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Remarks on the infinite product representations of holomorphic functions

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1. Introduction

Let f be holomorphic and different from zero for $|z| < r$, and assume $f(0) = 1$. Then, denote by g the logarithm of f for $|z| < r$, determined by $\exp\bigl(g(z)\bigr) = f(z)$ and $g(0) = 0$.
We use the representation

$$g(z) = \sum_{n=1}^{\infty} \alpha(n) \frac{z^n}{n} \quad \bigl(|z| < r\bigr) \tag{1}$$

and conclude

$$\alpha(n) \ll R^{-n} \quad \text{for} \quad 0 < R < r \ . \tag{2}$$

Now, let the sequences $\{\pi(m)\}_{m=1}^{\infty}$ and $\{\rho(m)\}_{m=1}^{\infty}$ be defined by

$$\alpha(n) = \sum_{m|n} m\pi(m) \tag{3}$$

and

$$\alpha(n) = -\sum_{m|n} m\bigl(-\rho(m)\bigr)^{m/n} \ , \tag{4}$$

respectively. Then, if

$$f(z) = 1 + \sum_{n=1}^{\infty} \gamma(n) z^n \quad \text{for} \quad |z| < r \ , \tag{5}$$

the formal infinite products

$$\prod_{m=1}^{\infty}(1-z^m)^{-\pi(m)} \tag{6}$$

and

$$\prod_{m=1}^{\infty}\big(1+\rho(m)z^m\big) \tag{7}$$

correspond via (3) and (4), respectively, to a formal power series with coefficients $\gamma(n)$.

In [3], [4] we investigated the analytic behaviour of the product (6) in connection with applications for additive arithmetical semigroups.

The aim of this paper is to prove some intrinsic connections between $\gamma(n)$ (or $\alpha(n)$) and $\rho(n)$ and to give a good estimate of the radius of convergence for the product (7). This will sharpen earlier results by Ritt [6] (see also [1], [2], [5]).

Furthermore, we show by examples, that our results are, in the given form, best possible.

2. Results

As a first consequence of (4), (5) and (7) we obtain

Proposition 1. *The following two assertions hold:*
(i) $\rho(m) \geq 0$ *for all* $m \in \mathbf{N} \Longrightarrow \gamma(n) \geq \rho(n)$ *for all* $n \in \mathbf{N}$.
(ii) $\rho(m) \in \mathbf{Z}$ *for all* $m \in \mathbf{N} \Longleftrightarrow \gamma(n) \in \mathbf{Z}$ *for all* $n \in \mathbf{N}$.

The proof of Proposition 1 is the same as that for the corresponding results about $\pi(m)$ (see Proposition 1 in [4]) and is left to the reader.

We define

$$y := \sup_{n\in\mathbf{N}} \sqrt[n]{|\alpha(n)|} \quad \text{and put}$$

$$\beta(n) := \alpha(n)y^{-n} \quad \sigma(n) := \rho(n)y^{-n} \tag{8}$$

Then (4) leads to

$$\beta(n) = -\sum_{m|n} m\big(-\sigma(m)\big)^{n/m} \tag{9}$$

with the assumption $|\beta(n)| \leq 1$.
The main result in this paper will be

Proposition 2. *If $\beta(n)$ and $\sigma(n)$ are given by (8) then the following holds:*

$$|\beta(n)| \le 1 \text{ for all } n \in \mathbf{N} \Longrightarrow |\sigma(n)| \le 1 \quad \text{for all } n \in \mathbf{N}.$$

The proof of Proposition 2 will be done by induction. In fact we prove sharper estimates (see Lemma 1 and Lemma 3).

The above result will lead to the following immediate consequences.

Corollary 1. *If $\alpha(n) = -\sum_{m|n} m\big(-\rho(m)\big)^{n/m}$ then*

$$\sup_{n\in\mathbf{N}} \sqrt[n]{|\rho(n)|} \le \sup_{n\in\mathbf{N}} \sqrt[n]{|\alpha(n)|}.$$

Corollary 2. *Let f be holomorphic and different from zero for $|z| < r$. Then, if $|\alpha(n)| \le R^{-n}$ for all $n \in \mathbf{N}$ with some $0 < R \le r$, then*

$$f(z) = \prod_{m=1}^{\infty} \big(1 + \rho(m) z^m\big)$$

holds for $|z| < R$.

Remark 1. Corollary 2 sharpens a result by Ritt [6], who proved the absolute convergence of the product only for $|z| < R/6$.

The proof of Proposition 2 can be used for mild generalizations of the above assertions. An example of such a modification is given by

Corollary 3. *Let y be positive such that $|\alpha(n)| \le y^n$ for $n = 1$ and $n \ge 7$. If*

$$|\rho(n)| \le \begin{cases} y^n & \text{for} \quad n = 2^j, \quad j = 1,2\ , \\ \frac{2}{3} y^n & \text{for} \quad n = 3\cdot 2^j, \quad j = 0,1,2\ , \\ \frac{2}{3} y^n & \text{for} \quad n = 5\cdot 2^j, \quad j = 0,1,2\ , \end{cases}$$

then

$$|\rho(n)| \le y^n \quad \text{for all } \ n \in \mathbf{N}\ .$$

3. Proofs

The easy part of the proof of Proposition 2 is given in the following

Lemma 1. *If* $|\beta(n)| \le 1$ *for all* $n \in \mathbf{N}$, *then*

$$|\sigma(n)| \le \begin{cases} 1 & , \ \textit{if} \quad n = 2^j, \ j \ge 0, \\ 2/3 & , \ \textit{if} \quad n = 3, \\ 2/5 & , \ \textit{if} \quad n = 2m+1, \ m \ge 2. \end{cases}$$

Proof. Formula (9) implies

$$\begin{aligned} &\sigma(1) = \beta(1) \\ &\sigma(n) = \frac{1}{n}\Big(\beta(n) + \sum_{\substack{m|n \\ m<n}} m\big(-\sigma(m)^{n/m}\big)\Big) \text{ for } \quad n > 1. \end{aligned} \tag{10}$$

From this follows

$$\begin{aligned} \sigma(2) &= \frac{1}{2}\left(\beta(2) + \beta(1)^2\right) \ , \\ \sigma(p) &= \frac{1}{p}\left(\beta(p) - \beta(1)^p\right) \qquad \text{for primes } p > 2 \ . \end{aligned} \tag{11}$$

First we show that

$$|\sigma\left(2^j\right)| \le 1 \quad \text{for} \quad j \ge 0 \ .$$

Obviously

$$\sigma(2^0) = \sigma(1) = \beta(1) \ .$$

Let $j \ge 1$. Then

$$\sigma(2^j) = 2^{-j}\left(\beta(2^j) + \sum_{i=0}^{j-1} 2^i \sigma(2^i)^{2^{j-i}}\right), \tag{12}$$

and therefore by induction

$$|\sigma(2^j)| \le 2^{-j}\left(1 + \sum_{i=0}^{j-1} 2^i\right) = 2^{-j}\left(1 + 2^j - 1\right) = 1. \tag{13}$$

Because of (11) the assertion of Lemma 1 holds for $n = 3, 5, 7$. Again by induction, we have for n odd, $n \ge 9$,

$$|\sigma(n)| \le \frac{1}{n}\left(|\beta(n)| + |\sigma(1)|^n + 3(2/3)^{n/3} + \sum_{\substack{m|n \\ 3<m<n \\ m \text{ odd}}} m(2/5)^{n/m}\right)$$

$$\leq \frac{1}{n}\left(2+3(2/3)^{n/3}\right) + \sum_{\substack{2<d, d|n \\ d \text{ odd}}} \frac{1}{d}\left(\frac{2}{5}\right)^d$$

$$< \frac{1}{9}\left(2+3(2/3)^3\right) + \frac{1}{3}(2/5)^3 + \frac{1}{5}(2/5)^5 \cdot \frac{1}{1-2/5} < \frac{2}{5} \ .$$

Remark 2. We observe that if $\beta(2^j) \equiv 1$ then $\sigma(2^j) \equiv 1$. Furthermore (see (12) and (13), if $|\sigma(2^{j_0})| = 1$, i.e. $\sigma(2^{j_0}) = e^{i\phi_0}$, for some $j_0 \geq 1$, then

$$\sigma(2^j)^{2^{j_0-j}} = e^{i\phi_0} \quad \text{for} \quad 0 \leq j \leq j_0.$$

There remain the cases $n = 2^j u$ with $j \geq 1$ and odd $u \geq 3$. In (10) the sum in parentheses starts with

$$S(n) := \beta(n) + \sum_{i=0}^{j} 2^i \left(-\sigma(2^i)\right)^{n2^{-i}} = \beta(n) + \top(n) \ , \tag{14}$$

and thus ($n = 2^j u$)

$$\top(n) = \sum_{i=0}^{j-1} 2^i \sigma(2^i)^{u2^{j-i}} - 2^j \sigma(2^j)^u \ . \tag{15}$$

Trivially

$$|\top(n)| \leq 2^{j+1} - 1,$$

and our aim is to find a better upper estimate for $|\top(n)|$.

We put

$$z_i := \sigma(2^i)^{2^{j-i}}, \quad w := \beta(2^j) \quad \text{for} \quad 0 \leq i \leq j-1 \ ,$$

and, using (15) and (12),

$$\top(n) = \sum_{i=0}^{j-1} 2^i z_i^u - 2^{j(1-u)} \left(w + \sum_{i=0}^{j-1} 2^i z_i\right)^u := f_u\left(z_0, \ldots, z_{j-1}, w\right) \ . \tag{16}$$

Then

$$|\top(n)| \leq C(j,u) \ ,$$

where

$$C(j,u) := \max_{\substack{|z_i|\leq 1 \\ |w|\leq 1}} |f_u(z_0, \ldots, z_{j-1}, w)| \ ,$$

Because of

$$\begin{aligned} f_u(z_0,\dots,z_{j-1},w) =& z_0^u + 2\left\{ z_1^u + 2z_2^u + \dots + 2^{j-2} z_{j-1}^u \right.\\ &\left. - 2^{(j-1)(1-u)} \left(\frac{w+z_0}{2} + z_1 + 2z_2 + \ldots + 2^{j-2} z_{j-1} \right)^u \right\} \\ =& z_0^u + 2 f_u\left(z_1,\dots,z_{j-1},\frac{w+z_0}{2}\right) , \end{aligned}$$

the assertion of the following lemma is obviously true.

Lemma 2. *For all $j,u \in \mathbf{N}$ the inequality*

$$C(j,u) \le 1 + 2C(j-1,u)$$

holds.

An easy consequence of Lemma 2 is

$$\begin{aligned} C(j,u) &\le 1 + 2 + \ldots + 2^{j-3} + 2^{j-2} C(2,u) \\ &\le 1 + 2 + \ldots + 2^{j-2} + 2^{j-1} C(1,u) \end{aligned}$$

which implies

$$C(j,u) \le 2^{j-2} - 1 + 2^{j-2} C(2,u) \tag{17}$$

and

$$C(j,u) \le 2^{j-1} - 1 + 2^{j-1} C(1,u) \ . \tag{18}$$

Values of some $C(j,u)$. Example 1. $j=1$

By the relation

$$2^{1-u}\left(e^{2i\phi}+1\right)^u = 2e^{iu\phi}\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)^u = 2e^{iu\phi}(\cos\phi)^u$$

we have

$$C(1,u) = \max_{|\phi|\le\frac{\pi}{2}} \left|1 - 2e^{iu\phi}(\cos\phi)^u\right| \ .$$

Now, if $x := \cos^2\phi$,

$$\begin{aligned} &|1 - 2e^{iu\phi}(\cos\phi)^u|^2 = 1 - 4\cos u\phi(\cos\phi)^u + 4(\cos\phi)^{2u} \\ =& 1 - 4\sum_{k=1}^{\frac{u-1}{2}} \binom{u}{2k} x^{u-k}(x-1)^k =: F_u(x) \ . \end{aligned}$$

Thus

$$(C(1,u))^2 = \max_{0\le x\le 1} F_u(x) \ .$$

A simple calculation shows

$$F_3(x) = 1 - 12x^2(x-1)$$
$$F_5(x) = 1 - 20x^2(x-1)(3x-1) \ ,$$

and we obtain

$$C(1,3) = 5/3, \quad C(1,5) = 1,974561\dots \ .$$

Example 2. $j = 2$
Here we consider only the case $u = 3$, i.e.

$$C(2,3) = \max_{\substack{|z|\le 1\\ |W|\le 1}} \left|1 + 2z^3 - 2^{-4}(1+w+2z)^3\right| \ ,$$

and obtain by numerical calculations $C(2,3) < 2,7$ which is enough for our purposes. The rest of the proof of Proposition 2 is contained in

Lemma 3. *Let $n = 2^j u$, where $j \ge 1$ and $u \ge 3$ is odd. Then*

$$|\sigma(n)| \le 2/3 \ .$$

PROOF. From (10) and (14) we get

$$|\sigma(n)| \le \frac{1+|\top(n)|}{n} + \sum_{\substack{d|n,\ d\ge 2\\ n/d\ne 2^1}} \frac{|\sigma(n/d)|^d}{d} \ .$$

Case (i). Let $u = 3$. Then, by example 1,

$$|\sigma(6)| \le \frac{1}{6}(1+5/3) + \frac{1}{2}(2/3)^2 = 2/3$$

and, by example 2,

$$|\sigma(12)| \le \frac{1}{12}(1+2,7) + \frac{1}{2}(2/3)^2 + \frac{1}{4}(2/3)^4 < 2/3.$$

Now, using (17) and example 2, we have for $j \ge 3$,

$$C(j,3) \le 2^{j-2} - 1 + 2^{j-2}\cdot 2,7 < 2^j - 1$$

and obtain by induction

$$|\sigma(2^j\cdot 3)| \le 1/3 + \frac{1}{2}(2/3)^2 + \frac{1}{4}(2/3)^4 + \frac{1}{8}(2/3)^8 \frac{1}{1-2/3} < 2/3$$

for all $j \geq 3$.

Case (ii). Let $u = 5$. Then, again by example 1,

$$|\sigma(10)| \leq \frac{1+1,98}{10} + \frac{1}{2}(2/5)^2 < 2/3 \ .$$

Further, using (18) and example 1, we have for $j \geq 2$,

$$C(j,5) \leq 2^{j-1} - 1 + 2^{j-1} \cdot 1,98 < \frac{3}{2} \cdot 2^j - 1,$$

and conclude by induction

$$|\sigma(2^j \cdot 5)| \leq 3/10 + \frac{1}{2}(2/3)^2 + \frac{1}{4}(2/3)^4 + \frac{1}{8}(2/3)^8 \cdot 3 < 2/3$$

for all $j \geq 2$.

Case (iii). Let $u = 7$. Now we use the trivial estimate $1 + |\top(n)| \leq 2^{j+1}$. Then

$$|\sigma(2 \cdot 7)| \leq 2/7 + \frac{1}{2}(2/5)^2 = 2/7 + 2/25 < 2/3$$

and, by induction,

$$|\sigma(2^j \cdot 7)| \leq 2/7 + \frac{1}{2}(2/3)^2 + \frac{1}{4}(2/3)^4 + \frac{1}{8}(2/3)^8 \cdot 3 < 2/3$$

for all $j \geq 2$.

Case (iv). Let $u \geq 9$ be odd. Then, using the trivial estimate $1 + |\top(n)| \leq 2^{j+1}$ and making induction with respect to the set $2\mathbf{N} \setminus \{2^j : j \in \mathbf{N}\}$, we conclude

$$|\sigma(2^j \cdot u)| < 2/9 + \sum_{d=2}^{\infty} \frac{(2/3)^d}{d} = 2/9 + \log 3 - 2/3 < 2/3,$$

which ends the proof of Proposition 2.

For the proof of Corollary 3 we put again $\beta(n) = \alpha(n)y^{-n}$ and $\sigma(n) = \rho(n)y^{-n}$ and obtain as above

$$\begin{aligned} |\sigma(2^j)| &\leq 1 \quad \text{for} \quad j \geq 3, \\ |\sigma(n)| &\leq 2/5 \quad \text{for odd} \quad n \geq 7. \end{aligned}$$

Furthermore, the inequalities

$$\begin{aligned} |S(n)| \leq 1 + |\top(n)| \leq 2^{j+1} &\quad \text{if} \quad n = 2^j u, \quad j \geq 1 \quad \text{and} \quad u \geq 5 \ \text{odd}, \\ |\top(n)| \leq C(j,u) &\quad \text{if} \quad j \geq 1. \end{aligned}$$

hold.

Therefore, using the proof of Proposition 2 step by step, we obtain the assertion of Corollary 3.

4. Examples

1) Let $f(z) = (1-z)^{-1}$. Then $\alpha(n) \equiv 1$ and

$$f(z) = \prod_{m=1}^{\infty} (1 + \rho(m) z^m) \quad (|z| < 1) \tag{19}$$

holds with

$$\rho(m) \begin{cases} 1 & \text{if} \quad m = 2^j, \quad j \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $y = \sup \sqrt[n]{|\alpha(n)|} = 1$, and obviously the radius of convergence of (19) equals 1.

2) Let

$$f(z) = 1 + \sum_{n=1}^{\infty} p(n) z^n$$

where p(·) denotes the partition function. It is well-known that

$$f(z) = \prod_{m=1}^{\infty} (1 - z^m)^{-1} \quad \text{for} \quad |z| < 1 \ .$$

From this we conclude

$$\alpha(n) = \sum_{m|n} m \ .$$

We have

$$\rho(1) = \alpha(1) = 1$$

and

$$\rho(n) = \frac{1}{n} \left(\sum_{m|n} m + \sum_{\substack{m|n \\ m<n}} m \big(-\rho(m) \big)^{n/m} \right) \quad \text{for} \ \ n > 1 \ .$$

This implies

$$\rho(n) = 1 + \frac{1}{n} \sum_{\substack{m|n \\ m<n}} m \left(1 + \big(-\rho(m) \big)^{n/m} \right)$$

and, by induction

$$\rho(n) = 1 \quad \text{for odd} \ \ n \ .$$

Next we prove

Lemma 4. *Let $y=\sqrt{2}$. Then for $n=1$ and $n\geq 7$ the estimate*

$$\alpha(n)=\sum_{m|n} m\leq y^n \quad \textit{holds.}$$

Proof. We have

$$\alpha(n)=n\sum_{d|n} 1/d<n\prod_{p|n}\left(1-p^{-1}\right)^{-1}\leq n2^{\omega(n)}\leq n^2$$

where $\omega(n)$ denotes the number of distinct prime divisors of n. Now $n^2\leq y^n$ if and only if

$$4\log n\leq n\log 2$$

and this holds for $n\geq 2^4=16$. By calculating the values of $\alpha(n)$ for $7\leq n\leq 15$ $\big(\alpha(15)=\alpha(14)=24;\ \alpha(13)=14;\ \alpha(12)=28;\ \alpha(11)=12;\ \alpha(10)=18;\ \alpha(9)=13,\ \alpha(8)=15;\ \alpha(7)=8\big)$ we obtain the assertion of Lemma 4. We have

$$\rho(2^j)=2+\sum_{i=1}^{j-1}2^{-i}\rho(2^{j-1})^{2^i}\ . \tag{20}$$

Putting $y=\sqrt{2}$ and $\sigma(n)=\rho(n)y^{-n}$, we see $\sigma(2)=1$, $\sigma(4)=1$ and conclude

$$|\sigma(2^j)|\leq 1 \quad \text{for all} \quad j\geq 0\ .$$

Thus, by Corollary 3,

$$|\sigma(n)|\leq 1 \quad \text{for all} \quad n\in\mathbf{N},$$

and

$$f(z)=1+\sum_{n=1}^{\infty}p(n)z^n=\prod_{m=1}^{\infty}\left(1+\rho(m)z^m\right) \tag{21}$$

holds for $|z|<1/\sqrt{2}$. On the other hand (20) shows

$$\rho(2^j)\geq 2^{-(j-1)}\rho(2)^{2^{j-1}}=\frac{2}{2^j}\cdot y^{2^j}\ ,$$

which says that $1/\sqrt{2}$ is the radius of convergence for (21).

3) Let $f(z)=\exp(z)$. Then $\alpha(1)=1$ and $\alpha(n)=0$ for $n>1$. We shall prove

$$\rho(n)=\frac{1}{n}\left((-1)^n+O(n^{-1})\right)\ . \tag{22}$$

A consequence of this will be that the radius of convergence of

$$f(z) = \exp(z) = \prod_{m=1}^{\infty} (1 + \rho(m) z^m)$$

equals 1. By definition

$$\rho(1) = 1$$
$$\rho(n) = \frac{1}{n}\left((-1)^n + \sum_{\substack{m|n \\ 1<m<n}} m\big(-\rho(m)\big)^{n/m}\right) \quad \text{for} \quad n > 1 \ .$$

We note that

$$\rho(2) = 1/2$$
$$\rho(p) = -1/p \quad \text{for all primes} \quad p > 2.$$

As a first step we show

$$|\rho(n)| \le 1/\sqrt{n} \ . \tag{23}$$

It is easy to check (23) for all $1 \le n \le 15$. The cases $n \ge 16$ will be handled by induction. For this we estimate $\rho(n)$ by

$$|\rho(n)| \le \frac{1}{n} \sum_{\substack{m|n \\ m<n}} m|\rho(m)|^{n/m} \le \frac{1}{n} \sum_{\substack{m|n \\ m<n}} m(1/\sqrt{m})^{n/m} = \frac{1}{n} \sum_{\substack{m|n \\ m<n}} m^{1-n/(2m)}$$

$$= \frac{1}{n}\left(\sum_{\substack{m|n \\ m\le n/6}} m^{1-n/(2m)} + \sum_{\substack{m|n \\ n/6<m<n}} m^{1-n/(2m)} \right) =: \frac{1}{n}\,(A(n) + B(n))$$

where

$$A(n) \le \sum_{m=1}^{\infty} m^{-2} = \pi^2/6 < 5/3 \ .$$

In $B(n)$ the conditions about m imply that only numbers $m = n/d$ with $2 \le d \le 5$ can occur. Therefore

$$B(n) \le 1 + (3/n)^{1/2} + 4/n + (5/n)^{3/2} \ .$$

For $n \ge 16$ we get

$$A(n) + B(n) \le 5/3 + 1 + 1/2 + 1/4 + 1/5 < 4,$$

and thus

$$|\rho(n)| \le 4/n \le 1/\sqrt{n}.$$

As a second step we prove

$$|\rho(n)| \le 4/n \quad \text{for all} \quad n \in \mathbf{N}\ . \tag{24}$$

For $n \ge 16$ we use (23) in the defining relation for $\rho(n)$ and get

$$|\rho(n)| \le \frac{1}{n} \sum_{\substack{m|n \\ m<n}} m(1/\sqrt{m})^{n/m} \le 4/n \quad (n \ge 16)\ .$$

In the remaining cases we use (23) directly and obtain

$$|\rho(n)| \le \frac{1}{\sqrt{n}} = \frac{4}{n} \cdot \frac{\sqrt{n}}{4} \le 4/n \qquad (n \le 16)\ .$$

Now we define $R(n)$ for $n \ge 1$ by

$$\rho(n) = \frac{1}{n}\big((-1)^n + R(n)\big).$$

Then

$$R(n) = n\rho(n) - (-1)^n = \sum_{\substack{m|n \\ 1<m<n}} m\,(-\rho(m))^{n/m}\ . \tag{25}$$

Let $K := 11 \cdot e^{125}$. We shall prove by induction that $n|R(n)| \le K$.
By (24) we have

$$n|R(n)| \le n(n|\rho(n)| + 1) \le 5n$$

which implies

$$n|R(n)| \le K \quad \text{for} \quad n \le [K/5]\ .$$

Let $n \ge [K/5] + 1$ and assume that $m|R(m)| \le K$ for $m < n$. A trivial estimate of $R(n)$ in (25) leads to

$$\begin{aligned} |R(n)| &\le \sum_{\substack{m|n \\ 1<m<n}} m|\rho(n)|^{n/m} \\ &= \sum_{\substack{m|n \\ 2\le m\le n/6}} m|\rho(n)|^{n/m} + \sum_{\substack{m|n \\ n/6<m<n}} m|\rho(n)|^{n/m} =: C(n) + D(n)\ . \end{aligned}$$

By (23)

$$C(n) \sum_{\substack{m|n \\ 2\le m\le n/6}} m(1/\sqrt{m})^{n/m} = \sum_{\substack{m|n \\ 2\le m\le n/6}} m^{1-n/(2m)}\ .$$

Now, if $n \geq 100$, the function $t \to t^{1-n/(2t)}$, $2 \leq t \leq n/6$, attains its maximum at $n/6$. Thus ($\tau(\cdot)$: divisor function)

$$C(n) \leq \tau(n)\ (n/6)^{-2} \leq 72n^{-3/2}.$$

The estimate of $D(n)$ runs as follows:

$$\begin{aligned} D(n) &\leq \sum_{\substack{m|n \\ n/6<m<n}} m\left(\frac{1}{m}\big(1+|R(m)|\big)\right)^{n/m} \\ &\leq \sum_{\substack{m|n \\ n/6<m<n}} m^{1-n/m}\exp\left(\frac{n}{m}|R(m)|\right) \leq \sum_{\substack{m|n \\ n/6<m<n}} m^{1-n/m}\exp\big(Kn/m^2\big) \\ &\leq \sum_{j=2}^{5}(j/n)^{j-1}\exp\big(Kj^2/n\big) \leq 10\cdot\frac{1}{n}\cdot\exp(25K/n)\ . \end{aligned}$$

Gathering the estimates gives

$$n|R(n)| \leq 72\sqrt{n} + 10\exp(25K/n)$$

and because of $n > K/5$ the right hand side becomes

$$\leq \frac{72\sqrt{5}}{\sqrt{K}} + 10e^{125} \leq 1 + 10e^{125} < 11e^{125} = K\ .$$

This ends the proof of our assertion (22).

Note. After this paper had been submitted to Publ. Math. Debrecen the article

A. Knopfmacher, L. Lucht. The radius of convergence of power product expansions. Analysis 11 (1991), 91–99 appeared, where the authors essentially prove our Proposition 2 and Corollaries 1,2, but in a totally different way.

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