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Summation averaging technique for the oscillation of second order linear difference equations

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Abstract. The present paper is concerned with the oscillation of the second order linear difference equation $\Delta(c_{n-1}\Delta u_{n-1}) + q_n u_n = 0$, $n \geq 1$, where Δ is the usual forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\{c_n\}$ and $\{q_n\}$ are sequences of real numbers with $c_n > 0$. Using summation averaging technique, some new oscillation criteria are obtained and the discrete analogue of known results due to Kamenev and Philos for the corresponding differential equations are established.

1. Introduction

Consider the second order difference equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n u_n = 0, \quad n \geq 1, \tag{E}$$

where Δ denotes the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\{c_n\}$ and $\{q_n\}$ are sequences of real numbers with $c_n > 0$ for all integers $n \geq 0$.

By a solution of equation (E) we mean a real sequence $\{u_n\}$, $n = 0, 1, \dots$ satisfying equation (E).

A nontrivial solution $\{u_n\}$ of equation (E) is said to be nonoscillatory if there exists $N \geq 0$ such that $u_{n+1}u_n > 0$ for all $n \geq N$ and oscillatory otherwise. Equation (E) is called oscillatory if all its solutions are oscillatory. It is known that if equation (E) has an oscillatory solution, then all its solutions are oscillatory (see [5, pp. 153]).

The oscillatory, nonoscillatory and asymptotic properties of equation (E) have been considered extensively by many authors. Among the papers

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dealing with these subjects the reader is referred to [3, 4, 6, 8–10, 12, 14, 15] and to [2, 5, 13] for the general theory of the difference equations.

The oscillation problem of the continuous version of equation (E) i.e., the second order linear differential equation

$$(\mathrm{E}_1) \qquad (c(t)u'(t))' + q(t)u(t) = 0, \quad t \geq 0 \quad ('= d/dt)$$

has been the subject of numerous investigations e.g., see [7, 11, 16–18, 20, 21]. Motivated by Kamenev [11], most of the recent and useful oscillation criteria regarding equation (E_1), particularly for the special case

$$(\mathrm{E}_2) \qquad u''(t) + q(t)u(t) = 0, \quad t \geq t_0 \geq 0$$

involve the behavior of the averaging function $A_m(t)$ where

$$A_m(t) = \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m q(s)ds, \quad m \text{ is a positive integer.}$$

Therefore it is of interest to obtain discrete analogues of such criteria and to note the similarities and differences which may arise. The main purpose of the present paper is to proceed further in this direction. To this end, we will make use of the averaging sum $S_m(n)$ defined by

$$S_m(n) = \frac{1}{n^{(m)}} \sum_{k=N}^{n} \Big\{ (n-k)^{(m)} q_n - m(n+1-k)^{(m-1)} c_{k-1} \Big\}, \quad n \geq N \geq 0$$

where

$$(n-k)^{(m)} = (n-k)(n-(k-1))(n-(k-2)) \cdots (n-(k-(m-1))),$$

for $n \geq k$ and an integer $m \geq 1$.

In fact, some new oscillation results are established as well as the discrete analogues of some well-known results due to Kamenev [11] and Philos [17] for equation (E_2).

Before stating and proving the main results in Section 2, we prove some preparatory lemmas which are interesting in their own right.

Lemma 1.1. *For any real sequence $\{x_n\}$ and any integer $N \geq 0$, we have*

$$\sum_{k=N}^{n}(n-k)^{(m)}\Delta x_k = -(n+1-N)^{(m)}x_N + m\sum_{k=N}^{n}(n+1-k)^{(m-1)}x_k,$$
$$n \geq N \geq 0.$$

Proof. Let
$$y_k = (n+1-k)^{(m)}x_k.$$

Differencing with respect to k,

$$\Delta y_k = (n-k)^{(m)}\Delta x_k - m(n+1-k)^{(m-1)}x_k.$$

Summing from N to n, we obtain

$$y_{n+1} - y_N = \sum_{k=N}^{n}(n-k)^{(m)}\Delta x_k - m\sum_{k=N}^{n}(n+1-k)^{(m-1)}x_k.$$

Thus, since $y_{n+1} = 0$, we get

$$\sum_{k=N}^{n}(n-k)^{(m)}\Delta x_k = -(n+1-N)^{(m)}x_N + m\sum_{k=N}^{n}(n+1-k)^{(m-1)}x_k.$$

Which is our desired result. □

Lemma 1.2 [1]**.** *Let $\{x_n\}$ be a sequence of real numbers. If*

$$\liminf_{n\to\infty}\Delta^r x_n > 0 \text{ for some integer } r \geq 1,$$

then
$$\lim_{n\to\infty}\Delta^i x_n = \infty \text{ for } 0 \leq i \leq r-1.$$

Lemma 1.3. *Assume that the sequences of real numbers $\{a_n\}$ and $\{b_n\}$ are such that the sequence $\{\Delta^r b_n\}$ is increasing for some integer $r \geq 0$, and $\Delta^r b_n \to \infty$ as $n \to \infty$. Then*

$$\limsup_{n\to\infty}\frac{\Delta^{r+1}a_n}{\Delta^{r+1}b_n} \geq \limsup_{n\to\infty}\frac{a_n}{b_n} \geq \liminf_{n\to\infty}\frac{a_n}{b_n} \geq \liminf_{n\to\infty}\frac{\Delta^{r+1}a_n}{\Delta^{r+1}b_n}. \tag{1.1}$$

Proof. First, we prove that

$$\limsup_{n\to\infty} \frac{\Delta^{r+1} a_n}{\Delta^{r+1} b_n} \ge \limsup_{n\to\infty} \frac{a_n}{b_n}, \tag{1.2}$$

and the remaining part of (1.1) goes the same way. Now, let

$$\limsup_{n\to\infty} \frac{\Delta^{r+1} a_n}{\Delta^{r+1} b_n} = L. \tag{1.3}$$

Clearly, (1.2) is satisfied if $L = \infty$. If $L \in (-\infty, \infty)$, (1.3) implies the existence of an integer $N(\varepsilon)$ such that

$$\frac{\Delta^{r+1} a_n}{\Delta^{r+1} b_n} < L + \varepsilon, \quad \text{for all } \varepsilon > 0 \quad \text{and } n \ge N(\varepsilon). \tag{1.4}$$

Since $\Delta^r b_n$ is increasing, the number $N(\varepsilon)$ can be chosen so large that $\Delta^{r+1} b_n > 0$ for all $n \ge N(\varepsilon)$. Therefore, (1.4) implies that

$$\Delta^{r+1} a_n < (L + \varepsilon)\Delta^{r+1} b_n, \qquad n \ge N(\varepsilon). \tag{1.5}$$

Summing (1.5) from N to n, we get

$$\Delta^r a_{n+1} - \Delta^r a_N < (L + \varepsilon)(\Delta^r b_{n+1} - \Delta^r b_N), \qquad n \ge N(\varepsilon). \tag{1.6}$$

Dividing both sides of (1.6) by $\Delta^r b_{n+1}$ and taking the upper limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{\Delta^r a_{n+1}}{\Delta^r b_{n+1}} \le L + \varepsilon, \quad \text{for all } \varepsilon > 0,$$

then

$$\limsup_{n\to\infty} \frac{\Delta^r a_{n+1}}{\Delta^r b_{n+1}} \le L,$$

or

$$\limsup_{n\to\infty} \frac{\Delta^{r+1} a_n}{\Delta^{r+1} b_n} \ge \limsup_{n\to\infty} \frac{\Delta^r a_n}{\Delta^r b_n}. \tag{1.7}$$

On the other hand, Lemma 1.2 implies that $\Delta^i b_n \to \infty$ as $n \to \infty$, for $0 \le i \le r$. Then, in a similar fashion as that implying (1.7), one can prove that

$$\limsup_{n\to\infty} \frac{\Delta^{i+1} a_n}{\Delta^{i+1} b_n} \ge \limsup_{n\to\infty} \frac{\Delta^i a_n}{\Delta^i b_n}, \qquad 0 \le i \le r - 1. \tag{1.8}$$

Combining (1.7) and (1.8), we obtain (1.2).

Finally, if $L = -\infty$, then for any $\overline{M} < 0$, we have

$$\limsup_{n\to\infty} \frac{\Delta^{r+1} a_n}{\Delta^{r+1} b_n} < \overline{M}.$$

We claim that $\limsup_{n\to\infty} a_n/b_n = -\infty$. Suppose not, then

$$\limsup_{n\to\infty} \frac{a_n}{b_n} \geq M > -\infty.$$

Using similar arguments as before, we obtain that $\overline{M} > M$ for any $\overline{M} < 0$. Thus, M can not be finite. This contradiction completes the proof. $\square$

Now, for some integers $m \geq 1$, define $\{a_n\}$ and $\{b_n\}$ as follows

$$a_n = \sum^{n} (n-k)^{(m)} f_k \quad \text{and} \quad b_n = n^{(m)}.$$

An easy computation gives

$$\Delta^m a_n = m! \sum^{n} f_k \quad \text{and} \quad \Delta^{m+1} a_n = m! f_{n+1},$$

and

$$\Delta^{m-1} b_n = m!(n+m-1),$$

then

$$\Delta^{m-1} b_n \to \infty \text{ as } n \to \infty \text{ and } \Delta^m b_n = m!\ .$$

Then, as an application of the preceding lemma, we obtain the following result:

Lemma 1.4. *For any integer $m \geq 1$ and real sequence $\{f_n\}$ we have,*

$$\begin{aligned} \limsup_{n\to\infty} \frac{f_n}{m} &\geq \limsup_{n\to\infty} \frac{1}{n^{(m)}} \sum^{n} (n-k)^{(m-1)} f_k \\ &\geq \liminf_{n\to\infty} \frac{1}{n^{(m)}} \sum^{n} (n-k)^{(m-1)} f_k \geq \liminf_{n\to\infty} \frac{f_n}{m}, \end{aligned} \tag{1.9}$$

and

$$\limsup_{n\to\infty}\sum^{n} f_k \geq \limsup_{n\to\infty}\frac{a_n}{b_n} \geq \liminf_{n\to\infty}\frac{a_n}{b_n} \geq \liminf_{n\to\infty}\sum^{n} f_k. \tag{1.10}$$

2. Main results

Theorem 2.1. *Suppose that there exist an integer $m \geq 1$ and a real sequence $\{\phi_s\}$ such that*

$$\limsup_{n\to\infty}\frac{1}{n^{(m)}}\sum_{k=s}^{n}\Big[(n-k)^{(m)}q_k - m(n+1-k)^{(m-1)}c_{k-1}\Big] \geq \phi_s, \qquad s \geq 1, \tag{2.1}$$

and

$$\lim_{n\to\infty}\frac{1}{n^{(m)}}\sum_{k=1}^{n}\Big[(n+1-k)^{(m-1)}\sqrt{n-k}\,\phi_k^{+}\Big] = \infty, \tag{2.2}$$

where $\phi_k^{+} = \max\{\phi_k, 0\}$. Then equation (E) *is oscillatory.*

Proof. Assume that $\{u_n\}$ is a nonoscillatory solution of equation (E). Without loss of generality, one can assume that u_n is eventually positive. Thus, $u_n > 0$ for all $n \geq N-1 \geq 0$, N is sufficiently large. Let

$$w_n = c_{n-1}\frac{\Delta u_{n-1}}{u_{n-1}}, \qquad n \geq N.$$

Then it follows from equation (E) that

$$\Delta w_k = -q_k - \frac{w_k^2}{w_k + c_{k-1}}, \qquad k \geq N. \tag{2.3}$$

Multiplying both sides of (2.3) by $(n-k)^{(m)}$, $m \geq 1$ and summing from N to n, we have

$$\sum_{k=N}^{n}(n-k)^{(m)}\Delta w_k = -\sum_{k=N}^{n}(n-k)^{(m)}q_k - \sum_{k=N}^{n}(n-k)^{(m)}\frac{w_k^2}{w_k + c_{k-1}}, \quad n \geq N.$$

But Lemma 1.1 yields

$$\sum_{k=N}^{n}(n-k)^{(m)}\Delta w_k = -(n+1-N)^{(m)}w_N + m\sum_{k=N}^{n}(n+1-k)^{(m-1)}w_k,$$

hence

$$-(n+1-N)^{(m)}w_N = -\sum_{k=N}^{n}(n-k)^{(m)}q_k \tag{2.4}$$
$$-\sum_{k=N}^{n}(n-k)^{(m)}\frac{w_k^2}{w_k+c_{k-1}} - m\sum_{k=N}^{n}(n+1-k)^{(m-1)}w_k,$$

or

$$(n+1-N)^{(m)}w_N \tag{2.5}$$
$$= \sum_{k=N}^{n}\Big[(n-k)^{(m)}q_k - m(n+1-k)^{(m-1)}c_{k-1}\Big] + \sum_{k=N}^{n}\Omega_{n,k},$$

where $\Omega_{n,k} = (n-k)^{(m)}\,w_k^2/(w_k+c_{k-1}) + m(n+1-k)^{(m-1)}(w_k+c_{k-1})$. Dividing both sides of (2.5) by $n^{(m)}$ and taking the upper limit as $n \to \infty$, we get

$$w_N \geq \phi_N + \liminf_{n\to\infty}\frac{1}{n^{(m)}}\sum_{k=N}^{n}\Omega_{n,k}.$$

Thus

$$w_n \geq \phi_n \quad \text{for all } n \geq N, \tag{2.6}$$

and

$$\liminf_{n\to\infty}\frac{1}{n^{(m)}}\sum_{k=N}^{n}\Omega_{n,k} < \infty. \tag{2.7}$$

By applying the Arithmetic-Geometric mean, we obtain

$$\begin{aligned}\Omega_{n,k} &\geq 2(n+1-k)^{(m-1)}[m(n-k)w_k^2]^{1/2}\\ &= 2(n+1-k)^{(m-1)}[m(n-k)]^{1/2}|w_k|.\end{aligned}$$

But (2.6) implies that $|w_k| \geq \phi_k^+$, for all $k \geq N$. Then

$$\Omega_{n,k} \geq 2[m(n-k)]^{1/2}(n+1-k)^{(m-1)}\phi_k^+, \qquad n \geq k \geq N.$$

Summing the above inequality from N to n, and dividing by $n^{(m)}$, we get

$$\frac{1}{n^{(m)}}\sum_{k=N}^{n}\Omega_{n,k} \geq \frac{2\sqrt{m}}{n^{(m)}}\sum_{k=N}^{n}\left[\sqrt{n-k}\,(n+1-k)^{(m-1)}\phi_k^+\right], \qquad n \geq k \geq N. \tag{2.8}$$

Now, using (2.2) and (2.8), we conclude that

$$\lim_{n\to\infty}\frac{1}{n^{(m)}}\sum_{k=N}^{n}\Omega_{n,k} = \infty,$$

which contradicts (2.7). This completes the proof. □

The following corollaries are immediate consequences of Theorem 2.1.

Corollary 2.1. *Suppose that there exists a positive integer $m \geq 1$ such that*

$$\limsup_{n\to\infty}\frac{1}{n^{(m)}}\sum_{k=1}^{n}\left[(n-k)^{(m)}q_k - m(n+1-k)^{(m-1)}c_{k-1}\right] = \infty. \tag{2.9}$$

Then equation (E) *is oscillatory.*

In the following result, we consider equation (E) with $c_n \equiv 1$, i.e., the equation

$$\Delta^2 u_{n-1} + q_n u_n = 0, \qquad n \geq 1. \tag{E$_3$}$$

Corollary 2.2. *If there exists an integer $m \geq 1$ such that*

$$\limsup_{n\to\infty}\frac{1}{n^{(m)}}\sum_{k=1}^{n}(n-k)^{(m)}q_k = \infty, \tag{2.10}$$

then equation (E_3) *is oscillatory.*

Remark 2.1.

1. In the above results, we do not require any explicit restrictions on $\sum^{\infty} c_k$ or $\sum^{\infty} c_k^{-1}$ as it is noted in [3, 8, 12].

2. Corollary 2.2 improves and unifies the discrete versions of the well-known oscillation criteria of Kamenev [11] and Wintner [18]. Furthermore, when $m = 1$, Corollary 2.2 does not have a continuous analogue as it was shown by Hartman [7], who proved that the condition

$$\limsup_{n\to\infty} \frac{1}{t}\int_{t_0}^{t}(t-s)q(s)ds = \infty,$$

can not, generally, insure the oscillation of equation (E_2).

Theorem 2.2. *Let condition* (2.1) *hold and assume that*

$$\lim_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=N}^{n} (n-k)^{(m)} \frac{(\phi_k^+)^2}{\phi_k^+ + c_{k-1}} + m(n+1-k)^{(m-1)}(\phi_k + c_{k-1}) = \infty. \tag{2.11}$$

Then equation (E) *is oscillatory.*

Proof. Let $\{u_n\}$ be a nonoscillatory solution of equation (E). As in the proof of Theorem 2.1, we obtain (2.7). On the other hand

$$\frac{w_k^2}{w_k + c_{k-1}} \geq \frac{w_k^2}{|w_k| + c_{k-1}} = c_{k-1}\frac{(w_k/c_{k-1})^2}{|w_k/c_{k-1}|+1}, \qquad k \geq N. \tag{2.12}$$

Using (2.6), we have

$$\left|\frac{w_k}{c_{k-1}}\right| \geq \frac{\phi_k^+}{c_{k-1}}, \qquad k \geq N,$$

but, $f(x) = x^2/(x+1)$ is an increasing function for the positive values of x. Then we get the following estimate of the right hand side of (2.12)

$$c_{k-1}\frac{(w_k/c_{k-1})^2}{|w_k/c_{k-1}|+1} \geq c_{k-1}\frac{(\phi_k^+/c_{k-1})^2}{(\phi_k^+/c_{k-1})+1}, \qquad k \geq N,$$

or

$$\frac{w_k^2}{w_k + c_{k-1}} \geq \frac{(\phi_k^+)^2}{\phi_k^+ + c_{k-1}}, \qquad k \geq N. \tag{2.13}$$

By the use of (2.6) and (2.13), (2.7) implies that

$$\liminf_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=N}^{n} (n-k)^{(m)} \frac{(\phi_k^+)^2}{\phi_k^+ + c_{k-1}} + m(n+1-k)^{(m-1)}(\phi_k + c_{k-1}) < \infty, \tag{2.14}$$

which contradicts (2.11), so the theorem follows. □

Corollary 2.3. *Suppose that* (2.1) *holds. If either*

$$\lim_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=N}^{n} (n+1-k)^{(m-1)}(\phi_k + c_{k-1}) = \infty, \tag{2.15}$$

or

$$\sum_{k=N}^{\infty} \frac{(\phi_k^+)^2}{\phi_k^+ + c_{k-1}} = \infty, \tag{2.16}$$

then equation (E) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.2 and hence is omitted. □

Theorem 2.3. *If condition* (2.1) *holds and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=N}^{n} c_k = \infty, \tag{2.17}$$

then equation (E) *is oscillatory.*

Proof. As in the proof of Theorems 2.1 and 2.2, we obtain (2.7) which yields

$$\liminf_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=N}^{n} (n-k)^{(m)} \frac{w_k^2}{w_k + c_{k-1}} < \infty,$$

and

$$\liminf_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=N}^{n} (n+1-k)^{(m-1)}(w_k + c_{k-1}) < \infty.$$

In view of Lemma 1.4, the above inequalities imply that

$$\sum_{k=N}^{\infty} \frac{w_k^2}{w_k + c_{k-1}} < \infty, \tag{2.18}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=N}^{n} (w_k + c_{k-1}) < \infty, \tag{2.19}$$

respectively. Using (2.18), we have

$$\lim_{n\to\infty} \frac{w_n^2}{w_n + c_{n-1}} = 0, \tag{2.20}$$

then there exists an integer $N_1 \geq N$ such that

$$\frac{w_n^2}{w_n + c_{n-1}} < 1, \qquad \text{for all } n \geq N_1,$$

hence

$$w_n^2 - w_n - c_{n-1} < 0, \quad \text{for all } n \geq N_1.$$

Completing the square of the left hand side of the above inequality, we get

$$-(1/4 + c_{n-1})^{1/2} < w_n - 1/2 < (1/4 + c_{n-1})^{1/2}, \qquad n \geq N_1.$$

From (2.19), we obtain

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \sum_{k=N_1}^{n} \left[c_{k-1} + 1/2 - (1/4 + c_{k-1})^{1/2}\right] \\ &\qquad \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{k=N_1}^{n} (w_k + c_{k-1}) < \infty, \end{aligned}$$

which implies that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=N_1}^{n} R_k < \infty, \tag{2.21}$$

where

$$R_k = c_{k-1} + 1/2 - (1/4 + c_{k-1})^{1/2} = \Big((1/4 + c_{k-1})^{1/2} - 1/2\Big)^2, \quad k \geq N_1.$$

But

$$\begin{aligned} R_k &= \Big((1/4 + c_{k-1})^{1/2} - 1/2\Big)^2 \frac{\big((1/4 + c_{k-1})^{1/2} + 1/2\big)^2}{\big((1/4 + c_{k-1})^{1/2} + 1/2\big)^2} \\ &= \frac{(c_{k-1})^2}{\big((1/4 + c_{k-1})^{1/2} + 1/2\big)^2}, \end{aligned}$$

and

$$(1/4 + c_{k-1})^{1/2} > 1/2, \qquad k \geq N_1,$$

therefore,

$$R_k = \frac{c_{k-1}}{4} - \frac{c_{k-1}}{16(1/4 + c_{k-1})} > \frac{c_{k-1}}{4} - \frac{1}{16}, \qquad k \geq N_1.$$

Now, summing up the above inequality from N_1 to n then dividing both sides of the resulting inequality by n, we get

$$\frac{1}{n} \sum_{k=N_1}^{n} R_k > \frac{1}{4n} \sum_{k=N_1}^{n} c_{k-1} - \frac{(n - N_1 + 1)}{16n}.$$

So that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=N_1}^{n} R_k \geq \liminf_{n\to\infty} \frac{1}{4n} \sum_{k=N_1}^{n} c_{k-1} - \frac{1}{16},$$

which is impossible in view of (2.17) and (2.21). This completes the proof. □

Next, by replacing condition (2.17) with

$$\limsup_{n\to\infty} \frac{1}{n} \sum^{n} c_k = \infty, \tag{2.22}$$

and using similar arguments as in the proof of Theorem 2.3, the following result can be proved.

Theorem 2.4. *Suppose that condition* (2.22) *holds and*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=N}^{n} [(n-k)q_k - c_{k-1}] > -\infty. \tag{2.23}$$

Then equation (E) *is oscillatory .*

Remark 2.2. Theorems 2.2–2.4 and Corollary 2.3 are applicable effectively when (2.2) is not satisfied. Furthermore, Theorems 2.3 and 2.4 complete, partially, all the results of [3] that require the restriction

$$\limsup_{n\to\infty} \frac{1}{n} \sum^{n} c_k < \infty.$$

Theorem 2.5. *If* $\{c_n\}$ *is bounded, and there exist a real sequence* $\{\lambda_n\}$ *and a positive integer* $m \geq 1$ *such that*

(2.24)
$$\limsup_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=s}^{n} (n-k)^{(m)} q_k \geq \lambda_s, \quad \textit{for every sufficiently large } s$$

and

$$\sum^{\infty} (\lambda_k^+)^2 = \infty, \tag{2.25}$$

where $\lambda_k^+ = \max\{\lambda_k, 0\}$, *then equation* (E) *is oscillatory.*

Proof. As in the proofs of Theorems 2.1 and 2.3, one can easily see that (2.20) holds. Then $w_n \to 0$ as $n \to \infty$ (since c_n is bounded). Consequently, (1.9) implies

$$\lim_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=N}^{n} (n+1-k)^{(m-1)} w_k = 0. \tag{2.26}$$

Dividing both sides of (2.4) by $n^{(m)}$, taking the upper limit as $n \to \infty$ and using (2.26), we have

$$w_N \geq \lambda_N + \liminf_{n\to\infty} \frac{1}{n^{(m)}} \sum_{k=N}^{n} (n-k)^{(m)} \frac{w_k^2}{w_k + c_{k-1}}, \tag{2.27}$$

then (2.18) holds and $w_n \geq \lambda_n$ for all $n \geq N$. But $(w_n + c_{n-1})$ is bounded, there exists a real number $M > 0$ such that

$$\frac{w_k^2}{w_k + c_{k-1}} \geq \frac{w_k^2}{M}, \qquad k \geq N,$$

hence

$$\frac{w_k^2}{w_k + c_{k-1}} \geq \frac{(\lambda_k^+)^2}{M}, \qquad k \geq N,$$

which, in view of (2.25), implies that

$$\infty > \sum_{k=N}^{\infty} \frac{w_k^2}{w_k + c_{k-1}} \geq \sum_{k=N}^{\infty} (\lambda_k^+)^2 = \infty.$$

This apparent contradiction, completes the proof. □

Remark 2.3. Theorem 2.5 when $c_n \equiv 1$ and $m > 1$ is the discrete analogue of a known result due to PHILOS [17] for equation (E_2). It, also, improves Theorem 2 of [15] especially when $q_n \geq 0$ eventually.

Some General Remarks. 1. The results of this paper are obtained in a form which is essentially new and are independent of those of [19].
2. For the sake of completeness, it is of interest to investigate the oscillatory behavior of equation (E) when the limit in condition (2.1) approaches $-\infty$.

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References

[1] R. P. AGARWAL, Properties of solutions of higher order nonlinear difference equations II, *An. Sti. Univ. Iasi* **29** (1983), 85–96.

[2] F. ATKINSON, Discrete and Continuous Boundary Value Problems, *Academic Press*, *New York, London*, 1964.

[3] S. CHEN and L. H. ERBE, Riccati Techniques and discrete oscillations, *J. Math. Anal. Appl.* **142** (1989), 468–487.

[4] S. S. CHENG, T. C. YAN and H. J. LI, Oscillation criteria for second order Difference Equations, *Funkcal. Ekvac.* **34** (1991), 223–239.

[5] T. FORT, Finite Differences and Difference Equations in the Real Domain, *Oxford Univ. Press*, *London*, 1948.

[6] S. R. GRACE, A. A. ABADEER and H. A. EL-MORSHEDY, On the oscillation of certain second order difference equations, *Comm. Appl. Anal.* **2** (1998), 447–456.

[7] P. Hartman, On nonoscillatory linear differential equations of second order, *Amer. J. Math.* **74** (1952), 389–400.

[8] D. B. Hinton and R. T. Lewis, Spectral analysis of second order difference equations, *J. Math. Anal. Appl.* **63** (1978), 421–438.

[9] J. W. Hooker, M. K. Kwong and W. Patula, Oscillatory second order linear difference equations, *SIAM J. Math. Anal.* **18** (1987), 54–63.

[10] J. W. Hooker and W. Patula, Riccati type transformations for second order linear difference equations, *J. Math. Anal. Appl.* **82** (1981), 451–462.

[11] I. V. Kamenev, An integral criterion for oscillation of linear differential equations of second order, *Mat. Zametki.* **23** (1978), 249–251.

[12] M. K. Kwong, J. W. Hooker and W. Patula, Riccati type transformations for second order linear difference equations, II, *J. Math. Anal. Appl.* **107** (1985), 182–196.

[13] V. Lakshmikantham and D. Trigiante, Theory of Difference Equations, Numerical methods and Applications, *Academic Press*, *New York*, 1988.

[14] W. Patula, Growth and oscillation properties of second order linear difference equations, *SIAM J. Math. Anal.* **10** (1979), 55–61.

[15] A. Peterson and J. Ridenhour, Oscillation theorems for second order scalar Difference Equations, Differential Equations: Stability and Control (S. Elaydi, ed.), pp. 415-421, Lecture Notes in Pure and Applied Mathematics, Vol. 127, *Dekker*, *New York*, 1991.

[16] Ch. G. Philos, Oscillation of second order linear ordinary differential equations with alternating coefficients, *Bull. Austral. Math. Soc.* **27** (1983), 307–313.

[17] Ch. G. Philos, Oscillation theorems for linear differential equations of second order, *Arch. Math.* **53** (1989), 482–492.

[18] A. Wintner, A criterion of oscillatory stability, *Quart. Appl. Math.* **7** (1949), 115–117.

[19] P. J. Y. Wong and R. P. Agarwal, Summation averages and the oscillation of second order nonlinear difference equations, *Mathl. Comput. Modelling* **24** (1996), 21–35.

[20] J. Yan, A note on an oscillation criterion for an equation with damped term, *Proc. Amer. Math. Soc.* **90** (1984), 277–280.

[21] J. Yan, Oscillation theorems for second order linear differential equations with damping, *Proc. Amer. Math. Soc.* **98** (1986), 276–282.

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