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On space-time manifolds carrying two exterior concurrent skew symmetric killing vector fields

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Abstract. We analyse the structural properties, from a geometrical point of view, of space-time manifolds carrying two exterior concurrent skew symmetric Killing vector fields.

1. Introduction

Let (M,g) be a general space-time with usual differentiability conditions, and normed hyperbolic metric g. We assume in this paper that M carries two space-like skew-symmetric Killing vector fields X and Y [R], such that:

a) X and Y are exterior concurrent [2],

b) both X and Y have as generative the timelike vector field e_4 of an orthonormal vector basis

$$\mathcal{O} = \mathrm{vect}\{e_A \mid A = 1, \dots, 4\}.$$

It is proved that any such manifold M is a space form of curvature -1 and that M is foliated by hypersurfaces M_S tangent to X, Y and e_4 and such that the normal of M_S is a timelike concircular vector field. In addition, the following properties are pointed out:

(i) If U is any vector field of M and V is any vector field of the exterior concurrent distribution $\{X, Y, e_4\}$, then the Ricci curvature $\mathcal{R}(U,V)$ satisfies:

$$\mathcal{R}(U,V) = 3g(U,V).$$

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(ii) The square of the length of X and Y define an isoparametric system [11] on M.

(iii) $(e_4)^\flat$ is a harmonic form and $X^\flat$ and $Y^\flat$ form an eigenspace $E^1(M)$ of eigenvalue 3.

Killing vector fields X (or infinitesimal isometries) play in many aspects a distinguished role in differential geometry [10]. They also play an important role when dealing with manifolds having indefinite metrics [20] (as for instance space-time C^∞-manifolds).

A vector field X whose covariant differential ∇X (∇ is supposed to be symmetric) satisfies $\nabla X = X \wedge \mathcal{U}$ ($\wedge$: wedge product of vector fields) has been defined [1] as a skew symmetric Killing vector field and $\mathcal{U}$ is called the generator of X. For manifolds M carrying such a vector field X, as for instance solutions to Einstein's equations containing massless fields, like electromagnetic or gravitational waves, this property typically reflects intrinsic features. We will see that in connection with the electromagnetic Faraday 2-form $\mathcal{F}$ when expressed in terms of PFAFFIANS [6] a skew symmetric Killing vector field appears in a quite natural manner.

It is well known that in special relativity, electromagnetism is described in the 4-vector formalism by the Maxwell tensor $(F^{\mu\nu})$, which incorporates both the electric and magnetic field [9]. Equivalently, in the language of forms, there thus exists a 2-form $\mathcal{F}$ on $\mathbb{R}^4$

$$\mathcal{F} = \sum_{\alpha=1}^{3} \mathcal{E}_a du^a \wedge du^4 + (\mathcal{B}_1 du^2 \wedge du^3 + \mathcal{B}_2 du^3 \wedge du^1 + \mathcal{B}_3 du^1 \wedge du^2),$$

where u^i $(i = 1, 2, 3, 4)$ are coordinates in Minkowski space [13].

Therefore, in general relativity on a space-time manifold M, electromagnetism is introduced by a 2-form on M

$$\mathcal{F} = (\mathcal{E}_a \omega^a) \wedge \omega^4 + \sum_{\text{cycl}} \mathcal{B}_a \omega^b \wedge \omega^c,$$

or, in intrinsic manner

$$\mathcal{F} = -(e_4)^\flat \wedge \mathcal{E}^\flat + *(e_4)^\flat \wedge \mathcal{B}^\flat,$$

where $\{\omega^A \mid A = 1, \ldots, 4\}$ is a local field of orthonormal coframes over M; $\mathcal{F}$ is called the generalized Faraday 2-form. On the space-time manifold (M, g) [6] ($\mathcal{E}_a$ and $\mathcal{B}_a$ represent the components of the electric and the magnetic vector fields respectively associated with $\mathcal{F}$; a, b, c are the spacelike indices).

Then if $\mathcal{E}$ and $\mathcal{B}$ coincide with X and Y respectively, one finds that $\mathcal{F}$ is a conformal symplectic form having $3(e_4)^\flat$ as covector of Lee and $3X^\flat$ as source form. One also finds that the Poynting covector $S^\flat$ is expressed by $S^\flat = *(\mathcal{B}^\flat \wedge \mathcal{E}^\flat \wedge (e_4)^\flat)$ and is an exterior recurrent form [19], having $2(e_4)^\flat$ as recurrence form.

Finally, some properties of the Lie algebra induced by X, Y and e_4 are pointed out. We are quoting here:

(i) X, Y and e_4 define a perfect symmetric group;

(ii) X and Y are affine Killing vector fields;

(iii) if $X = \mathcal{E}$, $Y = \mathcal{B}$, then $\mathcal{F}$ is a relative integral invariant of X and an invariant of Y. By interchanging the physical interpretations of X and Y one obtains similar results.

2. Preliminaries

Let (M, g) be a Riemannian or pseudo-Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined by the metric tensor g (we assume that ∇ is the Levi-Civita connection). Let ΓTM be the set of sections of the tangent bundle, and

$$TM \xrightarrow{\flat} T^*M \quad \text{and} \quad TM \xleftarrow{\sharp} T^*M$$

the classical isomorphism defined by g (i.e. $\flat$ is the index lowering operator, and $\sharp$ is the index raising operator).

Following [10], we denote by

$$A^q(M, TM) = \Gamma \operatorname{Hom}(\Lambda^q TM, TM)$$

the set of vector valued q-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM) \tag{1}$$

(it should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$ unlike $d^2 = d \circ d = 0$). If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the identity vector valued 1-form and is also called the soldering form of M [7]. Since ∇ is symmetric one has that $d^\nabla(dp) = 0$. A vector field Z which satisfies

$$d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM); \quad \pi \in \Lambda^1 M \tag{2}$$

is defined to be an exterior concurrent vector field [1] (see also [3], [4]). In (2) π is called the concurrence form and is defined by

$$\pi = \lambda Z^\flat, \quad \lambda \in \Lambda^0 M. \tag{3}$$

In this case, if $\mathcal{R}$ is the Ricci tensor, one has

$$\mathcal{R}(Z, V) = \epsilon(n-1)\lambda g(Z, V) \tag{4}$$

($\epsilon = \pm 1,\ V \in \Xi M,\ n = \dim M$).

If $f \in \Lambda^0 M$, then we set $\operatorname{grad} f = (df)^\sharp$.

Consider the function $F(f_1, \ldots, f_q)$. Then if

$$\begin{aligned} &\big\langle (df_i)^\sharp, (df_j)^\sharp \big\rangle = A_{ij}(F), \Delta f_i = B_i(F) \quad \text{and} \\ &\big[(df_i)^\sharp, (df_j)^\sharp\big] = \sum C_{ij}^k(F) \nabla f_k, \end{aligned} \tag{5}$$

where $A_{ij},\ B_i,\ C_{ij}^k$ are smooth functions, one says following [11], that $f_1, \ldots, f_q$ define an isoparametric system.

Let $\mathcal{O} = \{e_A \mid A = 1, \ldots, n\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \operatorname{covect}\{\omega^A\}$ be its associated coframe. Then E. Cartans structure equations written in indexless manner are

$$\nabla e = \theta \otimes e, \tag{6}$$

$$d\omega = -\theta \wedge \omega, \tag{7}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{8}$$

In the above equations θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature forms on M).

3. Exterior concurrent Killing vector fields

Let (M, g) be a space-time manifold with metric tensor g and let $\mathcal{O} = \{e_A \mid A = 1, \ldots, 4\}$ be a local field of orthonormal frames over M

and $\mathcal{O}^* = \text{covect}\{\omega^A\}$ its associated coframe. We assume that the indices $a, b \in \{1, 2, 3\}$ correspond to the spacelike vector fields of $\mathcal{O}$, whilst e_4 corresponds to the timelike vector field of $\mathcal{O}$. Then according to [12] (see also [4]), the soldering form dp is expressed by:

$$dp = -\sum \omega^a \otimes e_a + \omega^4 \otimes e_4. \tag{9}$$

By reference to [8] and in consequence of (9) one has the following structure equations:

$$\begin{aligned} \nabla e_a &= -\theta_a^b \otimes e_b + \theta_a^4 \otimes e_4, \\ \nabla e_4 &= -\theta_4^a \otimes e_a; \end{aligned} \tag{10}$$

$$\begin{aligned} d\omega^a &= -\omega^b \wedge \theta_b^a + \omega^4 \wedge \theta_4^a, \\ d\omega^4 &= -\omega^a \wedge \theta_a^4; \end{aligned} \tag{11}$$

and

$$\begin{aligned} d\theta_b^a &= \Theta_b^a - \theta_b^c \wedge \theta_c^a + \theta_b^4 \wedge \theta_4^a, \\ d\theta_4^a &= \Theta_4^a - \theta_4^b \wedge \theta_b^a. \end{aligned} \tag{12}$$

Let now

$$X = -X^a e_a, \quad \text{and} \quad Y = -Y^a e_a; \ X^a, \ Y^a \in \Lambda^0 M, \tag{13}$$

be two skewsymmetric Killing vector fields (in the sense of [1], see also [4]) and assume in a first step that both X and Y have as generative the unit timelike vector field e_4. Hence, the covariant differentials of X and Y are expressed by

$$\begin{aligned} \nabla X = X \wedge e_4 &\iff \nabla X = \omega^4 \otimes X - X^\flat \otimes e_4, \\ \nabla Y = Y \wedge e_4 &\iff \nabla Y = \omega^4 \otimes Y - Y^\flat \otimes e_4. \end{aligned} \tag{14}$$

In order to simplify, we agree in the following to set: $X^\flat = \alpha$, $Y^\flat = \beta$. On the other hand, if Z is any spacelike vector field on M, then making use of the structure equations (10), one finds:

$$\nabla Z = -(dZ^a - Z^b \theta_b^a) \otimes e_a - (Z^a \theta_a^4) \otimes e_4. \tag{15}$$

Then by (14) and (15) one calculates

$$
\begin{aligned}
dX^a &= X^b \theta_b^a + X^a \omega^4, \\
dY^a &= Y^b \theta_b^a + Y^a \omega^4,
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
d\alpha &= 2\omega^4 \wedge \alpha, \\
d\beta &= 2\omega^4 \wedge \beta,
\end{aligned} \tag{17}
$$

which shows that α and β are exterior recurrent forms [19] having ${e_4}^\flat = \omega^4$ as recurrence form. We recall [1] that equations (17) are consequences of (14). In addition, it can be seen from (17) that one has

$$
d\omega^4 = 0 \Rightarrow {\theta_4}^a = \lambda \omega^a, \quad \lambda \in \Lambda^0 M. \tag{18}
$$

With the help of (18) one quickly finds

$$
\nabla e_4 = \lambda (dp - \omega^4 \otimes e_4), \tag{19}
$$

which shows that e_4 is torse forming (see also [4]) [14].

In a second step, operating on ∇X and ∇Y by the covariant derivative operator d^∇, one derives by (17) and (19)

$$
\begin{aligned}
d^\nabla(\nabla X) &= \nabla^2 X = \lambda \alpha \wedge dp + (\lambda - 1)(\omega^4 \wedge \alpha) \otimes e_4, \\
d^\nabla(\nabla Y) &= \nabla^2 Y = \lambda \beta \wedge dp + (\lambda - 1)(\omega^4 \wedge \beta) \otimes e_4.
\end{aligned} \tag{20}
$$

Hence, by (2) the necessary and sufficient condition in order that X and Y be exterior concurrent vector fields is that $\lambda = 1$, and in this case equations (20) go over into

$$
\begin{aligned}
\nabla^2 X &= \alpha \wedge dp, \\
\nabla^2 Y &= \beta \wedge dp.
\end{aligned} \tag{21}
$$

Since $\alpha = X^\flat$, $\beta = Y^\flat$, one may develop equations (21) as:

$$
\begin{aligned}
X^a(\theta_a^b \otimes e_b - \theta_a^4 \otimes e_4) &= \Big(\sum_a X^a \omega^a\Big) \wedge (\omega^b \otimes e_b - \omega^4 \otimes_4), \\
Y^a(\theta_a^b \otimes e_b - \theta_a^4 \otimes e_4) &= \Big(\sum_a Y^a \omega^a\Big) \wedge (\omega^b \otimes e_b - \omega^4 \otimes_4),
\end{aligned} \tag{22}
$$

and one gets

$$\Theta_B^A = -\omega^A \wedge \omega^B. \tag{23}$$

Hence, according to a well known formula, equation (23) shows the fact that the manifold M under consideration is a space form of curvature -1. In this condition ($\lambda = 1$) one easily finds that e_4 is also an exterior concurrent vector field. Since this property is preserved by linearity, one may say that $\mathcal{D} = \{X, Y, e_4\}$ defines an exterior concurrent distribution on M. Now, one can check that $\mathcal{D}$ is also involutive. It should also be noticed that the existence of the exterior concurrent skew symmetric Killing vector fields X and Y is determined by the closed differential system defined by (17) and (18). Further, it is not hard to show that if $N = \sum N^a e_a$ is a vector field orthogonal to $\mathcal{D} = \{X, Y, e_4\}$, then the differentials of N^a satisfy

$$dN^a = N^b \theta_b^a.$$

Then with the help of the first equations (10) one derives by (15) that

$$\nabla N = -N^\flat \otimes e_4, \tag{24}$$

which shows that N is a timelike concircular vector field. Therefore, one may say that the manifold M under consideration is foliated by hypersurfaces M_D whose normals are timelike concircular.

Next, taking into account the signature of g, and making use of the general formula

$$\mathcal{R}(U, Z) = \sum \langle R(e_A, U \mid Z, e_A \rangle, \quad U, Z \in \Xi M,$$

($\mathcal{R}$ (resp. R) denotes the Ricci tensor field of ∇ (resp. the curvature tensor field)), one derives:

$$\mathcal{R}(U, V) = 3g(U, V), \tag{25}$$

where V is any vector field of the distribution D. On the other hand recall that the covariant derivative $\nabla\omega$ of a 1-form $\omega = \omega_A \omega^A$, $\omega_A \in \Lambda^0 M$ is expressed by

$$\nabla\omega = (d\omega_A - \omega_B \theta_A^B) \otimes \omega^A, \tag{26}$$

(see also [20]) and following [15], ω is a Killing form if

$$\nabla\omega = 0, \quad \delta\omega = 0. \tag{27}$$

Coming back to the case under discussion, one finds by (17) that α and β satisfy conditions (27), which means that they are Killing 1-forms.

Finally, setting

$$2l_x = \|X\|^2, \quad 2l_y = \|Y\|^2,$$

one gets by (14)

$$\begin{aligned} (d2l_x)^\sharp &= 4l_x e_4 \Rightarrow \|(d2l_x)^\sharp\|^2 = 16l_x^2, \\ (d2l_y)^\sharp &= 4l_y e_4 \Rightarrow \|(d2l_y)^\sharp\|^2 = 16l_y^2, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \nabla(d2l_x)^\sharp &= -4l_x\omega^4 \otimes e_4, \\ \nabla(d2l_y)^\sharp &= -4l_y\omega^4 \otimes e_4. \end{aligned} \tag{29}$$

Next, taking into account the signature of g, one derives from (28)

$$\begin{aligned} \operatorname{div}(d2l_x)^\sharp &= -10(2l_x), \\ \operatorname{div}(d2l_y)^\sharp &= -10(2l_y), \end{aligned} \tag{30}$$

and

$$\big[(d2l_x)^\sharp, (d2l_y)^\sharp\big] = 0. \tag{31}$$

Hence, by reference to (5), equations (28), (30) and (31) show that $\|X\|^2$ and $\|Y\|^2$ define an isoparametric system on M. Summarizing, we can formulate the following

Theorem 3.1. *Let (M, g) be a space-time manifold carrying two spacelike vector fields X and Y which have the property to be exterior concurrent skewsymmetric Killing vector fields. If the generatives of X and Y coincide with the unit timelike vector field e_4 of M, then M is a space form of curvature -1. In addition, one has the following properties:*

(i) (M, g) *is foliated by hypersurfaces* M_D *tangent to the exterior concurrent distribution* $\mathcal{D} = \{X, Y, e_4\}$ *and the normal* N *of* M_D *is a timelike concircular vector field, i.e.*

$$\nabla N = -N^\flat \otimes e_4;$$

(ii) *if* U *is any vector field of* M *and* $V \in \mathcal{D}$*, then the Ricci curvature* $\mathcal{R}(U, V)$ *satisfies:*

$$\mathcal{R}(U, V) = 3g(U, V);$$

(iii) $X^\flat$ *and* $Y^\flat$ *are Killing forms;*

(iv) *the square of the length of* X *and* Y *define an isoparametric system on* M.

4. Harmonic properties on M

Let σ be the volume element of the manifold M under discussion, and let $*$ be the star operator determined by a local orientation of M. One has

$$*\omega^4 = \omega^1 \wedge \omega^2 \wedge \omega^3, \tag{32}$$

and taking into account (18) (remember that $\lambda = 1$), one finds by (11)

$$d * \omega^4 = -3\sigma. \tag{33}$$

Next, according to the general formula

$$\delta u = (-1)^{n(p+1)+1} * d * u, \quad u \in \Lambda^p M, \tag{34}$$

one finds with (33) that $\delta\omega^4 = 0$ and therefore by (18) one derives that

$$\Delta\omega^4 = 0,$$

which shows that ω^4 is a harmonic form.

Next, since $\alpha = X^\flat = -\sum_a X^a \omega^a$, one deduces by (16) and with the help of (11) that

$$\delta\alpha = d * \alpha = 0,$$

and taking account of (17) one deduces

$$\Delta\alpha = \delta(2\omega^4 \wedge \alpha) = 3\alpha. \tag{35}$$

Clearly, in a similar way, one has

$$\Delta\beta = 3\beta, \tag{36}$$

and the above equations show that α and β are eigenfunctions of Δ having 3 as eigenvalue (see also [16]). Therefore, one may state that α and β define an eigenspace $E^1(M)$ of eigenvalue 3.

Consequently, we have

Proposition 4.1. *The dual forms $X^\flat, Y^\flat$ of the exterior concurrent skewsymmetric Killing vector fields X and Y of M, define an eigenspace $E^1(M)$ of eigenvalue 3.*

5. Generalized Faraday form

Following [6] the 2-form

$$\mathcal{F} = (\mathcal{E}_a\omega^a) \wedge \omega^4 + \sum_{\text{cycl}} \mathcal{B}_a\omega^b \wedge \omega^c, \tag{37}$$

where $\mathcal{E}_a$ (resp. $\mathcal{B}_a$) are the components of the electric field $\mathcal{E}$ (resp. the components of the magnetic vector field) is called the generalized Faraday form of a space-time manifold M. In order to agree with (13) one may express (37) in an intrinsic manner as

$$\mathcal{F} = -(e_4)^\flat \wedge \mathcal{E}^\flat + *(e_4)^\flat \wedge \mathcal{B}^\flat.$$

It should be noticed that the above expression of $\mathcal{F}$ is in accordance with the expression of $\mathcal{F}$ in case M is a Minkowski manifold [18].

Assume now that $\mathcal{E}(\mathcal{E}_a)$ and $\mathcal{B}(\mathcal{B}_a)$ coincide with the Killing vector field X and the Killing vector field Y, respectively. Then setting

$$\phi = \sum_{\text{cycl}} \mathcal{B}_a\omega^b \wedge \omega^c = \sum_{\text{cycl}} Y^a\omega^b \wedge \omega^c,$$

($\sum_{\text{cycl}}$: cyclic permutation of the spacelike indices a, b, c) one may write

$$\mathcal{F} = X \wedge \omega^4 + \phi. \tag{38}$$

Making use of the equations (11), one gets by the equations (16) that

$$d\phi = 3\omega^4 \wedge \phi, \tag{39}$$

i.e. ϕ is an exterior recurrent 2-form having $3\omega^4$ as recurrence form [19]. We notice that, since $d(\alpha \wedge \omega^4) = 0$, (see 17), one may write

$$d\mathcal{F} = d\phi \iff \mathcal{F} \sim \phi,$$

i.e. $\mathcal{F}$ and ϕ belong to the same class of homology. Therefore one may write

$$d\mathcal{F} = 3\omega^4 \wedge \mathcal{F}, \quad \text{(remember } \omega^4 = (e_4)^\flat\text{)}, \tag{40}$$

which shows that $\mathcal{F}$ is a conformal symplectic form, having $3\omega^4$ as covector of Lee.

Setting

$$\psi = \sum_{\text{cycl}} X^a \omega^b \wedge \omega^c,$$

one obtains by (34)

$$*\mathcal{F} = -\beta \wedge \omega^4 + \psi \quad (\beta = Y^\flat). \tag{41}$$

Then by a standard calculation and with the help of the equations (16) one gets

$$\delta\mathcal{F} = 3\alpha. \tag{42}$$

Consequently, as a generalization of the concept of source, in case M is a Minkowski manifold, one may consider in the case under discussion 3α as being the source of the considered generalized Faraday form. Clearly by interchanging the role of $X = \mathcal{E}$ and $Y = \mathcal{B}$, the corresponding Faraday form will have as source 3β. Further, as a generalization of the concept of Poynting covector $S^\flat$ associated with $\mathcal{F}$, one may write in an intrinsic manner

$$S^\flat = *(\mathcal{B}^\flat \wedge \mathcal{E}^\flat \wedge (e_4)^\flat. \tag{43}$$

Since in the case under discussion, one has

$$\mathcal{B}^\flat = \beta, \quad \mathcal{E}^\flat = \alpha, \quad (e_4)^\flat = \omega^4,$$

one finds by a standard calculation

$$S^\flat = \sum_{\text{cycl}} (X^a Y^b - X^b Y^a)\omega^c. \tag{44}$$

Further, it is worth to emphasize that one may write

$$i_X\mathcal{F} = -2l_x\omega^4 + S^\flat. \tag{45}$$

Making in additon use of the relations (16) and with the help of the equations (12), exterior derivation of (44) gives

$$dS^\flat = 2\omega^4 \wedge S^\flat. \tag{46}$$

This shows that the Poynting covector $S^\flat$ is exterior recurrent [19], and has $2\omega^4$ as recurrence form.

Theorem 5.1. *Let $\mathcal{F}$ be the generalized Faraday form, such that the electric vector field $\mathcal{E}$ (respectively the magnetic vector field $\mathcal{B}$) coincides with the Killing vector field X (respectively the Killing vector field Y). Then $\mathcal{F}$ is a conformal symplectic form, having $3\omega^4$ as covector of Lee, and 3α as associated source. Further if $S^\flat$ denotes the Poynting covector associated with $\mathcal{F}$, then $S^\flat$ is exterior recurrent and has $2\omega^4$ as recurrence form. By interchanging the physical interpretation of X and Y one finds similar results.*

6. Lie algebra induced by the Killing vector fields X and Y

As is known there exists an isomorphism between the Lie algebra $\mathcal{G}$ of the Lie group, and the space of Killing vector fields. If $\omega \in \Lambda^1 M$, then if U and V are any vector fields, we have

$$d\omega(V,U) = 2\omega\,[V,U]\,. \tag{47}$$

Coming back to the case under consideration, one derives from (14) and (19), since $\lambda = 1$

$$[X,Y] = 0, \quad [X,e_4] = 0, \quad [Y,e_4] = 0, \tag{48}$$

which shows that the vector fields X, Y and e_4 define a commutative triple. We notice by (47) that one has $d\omega(X,Y) = 0$, for any $\omega \in \Lambda^1 M$.

Moreover one finds by (17)

$$\mathcal{L}_X\alpha = 0, \quad \mathcal{L}_Y\beta = 0, \tag{49}$$

and since $\alpha = X^\flat$, $\beta = Y^\flat$ one may say that the dual form of X and Y are self-invariant.

Next, taking the Lie derivatives of ∇X and ∇Y, one gets by (14) and (49)

$$\mathcal{L}_X(\nabla X) = 0, \quad \mathcal{L}_Y(\nabla Y) = 0. \tag{50}$$

This proves the fact that X and Y are affine Killing vector fields (see also [10]).

Further, by (45) one calculates

$$\mathcal{L}_X \mathcal{F} = -3\omega^4 \wedge i_X \mathcal{F} = -\omega^4 \wedge S^\flat, \tag{51}$$

which by (46) gives

$$d(\mathcal{L}_X \mathcal{F}) = 0. \tag{52}$$

Hence according to [17] the above equation confirms the fact that $\mathcal{F}$ is a relative integral invariant of X. Moreover, let $\mathbb{L}$ be the (1.1) type operator on forms defined by [5], that is:

$$\mathbb{L}u = u_1 = u \wedge \Omega,$$

where Ω is an almost symplectic form. In the case under discussion we set

$$\alpha_1 = \alpha \wedge \mathcal{F},$$

and by (49) we deduce by exterior differentiation

$$d(\mathcal{L}_X \alpha_1) = 0.$$

Therefore, we may assert that the property of integral invariance of X is preserved by the operator $\mathbb{L}$.

Finally, setting $s = g(X, Y)$, one quickly derives by (14)

$$ds = 2s\omega^4. \tag{53}$$

Next, since by (38) and (51) we deduce

$$i_Y \mathcal{F} = s\omega^4 \Rightarrow d(i_Y \mathcal{F}) = 0, \tag{54}$$

it quickly follows by (40)

$$\mathcal{L}_Y \mathcal{F} = 0. \tag{55}$$

This proves the fact that the generalized Faraday form having X as associated electric vector field is invariant by the magnetic vector field Y. Moreover, making use of the operator $\mathbb{L}$, i.e.

$$\mathbb{L}\beta = \beta_1 = \beta \wedge \mathcal{F},$$

then one calculates that

$$\mathcal{L}_Y \beta_1 = 0.$$

Hence the property of $\mathcal{F}$ to be invariant under Y is preserved by the operator $\mathbb{L}$. Clearly, the above properties hold by interchanging the role of the Killing vector fields X and Y.

We state the

Theorem 6.1. *Let (M, g) be the space-time manifold carrying two spacelike Killing vector fields X and Y defined in Section 3. Then regarding the Lie algebra induced by X and Y, we have the following properties:*

(i) *$X^\flat$ and $Y^\flat$ are selfinvariant, i.e.*

$$\mathcal{L}_X X^\flat = 0, \quad \mathcal{L}_Y Y^\flat = 0;$$

(ii) *X and Y are affine Killing vector fields, i.e.*

$$\mathcal{L}_X(\nabla X) = 0, \quad \mathcal{L}_Y(\nabla Y) = 0;$$

(iii) *let $\mathcal{F}$ be the generalized Faraday form on M, and assume that X and Y coincide with the electric vector field $\mathcal{E}$ and the magnetic vector field $\mathcal{B}$ respectively, associated with $\mathcal{F}$.*
Then the following properties hold:
(a) *$\mathcal{F}$ is a relative integral invariant of X and this property is preserved by Weyl's operator $\mathbb{L}$, i.e.*

$$d(\mathcal{L}_X \mathcal{F}) = 0, \qquad d(\mathcal{L}_X(\mathbb{L}X^\flat)) = 0, \quad (\mathbb{L}X^\flat = X^\flat \wedge \mathcal{F});$$

(b) *$\mathcal{F}$ is invariant by Y and the same property is preserved by $\mathbb{L}$, i.e.*

$$\mathcal{L}_Y \mathcal{F} = 0, \qquad \mathcal{L}_Y(\mathbb{L}Y^\flat) = 0, \quad (\mathbb{L}Y^\flat = Y^\flat \wedge \mathcal{F}).$$

By interchanging the physical interpretations of X and Y one obtains similar properties.

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