right) + \lambda Q_{2k}}{\Delta_{2k}(0) + \lambda^2(\ldots)}$$

$$= \frac{\Delta'_{2k}(a_{2k-1},d_{2k-1};0)}{\Delta_{2k}(0)} + \frac{\sum\limits_{j=1}^{k-2} \left(a_{2j+1}Q_{2k,a_{2j+1}} + d_{2j+1}Q_{2k,d_{2j+1}} \right)}{\Delta_{2k}(0)} + \lambda(\ldots).$$

We denote

$$\tilde{\eta}_{2k} = \frac{\Delta'_{2k}(a_{2k-1},d_{2k-1};0)}{\Delta_{2k}(0)},$$

$$R_{2k,a_{2j+1}} = \frac{a_{2j+1}Q_{2k,a_{2j+1}}}{\Delta_{2k}(0)}, \quad R_{2k,d_{2j+1}} = \frac{d_{2j+1}Q_{2k,d_{2j+1}}}{\Delta_{2k}(0)}.$$

From the Propositions 2.2, 2.3, 2.4 and 2.5 we have (14), that is

$$\tilde{\eta}_{2k} = \frac{(2k-1)!!}{\sum_{j=0}^{k} \binom{k}{j}(2k-1-2j)!!(2j-1)!!}(a_{2k-1}+d_{2k-1}) \text{ with } (-1)!! = 1,$$

We remark that for all $k > \alpha + 1$, $\tilde{\eta}_{2k} = 0$. $\square$

Let $B(O, r^*)$ be an open disc centered in the origin and having radius $r^* > 0$ which satisfies the following two conditions:

(a) $V(x,y) \geq 0$, for all $(x,y) \in B(O,r^*)$ and $V(x,y) = 0 \iff x = y = 0$;

(b) the origin is the unique critical point of the system (4) in the disc $B(O,r^*)$.

Theorem 2.5. *Consider a system of differential equations of the form* (4) *and suppose that the assumptions* (I$_1$)–(I$_7$) *are satisfied. Then*

(i) *there exist at most α small amplitude limit cycles;*

(ii) *the coefficients $a_1, a_3, d_3, a_5, d_5, \ldots, a_m$ and d_m can be chosen so that the corresponding system of the form* (4) *has precisely α small amplitude limit cycles;*

(iii) *denoting by $\mathcal{H}_{(1.2.2)}$ the supremum of the number of limit cycles of systems of the form* (4), *we have*

$$\mathcal{H}_{(1.2.2)} \geq \alpha.$$

PROOF. (i) Let U be the neighborhood of the origin on which the first return map δ corresponding to the system (4) can be defined. Let $B(O, \bar{r}) \subset U$ be the open disc centered in the origin O and having radius $\bar{r} > 0$, with the property that there exists an $\nu \geq 0$ such that $\dot{V}(r) = \eta_{2\nu+2}r^{2\nu+2} + o(r^{2\nu+2})$ for every $0 < r < \bar{r}$. Then $\eta_2 = \eta_4 = \cdots = \eta_{2\nu} = 0$, $\eta_{2\nu+2} = \tilde{\eta}_{2\nu+2}$ and $\text{sgn}\,\dot{V}(r) = \text{sgn}\,\tilde{\eta}_{2\nu+2} = \text{sgn}(a_{2\nu+1} + d_{2\nu+1}) \neq 0$ for every $0 < r < \bar{r}$. Thus, from (7), $\delta(0) = \delta'(0) = \cdots = \delta^{(2\nu)}(0) = 0$ and $\delta^{(2\nu+1)}(0) = 2\pi(2\nu+1)!\tilde{\eta}_{2\nu+2} \neq 0$ ($b = b_1 = 1$). There exists an open disc $B(O, \bar{\bar{r}}) \subset B(O, \bar{r})$ such that $\delta^{(2\nu+1)}$ keeps a constant sign on the open interval $(-\bar{\bar{r}}, \bar{\bar{r}})$. Thus $\delta^{(2\nu)}$ has at most one root in the interval $(-\bar{\bar{r}}, \bar{\bar{r}})$; $\delta^{(2\nu-1)}$ has at most two roots in the interval $(-\bar{\bar{r}}, \bar{\bar{r}})$. Going on in this way we infer that δ has at most $2\nu+1$ roots in the interval $(-\bar{\bar{r}}, \bar{\bar{r}})$. But $\delta(0) = 0$, so there remain at most 2ν roots in this interval. Since to each periodic solution there correspond two roots of the first return map (a positive and

a negative one), it results that the disc $B(O, \bar{r})$ contains at most ν periodic solutions. Since $\nu \leq \alpha$ (if $\nu > \alpha$ and $\delta(0) = \delta'(0) = \cdots = \delta^{(2\nu)}(0) = 0$, then $\eta_2 = \cdots = \eta_{2\alpha+2} = 0$ and $\dot{V} \equiv 0$), it follows that at most α small amplitude limit cycles can appear.

(ii) We show that by a convenient choice of the coefficients of the polynomials F_1 and F_4 we can get precisely α small amplitude limit cycles.

Denote by $I_{a_i}^{(1)}$ an interval closed and symmetric with respect to the origin, which contains the coefficient a_i. Similarly, let $I_{d_i}^{(1)}$ be a closed and symmetric interval which contains the coefficient d_i. Fix $(a_{2\alpha+1}, d_{2\alpha+1}) \in I_{a_{2\alpha+1}}^{(1)} \times I_{d_{2\alpha+1}}^{(1)}$ with $a_{2\alpha+1} + d_{2\alpha+1} < 0$. From the representation (13) it follows that there exists a hyper-parallelepiped $J_1 = \prod_{i=1}^{\alpha} \left(I_{a_{2i-1}}^{(1)} \times I_{d_{2i-1}}^{(1)} \right)$ such that

$$|\eta_{2\alpha+2} - \tilde{\eta}_{2\alpha+2}| < \frac{1}{3} \quad \text{on } J_1.$$

Choose $r_{2\alpha+1}$ such that the following inequalities hold:

$$0 < r_{2\alpha+1} < \min\{r^*, \bar{\bar{r}}\},$$

$$\eta_{2\alpha+2} r^{2\alpha+2} + \cdots < 0 \quad \text{for every } r \in (0, r_{2\alpha+1}], \quad \text{on } J_1.$$

We choose $(a_{2\alpha-1}, d_{2\alpha-1}) \in I_{a_{2\alpha-1}}^{(1)} \times I_{d_{2\alpha-1}}^{(1)}$ with $a_{2\alpha-1} + d_{2\alpha-1} > 0$ such that if we denote $J_2 = \prod_{i=1}^{\alpha-1} \left(I_{a_{2i-1}}^{(2)} \times I_{d_{2i-1}}^{(2)} \right)$ with $I_{a_{2i-1}}^{(2)} \subset I_{a_{2i-1}}^{(1)}$ and $I_{d_{2i-1}}^{(2)} \subset I_{d_{2i-1}}^{(1)}$, $i \in \{1, 2, \ldots, \alpha - 1\}$, closed and symmetric intervals, then

$$|\eta_{2\alpha} - \tilde{\eta}_{2\alpha}| < \frac{1}{3} \quad \text{on } J_2,$$

$$\max_{J_2} |\eta_{2\alpha} r_{2\alpha+1}^{2\alpha}| < \frac{1}{\alpha} \min_{J_2} |\eta_{2\alpha+2} r_{2\alpha+1}^{2\alpha+2} + \cdots|.$$

Choose $r_{2\alpha-1}$ such that the following inequalities hold:

$$0 < r_{2\alpha-1} < r_{2\alpha+1}$$

$$\operatorname{sgn}_{J_2}[\eta_{2\alpha} r_{2\alpha-1}^{2\alpha} + \eta_{2\alpha+2} r_{2\alpha-1}^{2\alpha+2} + \cdots] = 1.$$

We choose $(a_{2\alpha-3}, d_{2\alpha-3}) \in I_{a_{2\alpha-3}}^{(2)} \times I_{d_{2\alpha-3}}^{(2)}$ with $a_{2\alpha-3} + d_{2\alpha-3} < 0$ such that if we denote $J_3 = \prod_{i=1}^{\alpha-2} \left(I_{a_{2i-1}}^{(3)} \times I_{d_{2i-1}}^{(3)} \right)$ with $I_{a_{2i-1}}^{(3)} \subset I_{a_{2i-1}}^{(2)}$

and $I_{d_{2i-1}}^{(3)} \subset I_{d_{2i-1}}^{(2)}$, $i \in \{1, 2, \ldots, \alpha - 2\}$, closed and symmetric intervals, then

$$|\eta_{2\alpha-2} - \tilde{\eta}_{2\alpha-2}| < \frac{1}{3} \quad \text{on } J_3,$$

$$\max_{J_3} |\eta_{2\alpha-2} r_{2\alpha+1}^{2\alpha-2}| < \frac{1}{\alpha} \min_{J_3} |\eta_{2\alpha+2} r_{2\alpha+1}^{2\alpha+2} + \cdots|,$$

$$\max_{J_3} |\eta_{2\alpha-2} r_{2\alpha-1}^{2\alpha-2}| < \frac{1}{\alpha-1} \min_{J_3} |\eta_{2\alpha} r_{2\alpha-1}^{2\alpha} + \eta_{2\alpha+2} r_{2\alpha-1}^{2\alpha+2} \cdots|.$$

Choose $r_{2\alpha-3}$ such that the following inequalities hold:

$$0 < r_{2\alpha-3} < r_{2\alpha-1},$$

$$\operatorname{sgn}_{J_3}[\eta_{2\alpha-2} r_{2\alpha-3}^{2\alpha-2} + \eta_{2\alpha} r_{2\alpha-1}^{2\alpha} + \cdots] = -1.$$

We choose $(a_{2\alpha-5}, d_{2\alpha-5}) \in I_{a_{2\alpha-5}}^{(3)} \times I_{d_{2\alpha-5}}^{(3)}$ with $a_{2\alpha-5} + d_{2\alpha-5} > 0$ such that if we denote $J_4 = \prod_{i=1}^{\alpha-3} \left(I_{a_{2i-1}}^{(4)} \times I_{d_{2i-1}}^{(4)} \right)$ with $I_{a_{2i-1}}^{(4)} \subset I_{a_{2i-1}}^{(3)}$ and $I_{d_{2i-1}}^{(4)} \subset I_{d_{2i-1}}^{(3)}$, $i \in \{1, 2, \ldots, \alpha - 3\}$, closed and symmetric intervals, then

$$|\eta_{2\alpha-4} - \tilde{\eta}_{2\alpha-4}| < \frac{1}{3} \quad \text{on } J_4,$$

$$\max_{J_4} |\eta_{2\alpha-4} r_{2\alpha+1}^{2\alpha-4}| < \frac{1}{\alpha} \min_{J_4} |\eta_{2\alpha+2} r_{2\alpha+1}^{2\alpha+2} + \cdots|,$$

$$\max_{J_4} |\eta_{2\alpha-4} r_{2\alpha-1}^{2\alpha-4}| < \frac{1}{\alpha-1} \min_{J_4} |\eta_{2\alpha} r_{2\alpha-1}^{2\alpha} + \eta_{2\alpha+2} r_{2\alpha-1}^{2\alpha+2} \cdots|,$$

$$\max_{J_4} |\eta_{2\alpha-4} r_{2\alpha-3}^{2\alpha-4}| < \frac{1}{\alpha-2} \min_{J_4} |\eta_{2\alpha-2} r_{2\alpha-3}^{2\alpha-2} + \eta_{2\alpha} r_{2\alpha-3}^{2\alpha} \cdots|.$$

We go on with the selections of the odd coefficients a_i, d_i and of the positive numbers r_i till we select λ and r_1.

From the assumption (I_7) for small variations of the coefficients of the polynomials F_1 and F_4 there exists a neighborhood of the origin such that the origin is the only critical point of (4) contained in it. Suppose that this neighborhood contains the ball $B(0, \bar{\bar{r}})$. If not, we choose a new $\bar{\bar{r}}$ such that this new ball be contained in the neighborhood.

Then we have the sequence of inequalities

$$\dot{V}(r_{2\alpha+1}) < 0, \ \dot{V}(r_{2\alpha-1}) > 0, \ \dot{V}(r_{2\alpha-3}) < 0, \ \ldots, \operatorname{sgn} \dot{V}(r_1) = (-1)^{\alpha+1}.$$

Thus, invoking the Bendixson–Poincaré theorem there exist α limit cycles in $B(0, \bar{\bar{r}})$. But from (i) there exist in $B(0, \bar{\bar{r}})$ at most α periodic solutions, hence the maximum number of periodic solutions is obtained.

(iii) Follows from (ii). $\square$

Remarks. (a) The method used in [4], [11], [12] and [13] is based on the existence of certain inequalities of the form

$$|\eta_{2k} - \tilde{\eta}_{2k}| < c$$

where c is an arbitrary positive constant, for a convenient selection of the odd coefficients of F_1 and F_4 having subscript less than $2k$. Then in the inequalities from the above proof we may take $\tilde{\eta}_{2k}$ instead of η_{2k}.

(b) In our case $\tilde{\eta}_{2k}$ depends only on $a_{2k-1} + d_{2k-1}$. Therefore the reasoning on the existence of limit cycles may be carried out in terms of $a_{2k-1} + d_{2k-1}$. These sums are called *Liapunov quantities*, [4], and they are denoted by $L(0) = \lambda$, $L(i) = a_{2i+1} + d_{2i+1}$, $i \geq 1$. Obviously, the expressions of the Liapunov quantities depend on the studied problem.

Example. We introduce below a system of differential equations having the form (4) which has the following two properties:

(a) it satisfies the hypotheses (I_1)–(I_6) (hence these are non-contradictory);

(b) it is not of the form considered by BLOWS and LLOYD in [4].

The system is the following one:

$$\dot{x} = \lambda x - y \quad + ax^3$$
$$\dot{y} = x \ + \lambda y \qquad + ay^3.$$

We immediately observe that the origin is a critical point. By easy manipulations it is clear that for any $(a, \lambda) \in \mathbb{R} \times [-1, 1]$ the origin is the only critical point.

Remarks. (a) Theorem 2.5 generalizes two of the results in [4], namely Theorem 2.3 and Theorem 3.1. Theorem 2.3 can be obtained by taking $F_3(y) = y$, $F_2(x) = -x$ and $F_4(\cdot) = 0$. In this case the origin is the unique critical point, and (I_7) is superfluous. Theorem 3.1 can be obtained by taking $F_3(y) = y$ and $F_4(\cdot) = 0$. In this case, in general, there are several critical points about whose behaviour there are no assumptions.

(b) From the proof of the Theorem 2.5 it can be seen that a main ingredient was that F_2 is an odd function. In [13] the case is considered in which F_2 is a second degree polynomial as well as the case deg $F_2 > 2$.

Let us comment briefly on the results of LYNCH in [13]. The most important feature lies in the fact that F_2 is no longer an odd function. This modifies significantly the whole machinery of computations, and it is no longer possible to get a simple formula for the Liapunov quantities. One of the systems considered by LYNCH in [13] is

$$(15) \qquad \begin{cases} \dot{x} = y - (a_1 x + a_2 x^2 + \cdots + a_i x^i) \\ \dot{y} = -(x + b_2 x^2), \end{cases}$$

with $b_2 \neq 0$. We consider as a Liapunov function

$$V = V_{2,0} x^2 + V_{1,1} xy + V_{0,2} y^2 + V_3 + V_4 + \cdots$$

where V_k, $k \geq 3$, are the homogeneous polynomials given in (8). The search for the coefficients of the function V is done so that the derivative of V with respect to the system (15) should have the form

$$\dot{V} = \eta_2 (x^2 + y^2) + \eta_4 (x^2 + y^2)^2 + \cdots .$$

Then we have the following result whose proof consists in linear algebra manipulations, hence we omit it.

Theorem 2.6. *There holds the equality*

$$\eta_{2n} = \frac{-\sum\limits_{k=1}^{n} \frac{2n-2k+1}{2k-1} \left[\sum\limits_{j=1}^{2n-2k+1} (2n+3-2k-j)a_j V_{2n+3-2k-j,2k-2} + b_2 V_{2n-2k,2k+1} \right]}{1 + \sum\limits_{k=1}^{n} \frac{2n-2k+1}{2k-1} \binom{n}{k-1}},$$

where $2n \leq i$.

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