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A general stability result in the class of Lipschitz functions

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Abstract. We prove that a large class of functional equations in the class of functions with bounded Lipschitz constant is stable in the Hyers–Ulam sense. However, we do not obtain uniqueness and bounds on approximation as the method we use is not constructive.

As a direct corollary we obtain stability of the isometry equation on compact metric spaces and a stability-type result of the Hosszú functional equation on the interval $[0,1]$.

1. Introduction

It is well known (cf. [2]) that every function $h:\mathbb{R}\to\mathbb{R}$ satisfying the Hosszú's equation, that is the equation

$$h(x+y-xy)+h(xy)-h(x)-h(y)=0 \quad \text{for } x,y\in\mathbb{R}$$

is a sum of an additive function and a constant.

Since the interval $[0,1]$ is closed under operations operations $(x,y)\to xy$, $(x,y)\to x+y-xy$ one can study the Hosszú's equation on $[0,1]$. K. Lajkó proved in [3] that if a function h satisfies the Hosszú's equation on the unit interval then h is a sum of an additive function and a constant on $(0,1)$.

After characterisation of the solutions of the Hosszú functional equation was given there naturally appeared the question of its stability. L. Losonczi proved in [4] that the Hosszú's equation on $\mathbb{R}$ is stable in the

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Hyers–Ulam sense, i.e. he obtained the following:

Theorem L. *If X is a Banach space and $f : \mathbb{R} \to X$ satisfies the functional inequality*

$$\|f(x+y-xy)+f(xy)-f(x)-f(y)\| \le \varepsilon \quad \text{for } x,y \in \mathbb{R}$$

with an $\varepsilon \ge 0$, then there exists a unique function $h : \mathbb{R} \to X$ satisfying the Hosszú's equation such that

$$\|f(x)-h(x)\| \le 20\varepsilon \quad \text{for } x \in \mathbb{R}.$$

We mention that the constant 20 appearing in the above Theorem was improved to 4 by P. Volkmann
(www.mathematik.uni-karlsruhe.de/∼semlv).

What is surprising, the Hosszú functional equation occured not to be stable on the interval $[0,1]$ (cf. [6]):

Theorem T. *Let $\varepsilon > 0$ be arbitrary. Then for every $\delta > 0$ there exists a continuous bounded function $h_\delta : [0,1] \to \mathbb{R}$ which satisfies the inequality*

$$|h_\delta(x+y-xy)+h_\delta(xy)-h_\delta(x)-h_d(y)| \le \delta \quad \text{for } x,y \in [0,1],$$

but such that for every solution $H : [0,1] \to \mathbb{R}$ of the Hosszú's functional equation

$$\sup_{x \in X} |h_\delta(x)-H(x)| \ge \varepsilon.$$

By examining the construction of the functions h_δ one can easily notice that with δ decreasing to zero the Lipschitz constant of the function h_δ increases to ∞.

The main motivation behind the present paper was to check whether this behaviour is really essential. In other words, can the functions h_δ in Theorem T be constructed in such a way that their Lipschitz constants are uniformly bounded from above?

We show in Corollary 3, that the answer is negative. What is even more interesting, Theorem 1 and Theorem 2 prove that analogous situation (that is Hyers–Ulam stability in the class of Lipschitz functions with Lipschitz constant bounded from above by a given real number) appears for quite a large family of functional equations. In particular as a corollary we obtain the stability of isometry equation and of Lipschitz functions on the compact sets.

2. Stability results

The following definition formalizes the notion of approximately double Lipschitz functions.

Definition 1. Let $\delta \geq 0$ and let $K, L \in \mathbb{R}_+$. Let X, Y be metric spaces and let $f : X \to Y$. We write that $f \in \mathrm{Lip}(\delta; K, L)$ iff

$$Kd(x,y) - \delta \leq d(f(x), f(y)) \leq Ld(x,y) + \delta \quad \text{for } x, y \in X.$$

One can easily notice that a 0-Lipschitz function is simply a Lipschitz function.

If X is a set, and Y a metric space then for functions $f, g : X \to Y$ we define the distance

$$d_{\sup}(f,g) := \sup_{x \in X} d(f(x), g(x)).$$

Notice that $d_{\sup(f,g)}$ can equal $+\infty$.

Proposition 1. *Let X, Y be a compact metric spaces. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence such that*

$$\lim_{n\to\infty} \delta_n = 0.$$

Let $K, L \in \mathbb{R}$ and let $f_n : X \to Y$ be a sequence of functions such that $f_n \in \mathrm{Lip}(\delta_n; K, L)$. Then there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of integers and a function $F : X \to Y$, $F \in \mathrm{Lip}(0; K, L)$ such that

$$\lim_{k\to\infty} d_{\sup}(f_{n_k}, F) = 0. \tag{1}$$

Proof. Let D be a dense countable subset of X. As Y is compact, applying the Cantor's diagonal procedure we can find an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of integers such that $\{f_{n_k}(d)\}_{k\in\mathbb{N}}$ is convergent for every $d \in D$. We define

$$F_D(d) := \lim_{k\to\infty} f_{n_k}(d) \quad \text{for } d \in D.$$

We show that $F_D : D \to Y$ is Lipschitz with constant L. Let $c, d \in D$ be arbitrary. Then

$$\begin{aligned} d(F_D(c), F_D(d)) &= \lim_{k\to\infty} d(f_{n_k}(c), f_{n_k}(d)) \\ &\le \lim_{k\to\infty} (Ld(c,d) + \delta_{n_k}) = Ld(c,d). \end{aligned}$$

As D is dense in X and F_D is Lipschitz with constant L there exists a unique continuous function $F : X \to Y$ such that

$$F|_D = F_D.$$

Moreover, F is Lipschitz with constant L.

Now we are going to prove that $F \in \mathrm{Lip}(0; K, L)$. Let $x, y \in X$ be arbitrary and let $\delta > 0$. Then there exist $x_\delta, y_\delta \in D$ such that $d(x, x_\delta) + d(y, y_\delta) \le \delta$. Then

$$\begin{aligned} d(F(x), F(y)) &\ge d(F(x_\delta), F(y_\delta)) - d(F(x_\delta), F(x)) - d(F(y_\delta), F(y)) \\ &\ge \liminf_{k\to\infty} d(f_{n_k}(x_\delta), f_{n_k}(y_\delta)) - L\delta \\ &\ge \liminf_{k\to\infty} (Kd(x_\delta, y_\delta) - \delta_{n_k}) - L\delta \ge Kd(x_\delta, y_\delta) - L\delta \\ &\ge Kd(x,y) - K(d(x, x_\delta) + d(y, y_\delta)) - L\delta \\ &\ge Kd(x,y) - (K+L)\delta. \end{aligned}$$

As $\delta > 0$ was arbitrary and F is Lipschitz with constant L it proves that $F \in \mathrm{Lip}(0; K, L)$.

Now we will show that F satisfies (1). Let $\delta > 0$ be arbitrary. We can find a finite subset D_δ of D which is δ net in X. As D_δ is finite there exists $k_0 \in \mathbb{N}$ such that

$$d(f_{n_k}(d), F(d)) \le \delta \quad \text{for } k \ge k_0,\ d \in D_\delta.$$

Obviously there exists $k_1 \in \mathbb{N}$ such that $\delta_{n_k} \le \delta$ for $k \ge k_1$. Then $f_{n_k} \in \mathrm{Lip}(\delta; 0, L)$ for $k \ge k_1$.

Let $x \in X$ and let $k \ge \max\{k_0, k_1\}$ be arbitrarily chosen. As D_δ is a δ-net there exists $d_x \in D_\delta$ such that

$$d(x, d_x) \le \delta.$$

Then

$$d(f_{n_k}(x),F(x)) \le d(f_{n_k}(x),f_{n_k}(d_x)){+}d(f_{n_k}(d_x),F(d_x)){+}d(F(d_x),F(x))$$
$$\le (Ld(x,d_x)+\delta)+\delta+Ld(x,d_x) \le (2L+2)\delta.$$

As δ was chosen arbitrarily we obtain (1). $\square$

By $S(X,Y)$ we denote the topological space of all functions from X to Y with the topology generated by $d_{\sup}$, that is $f_n \to f$ in $S(X,Y)$ iff $d_{\sup}(f_n,f) \to 0$.

Theorem 1. *Let X,Y be compact metric spaces and let S be a closed subspace of $S(X,Y)$. Let $\mathcal{F} : S \to [0,\infty]$ be a continuous function.*

Let $K,L \in \mathbb{R}_+$ be arbitrarily fixed. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every function $h \in S$, $h \in \mathrm{Lip}(\delta;K,L)$ satisfying

$$\mathcal{F}(h) \le \delta \tag{2}$$

there exists a function $H \in S$, $H \in \mathrm{Lip}(0;K,L)$ such that

$$\mathcal{F}(H) = 0$$

and

$$d_{\sup}(h,H) \le \varepsilon. \tag{3}$$

Proof. Suppose, for contradiction, that the assertion of Theorem 1 does not hold. Then there exist $\varepsilon > 0$, a sequence of functions $\{h_n\}_{n\in\mathbb{N}} \subset S$ and a sequence $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\lim\limits_{n\to\infty} \delta_n = 0$ such that $h_n \in \mathrm{Lip}(\delta_n;K,L)$,

$$\mathcal{F}(h_n) \le \delta_n$$

but

$$d_{\sup}(h_n,H) \ge \varepsilon \tag{5}$$

for every $n \in \mathbb{N}$ and every function $H \in S$, $H \in \mathrm{Lip}(0;K,L)$ satisfying $\mathcal{F}(H) = 0$.

We can apply Proposition 1 to the sequence h_n and conclude that there exists a function $H : X \to Y$, $H \in \mathrm{Lip}(0; K, L)$ and an increasing sequence $\{n_k\}$ of integers such that

$$\lim_{k\to\infty} d_{\sup}(H, h_{n_k}) = 0. \tag{6}$$

Since S is closed we obtain that $H \in S$. Moreover, by the continuity of $\mathcal{F}$ and (4) we get

$$\mathcal{F}(H) := \lim_{k\to\infty} \mathcal{F}(h_{n_k}) = 0.$$

Now, if k is large enough then

$$d_{\sup}(h_{n_k}, H) < \varepsilon \qquad (\text{and } \mathcal{F}(h_{n_k}) \le \delta_{n_k})$$

which contradicts to (5), proving Theorem 1. □

Theorem 2. *Let X be a compact metric space, let Y be a finite dimensional normed space. Let S be a closed subspace of $S(X, Y)$ and let $\mathcal{F} : S \to \mathbb{R}$ be a continuous function. We additionally assume that for every constant function $c : X \to Y$*

$$h + c \in S \text{ and } \mathcal{F}(h + c) = \mathcal{F}(h) \quad \text{for } h \in S. \tag{7}$$

Let $K, L \in \mathbb{R}_+$ be arbitrary. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in S$, $h \in \mathrm{Lip}(\delta; K, L)$ satisfying

$$\mathcal{F}(h) \le \delta \tag{8}$$

there exists a function $H \in S$, $H \in \mathrm{Lip}(0; K, L)$ such that

$$\mathcal{F}(H) = 0$$

and

$$d_{\sup}(h, H) \le \varepsilon. \tag{9}$$

Proof. Let $\varepsilon > 0$ be arbitrarily fixed and put $\tilde{S} := S \cap S(X, B(0, L\,\mathrm{diam}(X) + 1))$, where $B(0, r)$ denotes the closed ball with the center 0 and radius r. As Y is a finite dimensional normed vector space, $B(0, L\,\mathrm{diam}(X) + 1)$ is a compact set, and therefore by Theorem 1 there exists $\tilde{\delta} > 0$ such that for every function $\tilde{h} \in \tilde{S}$, $\tilde{h} \in \mathrm{Lip}(\tilde{\delta}; K, L)$ satisfying

$$\mathcal{F}(\tilde{h}) \leq \tilde{\delta} \tag{10}$$

there exists a function $\tilde{H} \in \tilde{S}$, $\tilde{H} \in \mathrm{Lip}(0; K, L)$ such that

$$\mathcal{F}(\tilde{H}) = 0 \tag{11}$$

and

$$d_{\sup}(\tilde{h}, \tilde{H}) \leq \varepsilon. \tag{12}$$

We put $\delta := \min\{1, \tilde{\delta}\}$. We show that δ satisfies the assertion of our Theorem. Let $x_0 \in X$ be fixed and let $h \in S$, $h \in \mathrm{Lip}(\delta; K, L)$ be arbitrary function such that

$$\mathcal{F}(h) \leq \delta.$$

Let $\tilde{h} := h - h(x_0)$. Then clearly by the assumptions $\tilde{h} \in \tilde{S} \subset S$ and

$$\mathcal{F}(\tilde{h}) = \mathcal{F}(h - h(x_0)) = \mathcal{F}(h) \leq \delta \leq \tilde{\delta}. \tag{13}$$

Obviously $\tilde{h} \in \mathrm{Lip}(\delta, K, L)$ and therefore for $x \in X$

$$\|\tilde{h}(x)\| = \|h(x) - h(x_0)\| \leq Ld(x, x_0) + \delta \leq L \operatorname{diam}(X) + 1$$

This implies that $\tilde{h} : X \to B(0, L \operatorname{diam}(X) + 1)$, and hence $\tilde{h} \in \tilde{S}$. Now by the first part of the proof we obtain that there exists $\tilde{H} \in \tilde{S}$, $\tilde{H} \in \mathrm{Lip}(0; K, L)$ such that $\tilde{H}$ satisfies (11) and (12). We put $H := \tilde{H} + h(x_0)$. Then one can easily check that H satisfies the assertions of the theorem. $\square$

3. Applications

At first we will show corollaries of Theorems 1 and 2 dealing with the stability of the Lipschitz and isometric functions.

Corollary 1. *Let $L \in \mathbb{R}_+$. Let X be a compact metric space and let Y be either a compact metric space or a finite dimensional normed vector space.*

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f : X \to Y$, $f \in \mathrm{Lip}(\delta; 0, L)$ there exists a Lipschitz function $F : X \to Y$ with constant L such that

$$d(f(x), F(x)) \leq \varepsilon \quad \textit{for } x \in X.$$

Proof. We put $K = 0$, $L = L$ and define the function $\mathcal{F} : S(X,Y) \to [0, \infty]$ by the formula

$$\mathcal{F}(h) = 0 \quad \text{for } h \in S(X,Y).$$

Theorems 1 and 2 make the proof complete. □

Before the next corollary we first have to introduce the definition of approximately isometric functions (see [1]):

Definition 2. Let X, Y be metric spaces and let $\varepsilon \geq 0$. We say that $f : X \to Y$ is an ε-isometry if

$$d(x,y) - \varepsilon \leq d(f(x), f(y)) \leq d(x,y) + \varepsilon \quad \text{for } x, y \in X.$$

For some recent results on ε-isometries we refer the reader to [5]. We would like to mention that without the surjectivity assumption the isometry equation is in general not stable in the Hyers–Ulam sense. In the following corollary we show that when the domain space is compact it is not the case.

Corollary 2. *Let X be a compact metric space and let Y be either a compact metric space or a finite dimensional normed vector space.*

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-isometric function $i : X \to Y$ there exists an isometry $I : X \to Y$ such that

$$d(i(x), I(x)) \leq \varepsilon \quad \textit{for } x \in X.$$

Proof. We put $K = L = 1$ and define the function $\mathcal{F}$ as in the previous corollary. One can easily notice that a function f is δ-isometry iff $f \in \mathrm{Lip}(\delta; 1, 1)$. Theorem 1 and 2 make the proof complete. □

As our results originated from the investigation of the Hosszú functional equation we will now show a corollary dealing with its stability.

Corollary 3. *Let $L \in \mathbb{R}_+$ be arbitrarily fixed and let Y be a finite dimensional normed vector space.*

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every function $h : [0,1] \to Y$, $h \in \mathrm{Lip}(\delta; 0, L)$ satisfying

$$\|h(x+y-xy) + h(xy) - h(x) - h(y)\| \leq \delta \quad \text{for } x, y \in [0,1],$$

there exists a Lipschitz with constant L function $H : [0,1] \to \mathbb{R}^n$ satisfying the Hosszú functional equation and such that

$$\|h - H\|_{\sup} \leq \varepsilon.$$

Proof. We show that Theorem 2 can be applied.

We put $X = [0,1]$, and define the function $\mathcal{F} : S(X,Y) \to [0,\infty]$ by the formula

$$\mathcal{F}(h) := \sup_{x,y\in[0,1]} \|h(x+y-xy) + h(xy) - h(x) - h(x)\|$$

for $h \in S(X,Y)$. Obviously a function h satisfies the Hosszú equation iff $\mathcal{F}(h) = 0$.

One can easily notice that $\mathcal{F}$ is a continuous function such that for every constant function $c : X \to Y$

$$\mathcal{F}(h + c) = \mathcal{F}(h) \quad \text{for } h \in S(X,Y).$$

Thus we have shown that all the assumptions of Theorem 2 are satisfied. Now the assertion of the Corollary follows trivially from Theorem 2. □

The following corollary deals with the stability of the Jensen functional equation, however we omit the proof as it is analogous to the previous one.

Corollary 4. *Let $L \in \mathbb{R}_+$ be arbitrary and let K be a compact convex set in a Banach space.*

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $j : K \to \mathbb{R}^n$, $j \in \mathrm{Lip}(\delta; 0, L)$ satisfying

$$\left\| j\left(\frac{x+y}{2}\right) - \frac{j(x)+j(y)}{2} \right\| \leq d \quad \text{for } x, y \in K,$$

there exists a Jensen function $J : K \to \mathbb{R}^n$ such that

$$\|j(x) - J(x)\| \leq \varepsilon \quad \text{for } x \in K.$$

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