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On the embeddability of a homeomorphism of the unit circle in disjoint iteration groups

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Abstract. In this note the problem of the embeddability of a homeomorphism F of the unit circle in disjoint iteration groups is studied. For F with an irrational rotation number a result concerning embeddability of F in an arbitrary iteration group is proved as a simple consequence of the obtained theorems.

1. Introduction

Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle with positive orientation and $F : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be a homeomorphism.

A family $\{F^t, t \in \mathbb{R}\}$ of homeomorphisms $F^t : \mathbb{S}^1 \mapsto \mathbb{S}^1$ such that

$$F^s \circ F^t = F^{s+t}, \quad s,t \in \mathbb{R}$$

is said to be a flow or an iteration group.

A homeomorphism F is said to be embeddable in an iteration group if there exists an iteration group $\{F^t, t \in \mathbb{R}\}$ such that $F^1 = F$. Then we will say that $\{F^t, t \in \mathbb{R}\}$ is an iteration group of F.

An iteration group $\{F^t, t \in \mathbb{R}\}$ is said to be continuous if for every $z \in \mathbb{S}^1$ the mapping $t \mapsto F^t(z)$ is continuous.

An iteration group $\{F^t, t \in \mathbb{R}\}$ such that for every $t \in \mathbb{R}$ the following condition holds: if F^t has a fixed point, then $F^t = id$ is said to be disjoint (see [1] and also [4]).

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According to Corollary 1 in [3], we get

Remark 1. Every continuous iteration group $\{F^t, t \in \mathbb{R}\}$ such that at least one F^t has not fixed points is disjoint. Thus, if a homeomorphism F without fixed points is embeddable in a continuous iteration group, then it is embeddable in a disjoint iteration group.

In [3] M. C. Zdun has proved a necessary and sufficient condition for the embeddability of a homeomorphism F of the unit circle in a continuous iteration group and he has given the general construction of all continuous iteration groups $\{F^t, t \in \mathbb{R}\}$ such that $F^1 = F$. This note deals with disjoint iteration groups.

Theorem 1 states that a homeomorphism $F : \mathbb{S}^1 \mapsto \mathbb{S}^1$ can always be embedded in such an iteration group whenever the rotation number of F is rational and one of its iterates equals identity. For F with an irrational rotation number two cases are considered. The required embedding of F is always possible if the limit set of F, which will be denoted by L_F, is equal to the whole circle $\mathbb{S}^1$. In this case the general form of all disjoint iteration groups of F is also obtained (Theorem 2). Theorem 3 shows that the embeddability of F with a nowhere dense limit set is equivalent to the existence of a sequence of non-negative integers satisfying some additional conditions. The proof of the theorem contains the general construction of all disjoint iteration groups of F. The paper is concluded with Theorem 4 which states that a homeomorphism F with an irrational rotation number is embeddable in an iteration group if and only if F is embeddable in a disjoint iteration group.

Throughout the paper the closure of the set A will be denoted by $\operatorname{cl} A$ and we write A^d for the set of all cluster points of A. $\sim p$ stands for the negation of p.

2. Preliminaries

Following [1] we introduce some notations and definitions.

Let $\tilde{\pi} : \mathbb{R} \ni t \mapsto e^{2\pi i t} \in \mathbb{S}^1$ and $\pi := \tilde{\pi}_{|[0,1)}$. The function π is a continuous bijection. Thus, if $v, w, z \in \mathbb{S}^1$, then there exist unique $t_1, t_2 \in [0, 1)$ such that $w\pi(t_1) = z$ and $w\pi(t_2) = v$. Define

$$v \prec w \prec z \quad \text{if and only if } 0 < t_1 < t_2$$

and

$$v \preceq w \preceq z \quad \text{if and only if } t_1 \leq t_2 \text{ or } t_2 = 0.$$

Now we give some useful characterizations and properties of "$\prec$" and "$\preceq$".

Remark 2. Let $v, w, z \in \mathbb{S}^1$. If $v \prec w \prec z$, then $v \neq w$, $w \neq z$, $v \neq z$.

If $v, z \in \mathbb{S}^1$, $v \neq z$, then there exist $t_v, t_z \in \mathbb{R}$ such that $t_v < t_z < t_v + 1$ and $v = \tilde{\pi}(t_v)$, $z = \tilde{\pi}(t_z)$. Put

$$\overrightarrow{(v, z)} := \{\tilde{\pi}(t),\ t \in (t_v, t_z)\}.$$

It is clear that $\overrightarrow{(v, z)}$ does not depend on the choice of t_v and t_z.

The straightforward proofs of the following three lemmas are omitted.

Lemma 1. *Let $v, w, z \in \mathbb{S}^1$. $v \prec w \prec z$ if and only if $w \in \overrightarrow{(v, z)}$.*

Lemma 2. *For every $v, w, z \in \mathbb{S}^1$ the following conditions are equivalent:*

(i) $v \prec w \prec z$,

(ii) $w \prec z \prec v$,

(iii) $z \prec v \prec w$.

Lemma 3. *For every $v, w, z \in \mathbb{S}^1$ the following conditions are equivalent:*

(i) $\sim (v \prec w \prec z)$,

(ii) $v = w$ *or* $w = z$ *or* $v = z$ *or* $z \prec w \prec v$,

(iii) $z \preceq w \preceq v$.

As an immediate consequence of Lemmas 3 and 2 we have

Remark 3. For every $v, w, z \in \mathbb{S}^1$ the following conditions are equivalent:

(i) $v \preceq w \preceq z$,

(ii) $w \preceq z \preceq v$,

(iii) $z \preceq v \preceq w$.

Let $A \subset \mathbb{S}^1$ be such that $\operatorname{card} A \geq 3$. We say that the function $\varphi : A \mapsto \mathbb{S}^1$ is increasing (respectively, strictly increasing) if for every v, w, z belonging to A such that $v \prec w \prec z$ we have $\varphi(v) \preceq \varphi(w) \preceq \varphi(z)$ (respectively, $\varphi(v) \prec \varphi(w) \prec \varphi(z)$). According to Lemma 1, the map φ is strictly increasing if $w \in \overrightarrow{(v, z)}$ yields $\varphi(w) \in \overrightarrow{(\varphi(v), \varphi(z))}$.

By Remark 2 we get

Remark 4. Every strictly increasing mapping is an injection.

A subset $A \subset \mathbb{S}^1$ is said to be an open arc if there exist $v, z \in \mathbb{S}^1$, $v \neq z$ such that $A = \overrightarrow{(v,z)} = \{\tilde{\pi}(t),\, t \in (t_v,\, t_z)\}$, where $\tilde{\pi}(t_v) = v$, $\tilde{\pi}(t_z) = z$ and $0 < t_z - t_v < 1$, that is, in the notation of "$\prec$", $A = \{w \in \mathbb{S}^1 : v \prec w \prec z\}$. Let us note that such a subset is non-empty, different from $\mathbb{S}^1$, open and connected.

It is a simple matter to show that strictly increasing mappings have the following property

Lemma 4. *Every strictly increasing function $G : \mathbb{S}^1 \mapsto \mathbb{S}^1$ such that the image of G is a dense subset of $\mathbb{S}^1$ is continuous.*

Let $A := \overrightarrow{(v,z)}$, where $v = \tilde{\pi}(t_v)$, $z = \tilde{\pi}(t_z)$, $0 < t_z - t_v < 1$, be an open arc. The mapping $F : A \mapsto \mathbb{S}^1$ is said to be linear if $F(e^{2\pi i x}) = e^{2\pi i(ax+b)}$ for some $a > 0$, $b \in \mathbb{R}$ and every $x \in (t_v, t_z)$.

It is known (see for instance [2]) that for every homeomorphism $F : \mathbb{S}^1 \mapsto \mathbb{S}^1$ there exists a homeomorphism $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$F \circ \tilde{\pi} = \tilde{\pi} \circ f$$

and

$$f(x+1) = f(x) + 1, \quad \text{if } f \text{ is strictly increasing}$$

and

$$f(x+1) = f(x) - 1, \quad \text{if } f \text{ is strictly decreasing.}$$

We will say that the function f represents the homeomorphism F. If f_1 and f_2 represent the same homeomorphism, then $f_1 = f_2 + k$ for a $k \in \mathbb{Z}$. If f is strictly increasing we will say that the homeomorphism F preserves orientation.

M. Bajger has proved that every orientation-preserving homeomorphism is strictly increasing (see [1]). It is easy to check that the converse statement is also true, so we have

Remark 5. A homeomorphism $F : \mathbb{S}^1 \mapsto \mathbb{S}^1$ preserves orientation if and only if F is strictly increasing.

It is easily seen that for every homeomorphism $F : A \mapsto B$, where A and B are open arcs, say $A = \{\tilde{\pi}(t),\ t \in (a,b)\}$, $B = \{\tilde{\pi}(t),\ t \in$

$(c,d)\}$ there exists a unique homeomorphism $f:(a,b)\mapsto(c,d)$ such that $(F\circ\tilde{\pi})(x)=(\tilde{\pi}\circ f)(x)$ for $x\in(a,b)$. We will say that the function f represents the homeomorphism F and if f is strictly increasing, then we will say that F preserves orientation.

The following lemma is easy to check

Lemma 5 (see also Lemma 4 in [3]). *Suppose that either A, B, C are open arcs or $A=B=C=\mathbb{S}^1$. If f represents a homeomorphism $F:A\mapsto B$ and g represents a homeomorphism $G:B\mapsto C$, then*

(i) *$g\circ f$ represents $G\circ F$,*

(ii) *f^{-1} represents F^{-1}.*

Let us note that if $\{F^t, t\in\mathbb{R}\}$ is an iteration group, then for every $t\in\mathbb{R}$, $F^t=F^{\frac{t}{2}}\circ F^{\frac{t}{2}}$, so every homeomorphism F^t preserves orientation. Thus, if F is embeddable in an iteration group, then it preserves orientation.

If $F:\mathbb{S}^1\mapsto\mathbb{S}^1$ is an orientation-preserving homeomorphism represented by a function f then the number $\alpha(F)\in[0,1)$ defined by

$$\alpha(F):=\lim_{n\to\infty}\frac{f^n(x)}{n}\pmod 1,\quad x\in\mathbb{R}$$

is said to be the rotation number of F. This limit always exists and does not depend on x and f. Moreover, $\alpha(F)$ is rational if and only if $F^n(z_0)=z_0$ for a $z_0\in\mathbb{S}^1$ and an $n\in\mathbb{Z}\setminus\{0\}$ (see [2]).

3. Main results

We begin with the case when F is an orientation-preserving homeomorphism such that $\alpha(F)\in\mathbb{Q}$. Let us note that if such an F is embeddable in a disjoint iteration group, then

$$F^n(z)=z\quad\text{for an } n\in\mathbb{Z}\setminus\{0\}\quad\text{and every } z\in\mathbb{S}^1. \tag{1}$$

Theorem 1. *Every orientation-preserving homeomorphism F fulfilling (1) is embeddable in a disjoint iteration group.*

Proof. Let $n\in\mathbb{N}$, which we may assume, be the smallest number satisfying (1). If $n=1$, then putting $F^t=id$, $t\in\mathbb{R}$ we get the desired iteration group. Suppose now that $n>1$. Set $g(x):=x+1$, $x\in\mathbb{R}$ and let a homeomorphism $f : \mathbb{R} \mapsto \mathbb{R}$ representing F be such that $0 < f(0) < 1$. Note that $f \circ g = g \circ f$ and f and g have no fixed points. By Lemma 6 in [3], $g^0 < f < g^1$ and there exists an $m \in \mathbb{N}$ such that $f^n = g^m$. In [4] it is proved that for such homeomorphisms there exist infinitely many disjoint iteration groups $\{f^t, t \in \mathbb{R}\}$ of homeomorphisms $f^t : \mathbb{R} \mapsto \mathbb{R}$ such that $f^1 = f$ and $g = f^s$ for an $s \in \mathbb{R}$. Since $g \in \{f^t,\ t \in \mathbb{R}\}$, $f^t \circ g = g \circ f^t$ for all $t \in \mathbb{R}$ and the functions $F^t(\tilde{\pi}(x)) := \tilde{\pi}(f^t(x))$, $t, x \in \mathbb{R}$ are homeomorphisms. Obviously, $\{F^t,\ t \in \mathbb{R}\}$ is an iteration group such that $F^1 = F$. What is left is to show that this iteration group is disjoint. Let $t \in \mathbb{R}$ and $z_0 = \tilde{\pi}(x_0) \in \mathbb{S}^1$ be such that $F^t(z_0) = z_0$, that is $\tilde{\pi}(x_0) = F^t(\tilde{\pi}(x_0)) = \tilde{\pi}(f^t(x_0))$. Hence there exists a $k \in \mathbb{Z}$ such that $f^t(x_0) = x_0 + k = g^k(x_0)$. Thus $f^t = g^k$, since f^t, g^k belong to the iteration group $\{f^t, t \in \mathbb{R}\}$, which is disjoint. This clearly forces $F^t = id$. □

From now on we assume that F is an orientation-preserving homeomorphism with the irrational rotation number $\alpha(F)$.

Let

$$C_F(z) := \{F^n(z),\ n \in \mathbb{Z}\}, \quad z \in \mathbb{S}^1.$$

The set $L_F := C_F(z)^d$ does not depend on z, is invariant with respect to F (that is $F[L_F] = L_F$) and either $L_F = \mathbb{S}^1$ or L_F is a perfect nowhere dense subset of $\mathbb{S}^1$ (see for instance [2]). In the second case we have the following unique decomposition

$$\mathbb{S}^1 \setminus L_F = \bigcup_{q \in M} L_q, \tag{2}$$

where L_q for $q \in M$ are open pairwise disjoint arcs and $\operatorname{card} M = \aleph_0$.

Lemma 6. *If $\alpha(F) \notin \mathbb{Q}$, then for every $z \in L_F$, $L_F = \operatorname{cl} C_F(z)$.*

Proof. It is evident that for every $z \in L_F$ we have $L_F = C_F(z)^d \subset \operatorname{cl} C_F(z)$. Let $z \in L_F$. Then $F^n(z) \in L_F$ for $n \in \mathbb{Z}$ since $F[L_F] = L_F$. Thus, $C_F(z) \subset L_F$ and consequently $\operatorname{cl} C_F(z) \subset \operatorname{cl} L_F = L_F$. □

Lemma 7. *If $\alpha(F) \notin \mathbb{Q}$, then there exists a unique up to a multiplicative constant continuous mapping φ from $\mathbb{S}^1$ onto $\mathbb{S}^1$ such that*

$$\varphi(F(z)) = \pi(\alpha(F))\varphi(z), \quad z \in \mathbb{S}^1. \tag{3}$$

This map is increasing. Moreover, φ is a homeomorphism if and only if $L_F = \mathbb{S}^1$.

Proof. It is well-known (see for instance [2]) that there exists a continuous function $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ fulfilling (3), which is a homeomorphism if and only if $L_F = \mathbb{S}^1$. This map is defined by

$$\varphi(z) := \tilde{\pi}(\mu(\overrightarrow{[1, z]})), \quad z \in \mathbb{S}^1,$$

where μ is a probability Borel measure on $\mathbb{S}^1$ invariant with respect to F and $\overrightarrow{[1, z]} := \overrightarrow{(1, z)} \cup \{1, z\}$. To prove that φ is increasing, take $v, w, z \in \mathbb{S}^1$ such that $v \prec w \prec z$. In view of Lemmas 1, 2 and Remark 3 we may assume that $\overrightarrow{[1, v]} \subset \overrightarrow{[1, w]} \subset \overrightarrow{[1, z]}$. This gives $0 \leq \mu(\overrightarrow{[1, v]}) \leq \mu(\overrightarrow{[1, w]}) \leq \mu(\overrightarrow{[1, z]}) \leq 1$ and, in consequence, $\varphi(v) \preceq \varphi(w) \preceq \varphi(z)$, which is our claim. Assume now that $\varphi_1, \varphi_2 : \mathbb{S}^1 \mapsto \mathbb{S}^1$ are continuous solutions of (3) and set $\psi := \frac{\varphi_1}{\varphi_2}$. Obviously, ψ is continuous and $\psi(F(z)) = \psi(z)$, $z \in \mathbb{S}^1$. Hence,

$$\psi(F^n(z)) = \psi(z), \quad z \in \mathbb{S}^1,\ n \in \mathbb{Z}. \tag{4}$$

Fix $z_0 \in L_F$, $z \in \mathbb{S}^1$. By the definition of L_F there exists a sequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{Z}$ such that $\lim_{k\to\infty} F^{n_k}(z) = z_0$. Since ψ is continuous we have $\psi(z_0) = \lim_{k\to\infty} \psi(F^{n_k}(z))$ and by (4), $\psi(z_0) = \psi(z)$. Thus ψ is constant and consequently $\varphi_1 = c\varphi_2$ for $c = \psi(z_0)$. □

For the convenience of the reader we quote, slightly modified, four lemmas from [1].

Lemma 8 (see [1]). *Let $\{F^t, t \in \mathbb{R}\}$ be an iteration group of F and $\alpha(F) \notin \mathbb{Q}$. Then there exist a continuous increasing function $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ and a function $c : \mathbb{R} \mapsto \mathbb{S}^1$ such that*

$$\varphi\left(F^t(z)\right) = c(t)\varphi(z), \quad z \in \mathbb{S}^1,\ t \in \mathbb{R}, \tag{5}$$

$$c(s + t) = c(s)c(t), \quad s, t \in \mathbb{R}, \tag{6}$$

$$\varphi[L_F] = \mathbb{S}^1 \tag{7}$$

and

$$c(1) = \pi(\alpha(F)). \tag{8}$$

The solution φ of (5) is a homeomorphism if and only if $L_F = \mathbb{S}^1$.

Lemma 9 (see [1]). *Let $\{F^t,\ t \in \mathbb{R}\}$ be an iteration group of F, $\alpha(F) \notin \mathbb{Q}$ and $L_F \neq \mathbb{S}^1$. If $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is a continuous increasing solution of* (5) *satisfying* (7) *with $c : \mathbb{R} \mapsto \mathbb{S}^1$, then:*

(a) *for every $q \in M$, φ is constant on L_q,*

(b) *if $V \subset \mathbb{S}^1$ is an open arc and φ is constant on V, then $V \subset L_q$ for some $q \in M$,*

(c) *if $p \neq q$, then $\varphi[L_p] \cap \varphi[L_q] = \emptyset$,*

(d) *for every $q \in M$ and every $t \in \mathbb{R}$, there exists a $p \in M$ such that $F^t[L_q] = L_p$,*

(e) *the sets $\operatorname{Im} c$, $\varphi[\mathbb{S}^1 \setminus L_F]$ are countable,*

(f) $\varphi[\mathbb{S}^1 \setminus L_F] \cdot \operatorname{Im} c = \varphi[\mathbb{S}^1 \setminus L_F]$,
where L_q, $q \in M$ are open arcs defined by (2).

Put

$$\Phi(q) := \varphi[L_q], \quad q \in M \tag{9}$$

and

$$T(q,t) := \Phi^{-1}(\Phi(q)c(t)), \quad q \in M,\ t \in \mathbb{R}, \tag{10}$$

where $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is a continuous solution of (5) with $c : \mathbb{R} \mapsto \mathbb{S}^1$.

Lemma 10 (see [1]). *Assume that $\{F^t, t \in \mathbb{R}\}$ is an iteration group of F, $\alpha(F) \notin \mathbb{Q}$ and $L_F \neq \mathbb{S}^1$. Then there exists a unique disjoint iteration group $\{R^t, t \in \mathbb{R}\}$ on $\mathbb{S}^1$ such that*

$$\begin{aligned} &R^t \text{ is linear on } L_q, && q \in M,\ t \in \mathbb{R}, \\ &R^t[L_q] = L_{T(q,t)}, && q \in M,\ t \in \mathbb{R}, \\ &\varphi\left(R^t(z)\right) = c(t)\varphi(z), && z \in \mathbb{S}^1,\ t \in \mathbb{R}, \end{aligned}$$

where $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$, $c : \mathbb{R} \mapsto \mathbb{S}^1$ are solutions of (5) *such that φ is a continuous increasing function.*

The iteration group $\{R^t,\ t \in \mathbb{R}\}$ determined above is said to be the generating iteration group of $\{F^t,\ t \in \mathbb{R}\}$.

Lemma 11 (see [1]). *Let $\{F^t, t \in \mathbb{R}\}$ be a disjoint iteration group of F, $\alpha(F) \notin \mathbb{Q}$, $L_F \neq \mathbb{S}^1$ and $\{R^t, t \in \mathbb{R}\}$ be its generating iteration group. Then there exists an orientation-preserving homeomorphism $\Gamma : \mathbb{S}^1 \mapsto \mathbb{S}^1$ such that*

$$\Gamma \circ F^t = R^t \circ \Gamma, \quad t \in \mathbb{R}$$

and $\Gamma(z) = z$, for all $z \in L_F$.

First, consider the case $L_F = \mathbb{S}^1$.

Theorem 2. *Every homeomorphism $F : \mathbb{S}^1 \mapsto \mathbb{S}^1$ such that $\alpha(F) \notin \mathbb{Q}$ and $L_F = \mathbb{S}^1$ is embeddable in a disjoint iteration group. The general form of all disjoint iteration groups $\{F^t, t \in \mathbb{R}\}$ such that $F^1 = F$ is given by the formula*

$$F^t(z) = \varphi^{-1}(c(t)\varphi(z)), \quad z \in \mathbb{S}^1,\ t \in \mathbb{R}, \tag{11}$$

where $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is a homeomorphism fulfilling (6) *and $c : \mathbb{R} \mapsto \mathbb{S}^1$ is a function satisfying conditions* (6) *and* (8).

Proof. It is easy to check that formula (11) with the above-mentioned functions φ and c defines the desired iteration group. On the other hand, from Lemma 8 we conclude that every disjoint iteration group $\{F^t, t \in \mathbb{R}\}$ such that $F^1 = F$ is of this form. □

We now turn to the case $L_F \neq \mathbb{S}^1$.

Proceeding analogously to the proof of Proposition 2 in [1] we obtain

Lemma 12. *Let $\alpha(F) \notin \mathbb{Q}$, $L_F \neq \mathbb{S}^1$ and $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be a continuous increasing solution of* (3). *Then conditions* (a), (b), (c) *hold and*

(d') *for every $q \in M$ there exists a $p \in M$ such that $F[L_q] = L_p$,*

(e') *the set*

$$K_F := \varphi\left[\mathbb{S}^1 \setminus L_F\right]$$

is countable.

It follows by Lemma 7 that for every homeomorphism F with irrational rotation number the set K_F is determined uniquely up to a multiplicative constant.

Lemmas 8 and 9 show that if a homeomorphism F such that $\alpha(F) \notin \mathbb{Q}$ is embeddable in an iteration group, then the set K_F has the following property

(C) there exists a function $c : \mathbb{R} \mapsto \mathbb{S}^1$ satisfying conditions (6) and (8) such that $\operatorname{card} \operatorname{Im} c = \aleph_0$ and

$$K_F \cdot \operatorname{Im} c = K_F. \tag{12}$$

Assuming further that K_F has this property we define a function Φ by (9). It follows from Lemma 12 that Φ is a bijection of M onto K_F. According to (12) we may define a map $T : M \times \mathbb{R} \mapsto M$ by (10). An easy computation shows that T satisfies the translation equation

$$T(T(q,s),t) = T(q,s+t), \quad q \in M,\ s,t \in \mathbb{R}. \tag{13}$$

At present we will construct some special disjoint iteration group $\{P^t,\ t \in \mathbb{R}\}$.

Lemma 13. *Assume that $\alpha(F) \notin \mathbb{Q}$, $L_F \neq \mathbb{S}^1$ and $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is a continuous solution of* (3). *Moreover, let K_F has property* (C). *Then there exists the unique family $\{P^t, t \in \mathbb{R}\}$ of continuous functions such that*

$$P^t \text{ is linear on } L_q, \quad q \in M,\ t \in \mathbb{R} \tag{14}$$

and

$$P^t[L_q] = L_{T(q,t)}, \qquad q \in M,\ t \in \mathbb{R}. \tag{15}$$

This family is a disjoint iteration group of homeomorphisms.

Proof. Let $x_0 \in [0,1)$ be such that $\pi(x_0) =: z_0 \in L_F$. Put $\nu(x) := \pi(x)z_0$, that is $\nu(x) = \tilde{\pi}(x + x_0)$, $x \in [0,1)$. Setting $L' := \nu^{-1}[L_F] \cap (0,1)$ we have the following decomposition

$$(0,1) \setminus L' = \bigcup_{q \in M} L'_q,$$

where $L'_q := \nu^{-1}[L_q]$, $q \in M$. Since $z_0 \notin L_q$, $q \in M$, L'_q are open intervals. Moreover, L'_q are pairwise disjoint. Let $l_{q,t}$ for $q \in M$, $t \in \mathbb{R}$ be the unique strictly increasing linear function such that

$$l_{q,t}\left[L'_q\right] = L'_{T(q,t)}. \tag{16}$$

Define

$$B_t(z) := \left(\nu \circ l_{q,t} \circ \nu^{-1}_{|L_q}\right)(z), \quad z \in L_q,\ t \in \mathbb{R}. \tag{17}$$

Hence, by (16), we have

$$B_t[L_q] = L_{T(q,t)}, \quad q \in M,\ t \in \mathbb{R} \tag{18}$$

and, taking in (18) $q := T(p, -t)$ and using (13),

$$B_t\left[\mathbb{S}^1 \setminus L_F\right] = \mathbb{S}^1 \setminus L_F, \quad t \in \mathbb{R}. \tag{19}$$

We first prove that $B_t : \mathbb{S}^1 \setminus L_F \mapsto \mathbb{S}^1 \setminus L_F$, $t \in \mathbb{R}$ is strictly increasing. Indeed, fix $t \in \mathbb{R}$, $v, w, z \in \mathbb{S}^1 \setminus L_F$ such that $v \prec w \prec z$ (that is, $w \in \overrightarrow{(v, z)}$) and consider three cases.

(i) $\{v, w, z\} \subset L_q$ for a $q \in M$.
Let $v = \nu(t_v)$, $w = \nu(t_w)$, $z = \nu(t_z)$. We can certainly assume that $t_v, t_w, t_z \in (0, 1)$, because $z_0 \notin L_q$. Hence,

$$\nu^{-1}(v) < \nu^{-1}(w) < \nu^{-1}(z),$$

and consequently by the fact that $l_{q,t}$ is strictly increasing and (17) we obtain

$$B_t(v) \prec B_t(w) \prec B_t(z).$$

(ii) $\mathrm{card}(\{v, w, z\} \cap L_q) = 2$ for a $q \in M$.
Using Lemma 2 assume, for example, that $v, w \in L_q$, $z \in L_p$ for a $p \in M$, $p \neq q$. Take $u \in L_q$ such that $w \in \overrightarrow{(v, u)}$. By (i) we get

$$B_t(w) \in \overrightarrow{(B_t(v), B_t(u))}. \tag{20}$$

From (18), $B_t(u), B_t(v), B_t(w) \in L_{T(q,t)}$, $B_t(z) \in L_{T(p,t)}$. Moreover, $T(q, t) \neq T(p, t)$. Hence and by (20) we obtain

$$B_t(w) \in \overrightarrow{(B_t(v), B_t(z))}.$$

(iii) $\mathrm{card}(\{v, w, z\} \cap L_q) \leq 1$ for every $q \in M$.
Suppose that $v \in L_q, w \in L_p, z \in L_r$ for $p, q, r \in M$, $p \neq q$, $q \neq r$, $p \neq r$. Let us note that $L_q \prec L_p \prec L_r$ (that is for every $v \in L_q$, $w \in L_p$, $z \in L_r$ we have $v \prec w \prec z$). Using the monotonicity of φ and Lemma 12 we get $\varphi[L_q] \prec \varphi[L_p] \prec \varphi[L_r]$. Thus by (9) and (10) we have

$$\varphi[L_{T(q,t)}] \prec \varphi[L_{T(p,t)}] \prec \varphi[L_{T(r,t)}]. \tag{21}$$

Suppose that $L_{T(r,t)} \preceq L_{T(p,t)} \preceq L_{T(q,t)}$, that is, by Remark 3, $L_{T(q,t)} \preceq L_{T(r,t)} \preceq L_{T(p,t)}$. Then

$$\varphi[L_{T(q,t)}] \preceq \varphi[L_{T(r,t)}] \preceq \varphi[L_{T(p,t)}],$$

which, by Lemma 3 and Remark 3, contradicts (21). Hence, again by Lemma 3 we have $L_{T(q,t)} \prec L_{T(p,t)} \prec L_{T(r,t)}$.

Our next goal is to extent B_t to the monotonic function P^t of the entire circle $\mathbb{S}^1$. For this purpose, fix $t \in \mathbb{R}$, $w \in L_F$ and choose a sequence $\{w_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^1 \setminus L_F$ such that

$$\overrightarrow{(w_0, w_n)} \subset \overrightarrow{(w_0, w)},\ \overrightarrow{(w_0, w_n)} \subset \overrightarrow{(w_0, w_{n+1})}, \quad \text{for } n \in \mathbb{N} \setminus \{0\}$$

and

$$\bigcup_{n=1}^{\infty} \overrightarrow{(w_0, w_n)} = \overrightarrow{(w_0, w)}.$$

Since $B_t : \mathbb{S}^1 \setminus L_F \mapsto \mathbb{S}^1 \setminus L_F$ is strictly increasing, $\bigcup_{n=1}^{\infty} \overrightarrow{(B_t(w_0), B_t(w_n))}$ is an open arc, say $\overrightarrow{(B_t(w_0), a)}$. It is easily seen that a does not depend on the choice of the sequence $\{w_n\}_{n\in\mathbb{N}}$. Define $P^t(w) := a$ and

$$P^t(z) := B_t(z), \quad z \in \mathbb{S}^1 \setminus L_F,\ t \in \mathbb{R}. \tag{22}$$

We will prove that

$$P^t[L_F] \subset L_F. \tag{23}$$

Suppose, contrary to our claim, that there exists a $w \in L_F$ such that $P^t(w) \in \mathbb{S}^1 \setminus L_F$. Then by (19), $B_t(z) = P^t(w)$ for a $z \in \mathbb{S}^1 \setminus L_F$. Take $v \in \mathbb{S}^1 \setminus L_F$ such that $w \notin \overrightarrow{(v, z)}$ and let $\{w_n\}_{n\in\mathbb{N}}$ be the above-described sequence. Hence

$$m := \operatorname{card}(\overrightarrow{(v, z)} \cap \{w_n\}_{n\in\mathbb{N}}) < \aleph_0$$

and

$$\operatorname{card}\big(\overrightarrow{(B_t(v), B_t(z))} \cap \{B_t(w_n)\}_{n\in\mathbb{N}}\big) = m,$$

since B_t is strictly increasing. This contradicts the fact that

$$\bigcup_{n=1}^{\infty} \overrightarrow{(B_t(w_0), B_t(w_n))} = \overrightarrow{(B_t(w_0), P^t(w))} = \overrightarrow{(B_t(w_0), B_t(z))}.$$

Now we shall show that P^t are strictly increasing. To do this, take $v, w, z \in \mathbb{S}^1$ such that $v \prec z \prec w$ and consider three cases:

(i) $\mathrm{card}(\{v, w, z\} \cap L_F) = 1$.
In view of Lemma 2 we may assume that $w \in L_F$. From (22), Remark 4, (19) and (23) we have

$$(24) \qquad P^t(v) \neq P^t(z),\ P^t(v) \neq P^t(w) \quad \text{and } P^t(z) \neq P^t(w),$$

since $v, z \in \mathbb{S}^1 \setminus L_F$, $w \in L_F$. Suppose that $P^t(w) \prec P^t(z) \prec P^t(v)$, that is by Lemmas 2 and 1, $P^t(w) \in \overrightarrow{(P^t(v), P^t(z))}$. Let $n \in \mathbb{N} \setminus \{0\}$ be such that $z \in \overrightarrow{(v, w_n)}$ and $P^t(w_n) \in \overrightarrow{(P^t(v), P^t(w))}$. Hence, by the fact that $\overrightarrow{(P^t(v), P^t(w))} \subset \overrightarrow{(P^t(v), P^t(z))}$ we get $P^t(w_n) \in \overrightarrow{(P^t(v), P^t(z))}$. On the other hand, $P^t(z) \in \overrightarrow{(P^t(v), P^t(w_n))}$, since $z, v, w_n \in \mathbb{S}^1 \setminus L_F$ and P^t is strictly increasing on $\mathbb{S}^1 \setminus L_F$. But, in view of Lemmas 1 and 2, this is impossible. Thus, using Lemma 3 and (24), we infer that $P^t(v) \prec P^t(z) \prec P^t(w)$.

(ii) $\mathrm{card}(\{v, w, z\} \cap L_F) = 2$.
According to Lemma 2 we can assume that $z, w \in L_F$. Choose $x, y \in \mathbb{S}^1 \setminus L_F$ such that $v \in \overrightarrow{(w, z)} \subset \overrightarrow{(x, y)}$. Hence, $w \in \overrightarrow{(x, v)}, z \in \overrightarrow{(v, y)}$ and, by the monotonicity of P^t on $\mathbb{S}^1 \setminus L_F$, $P^t(v) \in \overrightarrow{(P^t(x), P^t(y))}$. By proved case (i), we obtain $P^t(w) \in \overrightarrow{(P^t(x), P^t(v))}$ and $P^t(z) \in \overrightarrow{(P^t(v), P^t(y))}$. Consequently, $P^t(v) \in \overrightarrow{(P^t(w), P^t(z))}$, which, by Lemmas 1 and 2, is our claim.

(iii) $\{v, w, z\} \subset L_F$.
As in case (ii), choose $x, y \in \mathbb{S}^1 \setminus L_F$ such that $v \in \overrightarrow{(w, z)} \subset \overrightarrow{(x, y)}$. Hence, using proved case (i), $P^t(v) \in \overrightarrow{(P^t(x), P^t(y))}$ and, by (ii), $P^t(w) \in \overrightarrow{(P^t(x), P^t(v))}$ and $P^t(z) \in \overrightarrow{(P^t(v), P^t(y))}$. Therefore $P^t(v) \in \overrightarrow{(P^t(w), P^t(z))}$, which, by Lemmas 1 and 2, is our assertion.

Thus, we have shown that P^t is strictly increasing.

Note that by (22) and (18) we get (15). Moreover, (22) and (19) give

$$\mathbb{S}^1 \setminus L_F = P^t\left[\mathbb{S}^1 \setminus L_F\right] \subset P^t\left[\mathbb{S}^1\right], \quad t \in \mathbb{R}.$$

Since L_F is a perfect nowhere dense subset of $\mathbb{S}^1$, the set $P^t[\mathbb{S}^1]$ is dense in $\mathbb{S}^1$ and Lemma 4 shows that every P^t is continuous.

Fix now $q \in M$, $t \in \mathbb{R}$. Let $L_q = \overrightarrow{(v,z)}$, where $v = \tilde{\pi}(t_v)$, $z = \tilde{\pi}(t_z)$, $0 < t_z - t_v < 1$ and let $l_{q,t}(x) = ax + b$, $x \in L'_q$, where $a > 0$. Using (22) and (17) we have

$$P^t\left(\nu(x)\right) = \nu(l_{q,t}(x)), \quad x \in L'_q.$$

Putting $y := x + x_0$, we have

$$P^t(e^{2\pi i y}) = e^{2\pi i(ay - ax_0 + b + x_0)}, \quad y \in (t_v,\ t_z),$$

and (14) is proved.

Suppose now that P^t, R^t are continuous functions fulfilling (14) and (15). Then $P^t(e^{2\pi i x}) = e^{2\pi i p_{q,t}(x)}$ and $R^t(e^{2\pi i x}) = e^{2\pi i r_{q,t}(x)}$, where $p_{q,t}, r_{q,t} : (t_v, t_z) \mapsto \mathbb{R}$ are strictly increasing linear functions. Hence, by (15), $p_{q,t} = r_{q,t} + k$ for a $k \in \mathbb{Z}$ and consequently $P^t_{|L_q} = R^t_{|L_q}$. As P^t, R^t are continuous functions and $\mathbb{S}^1 \setminus L_F$ is a dense subset of $\mathbb{S}^1$ we have $P^t = R^t$.

Let us note that

$$l_{T(q,s),t} \circ l_{q,s} = l_{q,t+s}, \quad q \in M,\ s,t \in \mathbb{R}. \tag{25}$$

In fact, by (16) and (13),

$$\left(l_{T(q,s),t} \circ l_{q,s}\right)\left[L'_q\right] = l_{q,t+s}\left[L'_q\right], \quad q \in M,\ s,t \in \mathbb{R}$$

and (25) follows, since $l_{T(q,s),t} \circ l_{q,s}$ and $l_{q,t+s}$ are strictly increasing linear functions. Fix $q \in M$, $z \in L_q$, $s,t \in \mathbb{R}$. Using (22), (17), the fact that $P^s(z) \in L_{T(q,s)}$ and (25) we obtain

$$\begin{aligned}(P^t \circ P^s)(z) &= \left(B_t \circ (\nu \circ l_{q,s} \circ \nu^{-1})\right)(z) \\ &= \left(\nu \circ l_{T(q,s),t} \circ l_{q,s} \circ \nu^{-1}\right)(z) = \left(\nu \circ l_{q,s+t} \circ \nu^{-1}\right)(z) = P^{s+t}(z).\end{aligned}$$

Hence,

$$P^t \circ P^s = P^{s+t}, \quad s,t \in \mathbb{R}, \tag{26}$$

since every P^t is continuous and $\mathbb{S}^1 \setminus L_F$ is dense in $\mathbb{S}^1$.

Let φ be a continuous solution of (3) and $z \in \mathbb{S}^1 \setminus L_F$. Then $z \in L_q$ for a $q \in M$ and, by (15), $P^t(z) \in L_{T(q,t)}$. Hence, by Lemma 12, (9) and (10), we get

$$\varphi(z) = \varphi[L_q] = \Phi(q)$$

and

$$\varphi\left(P^t(z)\right) = \varphi\left[L_{T(q,t)}\right] = \Phi(T(q,t)) = \Phi(q)c(t).$$

Consequently, by the continuity of φ, P^t and the density of $\mathbb{S}^1 \setminus L_F$ in $\mathbb{S}^1$,

$$\varphi\left(P^t(z)\right) = \varphi(z)c(t), \quad z \in \mathbb{S}^1,\ t \in \mathbb{R}. \tag{27}$$

The proof is completed by showing that the family $\{P^t, t \in \mathbb{R}\}$ is a disjoint iteration group. To do this suppose that $P^t(v_0) = v_0$ for a $v_0 \in \mathbb{S}^1$ and a $t \in \mathbb{R}$. Then from (27), $c(t) = 1$ and by (10), $T(q,t) = q$, $q \in M$. Consequently, by (15), $P^t[L_q] = L_q$, $q \in M$. Hence, by the linearity of P^t on L_q, $P^t(z) = z$ for $z \in \mathbb{S}^1 \setminus L_F$ and further by the continuity of $P^t, P^t(z) \equiv z$ for $z \in \mathbb{S}^1$. Finally, P^t are homeomorphisms, since $\mathbb{S}^1 = P^t[P^{-t}[\mathbb{S}^1]] \subset P^t[\mathbb{S}^1] \subset \mathbb{S}^1$, $(P^t)^{-1} = P^{-t}$ and P^t are continuous for $t \in \mathbb{R}$. □

For every sequence $\{a_n\}_{n \in \mathbb{N}\setminus\{0\}}$ such that

$$a_n \in \{0, \dots, n-1\},\ a_{kn} = a_n \pmod{n}, \quad k, n \in \mathbb{N} \setminus \{0\} \tag{28}$$

define

$$A_s(\{a_k\}) := \left\{\tilde{\pi}\left(\frac{m}{n}s + \frac{m}{n}a_n\right),\ n \in \mathbb{N} \setminus \{0\},\ m \in \mathbb{Z}\right\}, \quad s \in \mathbb{R}.$$

With the notation $\mathbb{N}! := \{n!,\ n \in \mathbb{N}\}$, we have

Lemma 14. *The general form of all sequences $\{a_n\}_{n \in \mathbb{N}\setminus\{0\}}$ fulfilling* (28) *is given by taking*

$$a_1 := 0,$$

choosing inductively

$$a_{n!} := a_{(n-1)!} + (n-1)!k_n,\ k_n \in \{0, \dots, n-1\} \quad \text{for } n > 1$$

and defining

$$a_n \text{ as the rest from the division } a_{n!} \text{ by } n,\ \text{for } n \notin \mathbb{N}!\,.$$

The simple proof is omitted.

Lemma 15. *For every* $s \in \mathbb{R}$ *and every sequence* $\{a_n\}_{n \in \mathbb{N} \setminus \{0\}}$ *satisfying* (28) *there exists a function* $c : \mathbb{R} \mapsto \mathbb{S}^1$ *such that condition* (6) *holds,* $c(1) = \tilde{\pi}(s)$ *and*

$$\operatorname{Im} c = A_s(\{a_k\}).$$

Proof. Let $\{a_n\}_{n \in \mathbb{N} \setminus \{0\}}$ be an arbitrary sequence fulfilling (28). Denote by H a Hamel basis such that $1 \in H$. Then every real number t can be expressed in the form

$$t = \sum_{h \in H} r_h(t) h,$$

where $r_h(t) \in \mathbb{Q}$ and almost every $r_h(t) = 0$. It is evident that the function $r_1 : \mathbb{R} \mapsto \mathbb{Q}$ is additive and $r_1(1) = 1$.

Put

$$b\left(\frac{m}{n}\right) := \tilde{\pi}\left(\frac{m}{n} s + \frac{m}{n} a_n\right), \quad n \in \mathbb{N} \setminus \{0\},\ m \in \mathbb{Z}.$$

Using (28) it is easy to check that

$$b(x + y) = b(x) b(y), \quad x,\ y \in \mathbb{Q}.$$

A trivial verification shows that the function $c : \mathbb{R} \mapsto \mathbb{S}^1$ given by

$$c(t) := b(r_1(t)), \quad t \in \mathbb{R}$$

is such that $\operatorname{Im} c = A_s(\{a_k\})$, $c(1) = \tilde{\pi}(s)$ and condition (6) holds. □

Let us recall that for every homeomorphism F such that $\alpha(F) \notin \mathbb{Q}$ and $L_F \neq \mathbb{S}^1$ we have defined the set $K_F = \varphi[\mathbb{S}^1 \setminus L_F]$, where $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is a continuous solution of (3). Since φ is unique up to a multiplicative constant, so is K_F.

Theorem 3. *Let* $\alpha(F) \notin \mathbb{Q}$ *and* $L_F \neq \mathbb{S}^1$*. Then the homeomorphism* F *is embeddable in a disjoint iteration group if and only if there exists a sequence* $\{a_n\}_{n \in \mathbb{N} \setminus \{0\}}$ *satisfying* (28) *and such that*

$$K_F \cdot A_{\alpha(F)}(\{a_k\}) = K_F. \tag{29}$$

Proof. *Necessary condition.* Let F be embeddable in a disjoint iteration group $\{F^t,\ t \in \mathbb{R}\}$. Lemmas 8 and 9 show that K_F has property (C). By (6) we get

$$c(mt) = c(t)^m, \quad m \in \mathbb{Z},\ t \in \mathbb{R}. \tag{30}$$

Let $n \in \mathbb{N} \setminus \{0\}$. Using (30) we have $c(1) = c(\frac{1}{n})^n$, hence by (8) there exists an $a_n \in \{0, \ldots, n-1\}$ such that $c(\frac{1}{n}) = \tilde{\pi}(\frac{\alpha(F)}{n} + \frac{a_n}{n})$. Since $c(\frac{1}{n}) = c(\frac{1}{kn})^k = \tilde{\pi}(\frac{\alpha(F)}{n} + \frac{a_{kn}}{n})$, we see that $a_{kn} = a_n \pmod{n}$. Thus, the sequence $\{a_n\}_{n \in \mathbb{N}\setminus\{0\}}$ satisfies (28). Using again (30) we obtain $c(\mathbb{Q}) = A_{\alpha(F)}(\{a_k\})$. Hence, by the fact that $1 \in A_{\alpha(F)}(\{a_k\})$ and (12),

$$K_F \subset K_F \cdot A_{\alpha(F)}(\{a_k\}) \subset K_F \cdot \operatorname{Im} c = K_F,$$

which gives our assertion.

Sufficient condition. We shall construct a disjoint iteration group $\{F^t,\ t \in \mathbb{R}\}$ such that $F^1 = F$. To do this we give

The general construction of all disjoint iteration groups $\{F^t,\ t \in \mathbb{R}\}$ *such that* $F^1 = F$.

1° Let $c : \mathbb{R} \mapsto \mathbb{S}^1$ be a function such that $\operatorname{Im} c$ is a countable set and conditions (6), (8), (12) hold. It follows from our assumption that every function $c : \mathbb{R} \mapsto \mathbb{S}^1$ fulfilling (6), (8) and such that $\operatorname{Im} c = A_{\alpha(F)}(\{a_k\})$ has this property. Let us note that Lemma 15 ensures the existence of such a function c.

2° Define the function $\Phi : M \mapsto K_F$ by (9), where $\varphi : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is a continuous increasing solution of (3) and $L_q, q \in M$ are open pairwise disjoint arcs such that (2) holds. The function $T : M \times \mathbb{R} \mapsto M$ given by (10) satisfies the translation equation (13).

3° Let $\{P^t, t \in \mathbb{R}\}$ be an iteration group described by Lemma 13. We can apply this lemma because K_F has property (C), which follows from 1°.

4° Define the following relation on M

$$p \sim q \Leftrightarrow \exists n \in \mathbb{Z} \quad q = T(p, n).$$

It is easy to check that "$\sim$" is an equivalence relation. Let A be an arbitrary set which has exactly one point in common with every equivalence class. Put

$$\bar{p} := [p] \cap A, \quad p \in M.$$

Fix $n, m \in \mathbb{Z}$, $p \in M$. Let us note that

$$\text{if } T(p,n) = T(p,m), \quad \text{then } n = m. \tag{31}$$

Indeed, let $T(p,n) = T(p,m)$. Then we conclude from (10) that $c(n) = c(m)$, hence using (6) and (8), that $\pi(\alpha(F))^n = \pi(\alpha(F))^m$, and finally, that $n = m$, since $\alpha(F) \notin \mathbb{Q}$. By (31), for every $p \in M$ there exists exactly one $n_p \in \mathbb{Z}$ such that

$$\bar{p} = T(p, n_p). \tag{32}$$

5° Let $\Gamma_q : L_q \mapsto L_q$ for $q \in A$ be arbitrary orientation-preserving homeomorphisms. Define

$$\Gamma(z) := \begin{cases} (F^{-n_p} \circ \Gamma_{\bar{p}} \circ P^{n_p})(z), & z \in L_p, \\ z, & z \in L_F. \end{cases} \tag{33}$$

We will show that

$$F^n[L_p] = L_{T(p,n)}, \quad n \in \mathbb{Z},\ p \in M. \tag{34}$$

Fix $p \in M$. According to Lemma 12 there exists a $q \in M$ such that $F[L_p] = L_q$. Consequently by (9), (3) and (8) we have

$$\Phi(q) = \varphi[L_q] = \varphi[F[L_p]] = \pi(\alpha(F))\varphi[L_p] = c(1)\Phi(p)$$

and (10) now yields $F[L_p] = L_{T(p,1)}$. Hence by induction and (13) we get (34). Let us observe that by (15), (32), (34) and (13),

$$\Gamma[L_p] = L_p \quad \text{for every } p \in M. \tag{35}$$

$P^{n_p}_{|L_p}$ is an orientation-preserving homeomorphism, so Lemma 5 makes it obvious that for every $p \in M$, $\Gamma_{|L_p}$ is an orientation-preserving homeomorphism.

We shall prove that Γ is strictly increasing. To do this, take $v, w, z \in \mathbb{S}^1$ such that $w \in \overrightarrow{(v, z)}$ and consider the following cases

(i) $\{v, w, z\} \subset L_q$ for a $q \in M$.
If we put $t_v := \pi^{-1}(v)$, $t_w := \pi^{-1}(w)$, $t_z := \pi^{-1}(z)$, then $t_v < t_w < t_z$ or $t_w < t_z < t_v$ or $t_z < t_v < t_w$. Assume, for instance, that $t_v < t_w < t_z$ and let f_q represent $\Gamma_{|L_q}$. Since $\Gamma_{|L_q}$ preserves orientation, $f_q(t_v) < f_q(t_w) < f_q(t_z)$. Moreover, by (35), $f_q(t_v), f_q(t_w), f_q(t_z) \in (t_1, t_2)$ for some t_1, t_2 such that $0 < t_2 - t_1 < 1$. This clearly forces $\tilde{\pi}(f_q(t_w)) \in \overrightarrow{(\tilde{\pi}(f_q(t_v)), \tilde{\pi}(f_q(t_z)))}$ and, in consequence, $\Gamma(w) \in \overrightarrow{(\Gamma(v), \Gamma(z))}$.

(ii) $\operatorname{card}(\{v, w, z\} \cap L_q) = 2$ for a $q \in M$.
By Lemmas 1 and 2 we can assume that $v, w \in L_q$. Choose $u \in L_q$

such that $w \in \overrightarrow{(v,u)}$. By (i), (35), (33), $\Gamma(w) \in \overrightarrow{(\Gamma(v),\Gamma(u))} \subset L_q$ and $\Gamma(z) \notin L_q$, which gives $\Gamma(w) \in \overrightarrow{(\Gamma(v),\Gamma(z))}$.

(iii) $\operatorname{card}(\{v,w,z\} \cap L_q) = \operatorname{card}(\{v,w,z\} \cap L_p) = 1$ for some $p,q \in M$, $p \neq q$.

In view of Lemmas 1 and 2 we may assume that $v \in L_q$, $w \in L_p$ and $v \prec w \prec z$. This clearly gives $L_q \prec L_p \prec L$, where $L := L_r$ if $z \in L_r$ for an $r \in M$ and $L := \{z\}$ if $z \in L_F$. Hence, using (35) and (33), we get $\Gamma[L_q] \prec \Gamma[L_p] \prec \Gamma[L]$ and consequently $\Gamma(v) \prec \Gamma(w) \prec \Gamma(z)$, which is our claim.

(iv) $\operatorname{card}(\{v,w,z\} \cap L_F) = 2$.

Using again Lemmas 1 and 2 suppose that $v,z \in L_F$, $w \in L_q$ for a $q \in M$. As L_q is an open arc, $L_q \subset \overrightarrow{(v,z)}$. Hence, in view of (35) and (33), we have $\Gamma[L_q] = L_q \subset \overrightarrow{(v,z)} = \overrightarrow{(\Gamma(v),\Gamma(z))}$. From this we see that $\Gamma(w) \in \overrightarrow{(\Gamma(v),\Gamma(z))}$.

(v) $\{v,w,z\} \subset L_F$.

It is clear, since $\Gamma(v) = v$, $\Gamma(w) = w$ and $\Gamma(z) = z$.

Let us note that $\Gamma : \mathbb{S}^1 \mapsto \mathbb{S}^1$ is a bijection. Hence and by Lemma 4 we conclude that Γ is a homeomorphism. Clearly,

$$\overline{T(p,1)} = \bar{p}, \quad p \in M. \tag{36}$$

Hence by (32) and (13) we have

$$T(p,n_p) = \bar{p} = \overline{T(p,1)} = T\left(T(p,1),\ n_{T(p,1)}\right) = T\left(p, 1 + n_{T(p,1)}\right)$$

and, using (31),

$$n_p = 1 + n_{T(p,1)}. \tag{37}$$

Fix $p \in M$, $z \in L_p$. Then by (15), $P^1(z) \in L_{T(p,1)}$, so from (33), (36) and (37) we obtain

$$\begin{aligned}\left(\Gamma \circ P^1\right)(z) &= \left(F^{-n_{T(p,1)}} \circ \Gamma_{\overline{T(p,1)}} \circ P^{n_{T(p,1)}+1}\right)(z) \\ &= \left(F \circ F^{-n_p} \circ \Gamma_{\bar{p}} \circ P^{n_p}\right)(z) = (F \circ \Gamma)(z).\end{aligned}$$

As F, P^1 and Γ are continuous and $\mathbb{S}^1 \setminus L_F$ is a dense subset of $\mathbb{S}^1$ we see that

$$F = \Gamma \circ P^1 \circ \Gamma^{-1}.$$

6° Define

$$F^t := \Gamma \circ P^t \circ \Gamma^{-1}, \quad t \in \mathbb{R}. \tag{38}$$

A trivial verification shows that $\{F^t,\ t \in \mathbb{R}\}$ is a disjoint iteration group such that $F^1 = F$.

Note that (38) defines the general form of all disjoint iteration groups $\{F^t,\ t \in \mathbb{R}\}$ such that $F^1 = F$. Indeed, suppose that $\{F^t,\ t \in \mathbb{R}\}$ is such an iteration group. According to Lemmas 11 and 10 there exists an orientation-preserving homeomorphism $\Gamma : \mathbb{S}^1 \mapsto \mathbb{S}^1$ fulfilling (38) and such that $\Gamma(z) = z$ for $z \in L_F$, where $\{P^t,\ t \in \mathbb{R}\}$ is an iteration group described by Lemma 13. If we put $\Gamma_q := \Gamma_{|L_q}$ for $q \in M$, then by (38), (15) and (32) we obtain (33). Moreover, it is clear that Γ_q are orientation-preserving homeomorphisms. Thus our assertion is proved. □

Finally, we get

Theorem 4. *Let $F : \mathbb{S}^1 \mapsto \mathbb{S}^1$ be an orientation-preserving homeomorphism with an irrational rotation number. Then F is embeddable in an iteration group if and only if F is embeddable in a disjoint iteration group.*

Proof. If $L_F = \mathbb{S}^1$, then by Theorem 2 we get our assertion. Let $L_F \neq \mathbb{S}^1$ and $\{F^t, t \in \mathbb{R}\}$ be an iteration group such that $F^1 = F$. By Lemmas 8 and 9 the set K_F has property (C). Further, similarly as in the first part of the proof of Theorem 3, we show that there exists a sequence $\{a_n\}_{n\in\mathbb{N}\setminus\{0\}}$ fulfilling (28) such that (29) holds. Theorem 3 completes the proof. □

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