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On an Einstein structure on the tangent bundle of a space form

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Abstract. It is studied the existence of an Einstein structure on the tangent bundle of a space form, endowed with a certain Riemannian metric G which is no longer a metric of Sasaki type. It is obtained that, if (M, g) is an n-dimensional space form, then (TM, G) is an Einstein manifold if and only if $n = 2$ and the almost Kaehler structure naturally defined on TM, is a Kaehler structure (see [15]). Moreover, the obtained Einstein manifold (TM, G) is Ricci flat.

Introduction

It is known (see [2], [7], [18]) that the tangent bundle TM of an n-dimensional Riemannian manifold (M, g) can be organized as an almost Kaehlerian manifold by using the Sasaki metric and an almost complex structure defined by the splitting of the tangent bundle to TM into the vertical and horizontal distributions VTM, HTM (the last one being determined by the Levi Civita connection on M) (see also [16], [17]). However, this structure is Kaehler only in the case where the base manifold is locally Euclidean.

In [14] V. Oproiu and the present author, inspired by an idea of Calabi (see [1]) to define a hyper-Kaehler structure on the cotangent bundle of a Kaehler manifold of positive constant holomorphic sectional curvature, have considered a Lagrangian on a Riemannian manifold (M, g), defined by a real valued smooth function depending on the energy density only. They have shown that the nonlinear connection defined by the Euler–Lagrange

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equations associated to the considered Lagrangian does coincide with the nonlinear connection defined by the Levi Civita connection of g and have obtained a Riemannian metric G and an almost complex structure J, defined on TM, such that (TM, J, G) is an almost Kaehler manifold (like in the case of the Sasaki metric). Further, if (M, g) has positive constant curvature, then there exists a Lagrangian defined on the bundle of nonzero tangent vectors T_0M, such that (T_0M, J, G) is Kaehlerian. Remark that (T_0M, J, G) cannot be an Einstein manifold.

In [11], V. Oproiu is interested in finding a Kaehler Einstein structure on the tangent bundle of a space form. In this purpose, he has changed the metric G on the tangent bundle (so that it is no longer obtained from a Lagrangian) in the following way. Let t be the energy density defined on TM by the Riemannian metric g on M:

$$t = \frac{1}{2} g_{ij}(x) y^i y^j.$$

Then V. Oproiu considers the symmetric M-tensor field of type $(0, 2)$ on TM defined by the components

$$\tilde{G}_{ij} = u(t)\ g_{ij} + \ v(t)\ g_{0i} g_{0j},$$

where $g_{0i} = g_{ki} y^k$ and $u, v : [0, \infty) \longrightarrow \mathbb{R}$ are smooth real valued functions depending on t only such that $u(t) > 0, u(t) + \ 2tv(t) > 0$ for all $t \in [0, \infty)$. The matrix $(\tilde{G}_{ij})$ is symmetric and positive definite and has the inverse with the entries $\tilde{H}^{kl} = \frac{1}{u} g^{kl} - \frac{v}{u(u+2tv)} y^k y^l$, where g^{kl} are the components of the inverse of the matrix (g_{ij}). Next, V. Oproiu considers the Riemannian metric $\tilde{G}$ on TM defined by

$$\tilde{G} = \tilde{G}_{ij} dx^i dx^j + \tilde{H}_{ij} \dot{\nabla} y^i \dot{\nabla} y^j,$$

where $\tilde{H}_{ij} = g_{ik} \tilde{H}^{kl} g_{lj}$ and $\dot{\nabla} y^i = dy^i + \Gamma^i_{j0} dx^j$ is the absolute differential of y^i with respect to the Levi Civita connection $\dot{\nabla}$ of g. He defines also an almost complex structure J on TM, related to the considered metric $\tilde{G}$ and determines the expressions for the functions u, v in order to obtain a Kaehler Einstein structure on TM. He obtains in fact, a Kaehler Einstein structure on TM with constant holomorphic sectional curvature in the case where (M, g) has constant (negative) sectional curvature.

Remark that in [11], the author excludes three important cases which appeared, in a certain sense, as singular cases. One of them is the case where the metric on TM is obtained from a regular Lagrangian depending on the energy density (in this case $v(t)$ is the derivative of $u(t)$). This case has been studied by V. Oproiu and the present author in [14]. The second special case is the case where $u(t) = 1$, obtaining $v = -c$, where c is the constant sectional curvature of the Riemannian manifold (M, g). This case has been studied by V. Oproiu in [10] and he obtains that in this case the found Kaehler Einstein structure on TM is locally symmetric.

In [15] we have studied the third singular case which appeared in [11], namely, the case where $u(t)$ is a smooth real valued function such that $u(t) > 0$ for all $t \in [0, \infty)$ and $v(t) = 0$. We have obtained a Kaehler structure on the tangent bundle TM if and only if (M, g) has positive constant sectional curvature c and the function $u(t)$ is given by $u(t) = \sqrt{2ct + A}$, where A is an arbitrary positive real constant. This Kaehler structure is Ricci flat if $n = 2$ but, generally, it is not an Einstein structure.

The purpose of the present paper is to obtain the necessary and sufficient conditions in order to the Riemannian manifold (TM, G) to be an Einstein manifold, where G is defined as in above third singular case. The main result is given by: If (M, g) is a space form, then (TM, G) is an Einstein manifold if and only if $n = 2$, M has positive sectional curvature c and the function $u(t)$ is defined by $u(t) = \sqrt{2ct + A}$, where A is an arbitrary positive real constant, i.e., if and only if $n = 2$ and the almost Kaehler manifold (TM, J, G) is a Kaehler manifold, where the almost complex structure J, related to the considered metric G, is naturally defined on TM. Moreover, in this case, the Einstein manifold (TM, G) is Ricci flat. Similar results are also obtained for some tubes in TM defined in the situations $c < 0, A > 0$ and respectively $c > 0, A \leq 0$.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart, but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, r being always$\{1, \ldots, n\}$ (see [4], [3], [12], [13]). We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

1. A Kaehler structure on the tangent bundle

Let (M,g) be a smooth n-dimensional Riemannian manifold, $n>1$, and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of $2n$-dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U,\varphi) = (U, x^1, \dots, x^n)$ on M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n,\ y^1, \dots, y^n)$ on TM where the local coordinates $x^i, y^i;\ i = 1, \dots, n$ are defined as follows. The first n local coordinates $x^i = x^i \circ \tau;\ i = 1, \dots, n$ on TM are the local coordinates in the local chart (U,φ) of the base point of a tangent vector from $\tau^{-1}(U)$. The last n local coordinates $y^i; i = 1, \dots, n$ are the vector space coordinates of the same tangent vector, with respect to the natural local basis in the corresponding tangent space, defined by the local chart (U,φ).

This special structure of TM allows us to introduce the notion of M-tensor field on it (see [5]). An M-tensor field of type (p,q) on TM is defined by sets of functions

$$T^{i_1 \dots i_p}_{j_1 \dots j_q}(x,y);\ i_1, \dots, i_p, j_1, \dots, j_q = 1, \dots, n$$

assigned to any induced local chart $(\tau^{-1}(U), \Phi)$ on TM, such that the change rule is that of the components of a tensor field of type (p,q) on the base manifold, when a change of local charts on the base manifold is performed. Remark that any M-tensor field on TM may be thought of as an ordinary tensor field T with the expression

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial y^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}.$$

Remark also that any ordinary tensor field on the base manifold may be thought of as an M-tensor field on TM, having the same type and with the components in the induced local chart on TM, equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold. In the case of a covariant tensor field on the base manifold M the corresponding M-tensor field on the tangent bundle TM may be thought of as the pullback of the initial tensor field defined on the base manifold, by the smooth submersion $\tau : TM \longrightarrow M$.

The tangent bundle TM of a Riemannian manifold (M,g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metric on TM defined by g (see [16], [2]) and the complete lift type pseudo-Riemannian metric defined by g (see [17], [18], [8], [9]). Recall that the Levi Civita connection $\dot{\nabla}$ of g defines a direct sum decomposition

$$TTM = VTM \oplus HTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \operatorname{Ker} \tau_*$ and the horizontal distribution HTM. The vector fields $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ define a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0} \frac{\partial}{\partial y^h}; \quad \Gamma^h_{i0} = \Gamma^h_{ik} y^k$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric g.

The distributions VTM and HTM are isomorphic each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines a structure of almost Kaehlerian manifold on TM (see [2]).

Consider now the energy density:

$$t = \frac{1}{2} g_{ik}(x) y^i y^k \tag{2}$$

defined on TM by the Riemannian metric g of M, where g_{ik} are the components of g in the local chart (U, φ). Let $u : [0, \infty) \longrightarrow \mathbb{R}$ be a real smooth function such that $u(t) > 0$ for all $t \in [0, \infty)$. Then we may consider the following symmetric M-tensor field of type (0,2) on TM, defined by the components (see [15]):

$$G_{ij} = u(t) g_{ij}.$$

The matrix (G_{ij}) is symmetric and positive definite and has the inverse with the entries

$$G^{kl} = \frac{1}{u} g^{kl},$$

where g^{kl} are the componets of the inverse of the matrix (g_{ij}). The components $G^{kl}(x, y)$ define a symmetric M-tensor field of type $(2, 0)$ on TM.

We shall use also the components $H_{ij}(x,y)$ of a symmetric M-tensor field of type $(0,2)$ on TM defined by the components:

$$H_{ij} = g_{ik}G^{kl}g_{lj} = \frac{1}{u}g_{ij}.$$

The following Riemannian metric may be considered on TM:

$$G = G_{ij}dx^i dx^j + H_{ij}\dot{\nabla}y^i\dot{\nabla}y^j = ug_{ij}dx^i dx^j + \frac{1}{u}g_{ij}\dot{\nabla}y^i\dot{\nabla}y^j, \tag{3}$$

where $\dot{\nabla}y^i = dy^i + \Gamma^i_{j0}dx^j$ is the absolute differential of y^i with respect to the Levi Civita connection $\dot{\nabla}$ of g. Equivalently, we have

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = ug_{ij}, \quad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{1}{u}g_{ij},$$
$$G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = 0.$$

Remark that HTM, VTM are orthogonal each other with respect to G but the Riemannian metrics induced from G on HTM, VTM are not the same, so the considered metric G on TM is no longer a metric of Sasaki type. Remark also that the system of 1-forms $(dx^1, \dots, dx^n, \dot{\nabla}y^1, \dots, \dot{\nabla}y^n)$ defines a local frame of T^*TM, dual to the local frame $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ adapted to the direct sum decomposition (1).

An almost complex structure J may be defined on TM by:

$$J\frac{\delta}{\delta x^i} = u(t)\frac{\partial}{\partial y^i}; \quad J\frac{\partial}{\partial y^i} = -\frac{1}{u(t)}\frac{\delta}{\delta x^i}. \tag{4}$$

In [15] we obtain the following results:

Theorem 1. *(TM, J, G) is an almost Kaehlerian manifold.*

Theorem 2. *The almost complex structure J on TM is integrable if and only if the base manifold (M, g) has constant sectional curvature c and the function $u(t)$ satisfies the ordinary differential equation*

$$uu' = c. \tag{5}$$

From (5) it follows

$$u(t) = \sqrt{2ct + A}, \tag{6}$$

where A is an arbitrary real constant.

Theorem 3. *Assume that (M, g) has positive constant sectional curvature c and the function $u(t)$ is given by (6) where $A > 0$. Then we have:*

(a) *(TM, J, G) is a Kaehlerian manifold.*

(b) *If $n \neq 2$, then the Kaehlerian manifold (TM, J, G) cannot be an Einstein manifold and cannot have constant holomorphic sectional curvature.*

(c) *If $n = 2$, then the Kaehlerian manifold (TM, J, G) is Ricci flat.*

In the case when (M, g) has negative constant sectional curvature c and the function $u(t)$ is given by (6) where $A > 0$, we denote by T_1M the tube around the zero section in TM defined by the conditions:

$$0 \leq t < -\frac{A}{2c}.$$

Also, in the case when (M, g) has positive constant sectional curvature c and the function $u(t)$ is given by (6) where $A \leq 0$, we denote by T_2M the tube in TM defined by the condition:

$$t > -\frac{A}{2c}.$$

Then, we may state

Theorem 4. (a) *Assume that (M, g) has negative constant sectional curvature c and the function $u(t)$ is given by (6) where $A > 0$. Then (T_1M, J, G) has a structure of Kaehler manifold.*

(b) *Assume that (M, g) has positive constant sectional curvature c and the function $u(t)$ is given by (6) where $A \leq 0$. Then (T_2M, J, G) has a structure of Kaehler manifold.*

2. The existence of an Einstein structure on (TM, G)

In this section we shall study the necessary and sufficient conditions in order to the Riemannian manifold (TM, G) to be an Einstein manifold, assuming that (M, g) is a space form. To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^i}$; $i = 1, \ldots, n$:

$$(7)\quad \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0;\ \left[\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right] = -\Gamma^h_{ij}\frac{\partial}{\partial y^h};\ \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^h_{0ij}\frac{\partial}{\partial y^h},$$

where $R^h_{0ij} = R^h_{kij} y^k$ and R^h_{kij} are the local coordinate components of the curvature tensor field of $\dot{\nabla}$ on M. Also, we shall denote $\delta^i_0 = \delta^i_j y^j = y^i$ and $R^h_{i0j} = R^h_{ikj} y^k$.

In the following we determine the Levi Civita connection ∇ of the Riemannian metric G on TM, where G is defined by (3). Recall that the Levi Civita connection ∇ on the Riemannian manifold (TM, G) is obtained from the formula

$$2G(\nabla_X Y, Z) = X(G(Y,Z)) + Y(G(X,Z)) - Z(G(X,Y))$$
$$+G([X,Y],Z) - G([X,Z],Y) - G([Y,Z],X); \quad \forall X,Y,Z \in \Gamma(TM).$$

We shall use this formula in order to obtain the expression of the Levi Civita connection ∇ on TM, determined by the conditions

$$\nabla G = 0, \qquad T = 0,$$

where T is the torsion tensor of ∇ [3].

Proposition 5. *The Levi Civita connection ∇ of the Riemannian manifold (TM, G) has the following expression in the local adapted frame $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^i})$:*

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = Q^h_{ij} \frac{\partial}{\partial y^h}; \qquad \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij} \frac{\partial}{\partial y^h} + P^h_{ji} \frac{\delta}{\delta x^h};$$
$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = P^h_{ij} \frac{\delta}{\delta x^h}; \qquad \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} + S^h_{ij} \frac{\partial}{\partial y^h},$$

where the M-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ are given by:

$$P^h_{ij} = \frac{1}{2u^2}(uu' g_{0i} \delta^h_j + R^h_{j0i});$$
$$Q^h_{ij} = \frac{u'}{2u}(g_{ij} y^h - g_{0i} \delta^h_j - g_{0j} \delta^h_i);$$
$$S^h_{ij} = -\frac{1}{2}(R^h_{0ij} + uu' g_{ij} y^h),$$

and where $g_{0i} = g_{ki} y^k$.

Denote by K the curvature tensor field of the Levi Civita connection ∇ of the Riemannian metric G on TM. Then we get by a straightforward computation

Proposition 6. *The local coordinate expression of the curvature tensor field K of the Riemannian manifold (TM, G) is given in the adapted local frame $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^i})$ by*

$$K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} = \Bigg[\frac{2uu' - t(u')^2}{2u^2}(g_{jk}\delta_i^h - g_{ik}\delta_j^h)$$
$$+\frac{2uu'' - (u')^2}{4u^2}(g_{0i}g_{jk}y^h - g_{0j}g_{ik}y^h + g_{0j}g_{0k}\delta_i^h - g_{0i}g_{0k}\delta_j^h)\Bigg]\frac{\partial}{\partial y^h};$$

$$K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} = \Bigg[\frac{1}{u^2}R^h_{kij} + \frac{u'}{u^3}(g_{0j}R^h_{k0i} - g_{0i}R^h_{k0j})$$
$$+\frac{1}{4u^4}(R^h_{l0i}R^l_{k0j} - R^h_{l0j}R^l_{k0i})\Bigg]\frac{\delta}{\delta x^h};$$

$$K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = \Bigg[\frac{1}{2u^2}R^h_{jik} + \frac{uu' - (u')^2 t}{2u^2}g_{ik}\delta_j^h$$
$$+\frac{2uu'' + (u')^2}{4u^2}g_{0i}g_{0k}\delta_j^h - \frac{u'}{2u^3}g_{0i}R^h_{j0k} + \frac{u'}{2u^3}g_{0k}R^h_{j0i}$$
$$+\frac{1}{4u^4}R^h_{l0i}R^l_{j0k}\Bigg]\frac{\delta}{\delta x^h};$$

$$K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = -\frac{1}{2}\Bigg[R^h_{ijk} - (t(u')^2 - uu')g_{jk}\delta_i^h$$
$$+\frac{(u')^2 + 2uu''}{2}g_{0i}g_{jk}y^h + \frac{u'}{2u}g_{il}R^l_{0jk}y^h - \frac{u'}{u}g_{0i}R^h_{0jk}$$
$$-\frac{u'}{2u}g_{jl}R^l_{k0i}y^h - \frac{1}{2u^2}R^h_{0jl}R^l_{k0i}\Bigg]\frac{\partial}{\partial y^h};$$

$$K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = \Bigg[R^h_{kij} - \frac{u'}{u}g_{0k}R^h_{0ij} + \frac{u'}{2u}g_{kl}R^l_{0ij}y^h - \frac{u'}{4u}g_{il}R^l_{j0k}y^h$$
$$+\frac{u'}{4u}g_{jl}R^l_{i0k}y^h + \frac{1}{4u^2}R^h_{0jl}R^l_{i0k} - \frac{1}{4u^2}R^h_{0il}R^l_{j0k}\Bigg]\frac{\partial}{\partial y^h};$$

$$K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k}=\Bigg[R^h_{kij}+\frac{1}{4u^2}(2R^h_{k0l}R^l_{0ij}-R^h_{i0l}R^l_{0jk}+R^h_{j0l}R^l_{0ik})$$
$$-\frac{t(u')^2}{2}(g_{jk}\delta^h_i-g_{ik}\delta^h_j)\Bigg]\frac{\delta}{\delta x^h}.$$

From the above formulas, we get by a straightforward computation that the local coordinate expression of the Ricci tensor $S(Y,Z)=$ trace $(X\longrightarrow K(X,Y)Z)$ in the local frame adapted to the direct sum decomposition (1) is given by

$$S\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=\frac{(n-2)uu'+2tuu''}{2u^2}g_{ij}-\frac{(n-1)(u')^2+2uu''}{2u^2}g_{0i}g_{0j}$$
$$-\frac{1}{4u^4}R^k_{l0i}R^l_{k0j};$$
$$S\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)=-\frac{nuu'+2tuu''}{2}g_{ij}+R_{ij}+\frac{1}{4u^2}(R^k_{j0l}R^l_{0ki}+R^k_{i0l}R^l_{0kj});$$
$$S\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)=0;$$

where R_{ij} denote the local coordinate components of the Ricci tensor field on the Riemannian manifold (M,g).

In the following we assume that (M,g) has constant sectional curvature $c\neq 0$. Then, from the above formulas, we get that the local coordinate expression of the Ricci tensor S on (TM,G) is given by

$$S\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=\left[\frac{(n-2)uu'+2tuu''}{2u^2}+\frac{tc^2}{u^4}\right]g_{ij}$$
$$-\left[\frac{(n-1)(u')^2+2uu''}{2u^2}+\frac{c^2}{2u^4}\right]g_{0i}g_{0j};$$
$$S\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)=\left[(n-1)c-\frac{nuu'+2tuu''}{2}-\frac{c^2t}{u^2}\right]g_{ij}-\frac{(n-2)c^2}{2u^2}g_{0i}g_{0j};$$
$$S\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)=0.$$

Comparing the obtained expressions of the Ricci tensor with the expressions of the components of G, we get that the necessary conditions in order to the Riemannian manifold (TM, G) to be an Einstein manifold, assuming that (M, g) has constant sectional curvature c, are given by:

(i) $n = 2$ and

(ii) the function $u(t)$ satisfies the ordinary differential equation

$$2u^3u'' + u^2(u')^2 + c^2 = 0. \tag{8}$$

By taking into account of the above conditions (i) and (ii), we obtain that the local coordinate expression of the Ricci tensor S on (TM, G) becomes:

$$\begin{cases} S\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{t(u^3u'' + c^2)}{u^3} G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right), \\ S\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \frac{cu^2 - u^3u' - tu^3u'' - c^2t}{u^3} G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right), \\ S\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0. \end{cases} \tag{9}$$

From (9) it follows that, if $n = 2$ and (M, g) has constant sectional curvature c, then the Riemannian manifold (TM, G) is an Einstein manifold if and only if the function $u(t)$ satisfies both the differential equations (8) and

$$2tu^3u'' + u^3u' - cu^2 + 2tc^2 = 0. \tag{10}$$

Eliminating $2u^3u''$ from (8) and (10), we obtain that, if $n = 2$ and (M, g) has constant sectional curvature c, then the Riemannian manifold (TM, G) is an Einstein manifold only if the function $u(t)$ satisfies the relation

$$(uu' - c)(tuu' - u^2 + ct) = 0,$$

i.e. $u(t)$ is a solution of one from the following two differential equations:

$$uu' = c \tag{11}$$

$$tuu' - u^2 + ct = 0. \tag{12}$$

From (11), by imposing the condition $u(t) > 0, \forall t \geq 0$, i.e. for G to be a Riemannian metric on TM, it follows $c > 0$ and then it can be checked easily that the general solution $u(t) = \sqrt{2ct + A}$ of the differential equation (11), where A is an arbitrary positive real constant, satisfies also both the differential equations (8) and (10).

Next, we get that the general solution of the differential equation (12) is given by

$$u(t) = \sqrt{Bt^2 + 2ct} \tag{13}$$

where B is an arbitrary real constant. It is easy to check that the function $u(t)$ defined by (13), where $B \neq 0$, does not satisfy the differential equations (8) and (10). Remark that for $B = 0$, from (13) we have $u = \sqrt{2ct}$, $c > 0$, and in this case $u(t)$ is also a particular solution of the differential equation (11) (obtained for $A = 0$). In this case it follows that G is a Riemannian metric only on the manifold $T_0M =$ the tangent bundle to M minus the zero section.

Hence, we state

Theorem 7. *Let (M, g) be an n-dimensional space form of sectional curvature $c \neq 0$. Then the Riemannian manifold (TM, G) is an Einstein manifold if and only if the following conditions are satisfied:*

(i) $n = 2$,

(ii) $c > 0$,

(iii) $u(t) = \sqrt{2ct + A}$, *where A is a positive real constant.*

Moreover, in this case, the Einstein manifold (TM, G) is Ricci flat.

From Theorem 3 and Theorem 7, we also have

Corollary 8. *Let (M, g) be an n-dimensional space form of sectional curvature $c \neq 0$. Then the Riemannian manifold (TM, G) is an Einstein manifold if and only if $n = 2$ and the almost Kaehler manifold (TM, J, G) is a Kaehler manifold.*

Remark. Similar results are also valid for the Riemannian manifolds (T_1M, G) and (T_2M, G) where T_1M and T_2M are defined at the end of section 1, analysing the situations when $c < 0, A > 0$ and respectively $c > 0$, $A \leq 0$.

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