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On some tangency relation of sets

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Abstract. In this paper the problem of the tangency relation $T_l(a,b,k,p)$ of sets of the classes $A^*_{p,k}$ having the Darboux property in a generalized metric space (E,l) is considered. Some conditions for this tangency relation will be given here.

Introduction

Let E be an arbitrary non-empty set and let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let l_0 be a function defined by the formula

$$l_0(x,y) = l(\{x\},\{y\}) \quad \text{for } x,y \in E. \tag{1}$$

If we put some conditions on the function l, then the function l_0 defined by (1) will be a metric of the set E. For this reason the pair (E,l) can be treated as a certain generalization of a metric space and we shall call it (see [9]) generalized metric space. Using (1) we may define in the space (E,l), similarly as in a metric space, the following notions: the sphere $S_l(p,r)$ and the ball $K_l(p,r)$ with centre at the point p and radius r.

Let $S_l(p,r)_u$ denote the so-called u-neighbourhood of the sphere $S_l(p,r)$ in the space (E,l) defined by the formula:

$$S_l(p,r)_u = \begin{cases} \bigcup\limits_{q \in S_l(p,r)} K_l(q,u) & \text{for } u > 0 \\ S_l(p,r) & \text{for } u = 0. \end{cases} \tag{2}$$

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Let k be a fixed positive real number and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r\to 0^+]{} 0. \tag{3}$$

We say that the pair (A, B) of the sets $A, B \in E_0$ is (a, b)-clustered at the point p of the space (E, l), if 0 is a cluster point of the set of all real numbers $r > 0$ such that $A \cap S_l(p, r)_{a(r)} \neq \emptyset$ and $B \cap S_l(p, r)_{b(r)} \neq \emptyset$.

Let (see [9])

$$\begin{aligned} T_l(a, b, k, p) = \Big\{&(A, B) : A, B \in E_0,\ \text{the pair } (A, B) \\ &\text{is } (a, b)\text{-clustered at the point } p \text{ of the space } (E, l) \\ &\text{and } \frac{1}{r^k}\, l\left(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}\right) \xrightarrow[r\to 0^+]{} 0\Big\}. \end{aligned} \tag{4}$$

If $(A, B) \in T_l(a, b, k, p)$, then we say that the set $A \in E_0$ is (a, b)-tangent of order k to the set $B \in E_0$ at the point p of the space (E, l).

We call $T_l(a, b, k, p)$ defined by (4) the (a, b)-tangency relation of order k at the point $p \in E$ or shortly: the tangency relation of sets in the generalized metric space (E, l).

Let ρ be an arbitrary metric of the set E. By A' we shall denote the set of all cluster points of the set $A \in E_0$, and let

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\} \quad \text{for } x \in E. \tag{5}$$

Let us put (see [4])

$$\begin{aligned} A^*_{p,k} = \Big\{&A \in E_0 : p \in A' \text{ and there exists a number } \lambda > 0 \\ &\text{such that } \limsup_{[A,p;k]\ni(x,y)\to(p,p)} \frac{\rho(x, y) - \lambda\rho(x, A)}{\rho^k(p, x)} \leq 0\Big\}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} [A, p; k] = \big\{&(x, y) : x \in E,\ y \in A \\ &\text{and } \rho(x, A) < \rho^k(p, x) = \rho^k(p, y)\big\}. \end{aligned} \tag{7}$$

We shall denote by $d_\rho A$ the diameter of the set $A \in E_0$, and by $\rho(A, B)$ the distance of sets $A, B \in E_0$ in the metric space (E, ρ).

We say that the set $A \in E_0$ has the Darboux property at the point p of the metric space (E,ρ), written $A \in D_p(E,\rho)$ (see [4]), if there exists a number $\tau > 0$ such that the set $A \cap S_\rho(p,r)$ is non-empty for $r \in (0,\tau)$.

Let f be any subadditive increasing real function defined for all real non-negative numbers, such that $f(0)=0$. By $\overline{F}_f$ we denote the class of all functions l fulfilling the conditions:

$$1^\circ \qquad l : E_0 \times E_0 \longrightarrow \langle 0,\infty),$$

$$2^\circ \qquad f(\rho(A,B)) \le l(A,B) \le f(d_\rho(A \cup B)) \quad \text{for } A,B \in E_0.$$

Now

$$f(\rho(x,y)) = f(\rho(\{x\},\{y\})) \le l(\{x\},\{y\}) \le f(d_\rho(\{x\} \cup \{y\})) = f(\rho(x,y));$$

from here and from (1) it follows that

$$l_0(x,y) = l(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for } l \in F_f \text{ and } x,y \in E. \tag{8}$$

It is easy to check that the function l_0 defined by (8) is a metric of the set E. From this and from Lemma 1 of the paper [4] it follows that setting $d_l = d_{l_0}$ we get

$$\frac{1}{r^k} d_l(A \cap S_l(p,r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0, \tag{9}$$

if $l \in \overline{F}_f$, $A \in A^*_{p,k} \cap D_p(E,l)$ and

$$\frac{a(r)}{r^k} \xrightarrow[r \to 0^+]{} 0. \tag{10}$$

In this paper the problem of the tangency relation $T_l(a,b,k,p)$ of sets of the classes $A^*_{p,k}$ having the Darboux property in the generalized metric space (E,l) for $l \in \overline{F}_f$ is considered. Some conditions for this relation have been given here.

1. Two lemmas

Let l_1, l_2 be arbitrary functions belonging to the class $l \in \overline{F}_f$. We put by definition

$$(l_1 + l_2)(A,B) = l_1(A,B) + l_2(A,B) \quad \text{for } A,B \in E_0. \tag{11}$$

Let $S_l(p,r)_u$ be a u-neighbourhood of the sphere $S_l(p,r)$ in the space (E,l) defined by the formula (2).

Lemma 1. *If the functions $l_1, l_2 \in \overline{F}_f$, then*

$$S_{l_1+l_2}(p,r)_u = S_l(p,r/2)_{u/2} \quad \text{for } l \in \overline{F}_f. \tag{12}$$

Proof. From the equalities (8) and (11), for $l \in \overline{F}_f$ we have

$$\begin{aligned} S_{l_1+l_2}(p,r) &= \{x \in E:\ (l_1+l_2)(\{p\},\{x\}) = r\} \\ &= \{x \in E:\ l_1(\{p\},\{x\}) + l_2(\{p\},\{x\}) = r\} \\ &= \{x \in E : 2l_0(p,x) = r\} \\ &= \{x \in E:\ l(\{p\},\{x\}) = r/2\} = S_l(p,r/2). \end{aligned}$$

Therefore if $l_1, l_2 \in \overline{F}_f$ then

$$S_{l_1+l_2}(p,r) = S_l(p,r/2) \quad \text{for } l \in \overline{F}_f. \tag{13}$$

Analogously

$$K_{l_1+l_2}(p,r) = K_l(p,r/2) \quad \text{for } l \in \overline{F}_f. \tag{14}$$

From the definition (2) of the u-neighbourhood of the sphere $S_l(p,r)$, and from (13) and (14) we obtain

$$S_{l_1+l_2}(p,r)_u = S_l(p,r/2)_{u/2},$$

which yields the statement (12) of this lemma.

An immediate consequence of (12) is the following equality

$$S_{l_1+\dots+l_n}(p,r)_u = S_l(p,r/n)_{u/n} \quad \text{for } l, l_1, \ldots = l_n \in \overline{F}_f. \tag{15}$$

□

Lemma 2. *If $l \in \overline{F}_f$ and the function a fulfils the condition* (10), *then*

$$\frac{1}{r^k} d_l(A \cap S_l(p,r/n)_{a(r)/n}) \xrightarrow[r\to 0^+]{} 0 \tag{16}$$

for $n \in \mathbb{N}$ *and* $A \in A^*_{p,k} \cap D_p(E,l)$.

Proof. From the assumption (10) it follows that

$$\frac{1}{t^k} a(nt) \xrightarrow[t\to 0^+]{} 0. \tag{17}$$

Hence and from (9) for $A \in A^*_{p,k} \cap D_p(E,l)$ it results

$$\frac{1}{t^k} d_l \left(A \cap S_l(p,t)_{a(nt)}\right) \xrightarrow[t\to 0^+]{} 0. \tag{18}$$

Putting $r = nt$, from (18) we get

$$\frac{1}{r^k} d_l (A \cap S_l(p,r/n)_{a(r)}) \xrightarrow[r\to 0^+]{} 0. \tag{19}$$

Since $A \cap S_l(p,r/n)_{a(r)/n} \subset A \cap S_l(p,r/n)_{a(r)}$, we have

$$0 \le d_l(A \cap S_l(p,r/n)_{a(r)/n}) \le d_l(A \cap S_l(p,r/n)_{a(r)}).$$

Hence and from (19) we get the statement (16) of this lemma. □

2. The main result

From (16) it follows in particular that

$$\frac{1}{r^k} d_l (A \cap S_l(p,r/2)_{a(r)/2}) \xrightarrow[r\to 0^+]{} 0, \tag{20}$$

if the function a fulfils the condition (10), $l \in \overline{F}_f$ and $A \in A^*_{p,k} \cap D_p(E,l)$.

From the definition

$$\begin{gathered} (A,B) \in (T_{l_1}(a,b,k,p) \cup T_{l_2}(a,b,k,p)) \\ \iff \left[(A,B) \in (T_{l_1}(a,b,k,p) \vee (A,B) \in T_{l_2}(a,b,k,p)\right], \end{gathered} \tag{21}$$

for $A, B \in E_0$ and $l_1, l_2 \in \overline{F}_f$.

Theorem 1. *If the non-decreasing functions* a, b *fulfil the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{22}$$

and $l_1, l_2 \in \overline{F}_f$, *then* $(A,B) \in T_{l_1}(a,b,k,p) \cup T_{l_2}(a,b,k,p)$ *iff* $(A,B) \in T_{l_1+l_2}(a,b,k,p)$ *for* $A,B \in A^*_{p,k} \cap D_p(E,l)$, *where* $l \in \overline{F}_f$.

PROOF. We assume that $(A,B) \in T_{l_1}(a,b,k,p) \cup T_{l_2}(a,b,k,p)$. From here and from (21) for instance it follows that $(A,B) \in T_{l_1}(a,b,k,p)$ for $A,B \in A^*_{p,k} \cap D_p(E,l)$. Hence and from the fact that (see (8))

$$S_{l_1}(p,r) = S_{l_2}(p,r) = S_l(p,r) \quad \text{for } l, l_1, l_2 \in \overline{F}_f, \tag{23}$$

it follows

$$\frac{1}{t^k} l_1(A \cap S_l(p,t)_{a(t)}, B \cap S_l(p,t)_{b(t)}) \xrightarrow[t \to 0^+]{} 0. \tag{24}$$

Hence, from the fact that $l_1, l_2 \in \overline{F}_f$, from the properties of the function f and from the equality

$$f(d_\rho A) = d_l A \quad \text{for } A \in E_0, \tag{25}$$

we obtain

$$\begin{aligned}
&(l_1+l_2)(A \cap S_{l_1+l_2}(p,r)_{a(r)}, B \cap S_{l_1+l_2}(p,r)_{b(r)}) \\
&\quad = l_1(A \cap S_{l_1+l_2}(p,r)_{a(r)}, B \cap S_{l_1+l_2}(p,r)_{b(r)}) \\
&\qquad + l_2(A \cap S_{l_1+l_2}(p,r)_{a(r)}, B \cap S_{l_1+l_2}(p,r)_{b(r)}) \\
&\quad = l_1(A \cap S_l(p,r/2)_{a(r)/2}, B \cap S_l(p,r/2)_{b(r)/2}) \\
&\qquad + l_2(A \cap S_l(p,r/2)_{a(r)/2}, B \cap S_l(p,r/2)_{b(r)/2}) \\
&\quad \le 2f(d_\rho((A \cap S_l(p,r/2)_{a(r)/2}) \cup (B \cap S_l(p,r/2)_{b(r)/2}))) \\
&\quad \le 2f(d_\rho((A \cap S_l(p,r/2)_{a(r)/2})) + 2f(d_\rho((B \cap S_l(p,r/2)_{b(r)/2})) \\
&\qquad + 2f(\rho(A \cap S_l(p,r/2)_{a(r)/2}, B \cap S_l(p,r/2)_{b(r)/2})) \\
&\quad \le 2d_l(A \cap S_l(p,r/2)_{a(r)/2}) + 2d_l(B \cap S_l(p,r/2)_{b(r)/2}) \\
&\qquad + 2l_1(A \cap S_l(p,r/2)_{a(r)/2}, B \cap S_l(p,r/2)_{b(r)/2}),
\end{aligned}$$

whence

$$\begin{aligned}
&\frac{1}{r^k}(l_1+l_2)(A \cap S_{l_1+l_2}(p,r)_{a(r)}, B \cap S_{l_1+l_2}(p,r)_{b(r)}) \\
&\quad \le \frac{2}{r^k} d_l(A \cap S_l(p,r/2)_{a(r)/2}) + \frac{2}{r^k} d_l(B \cap S_l(p,r/2)_{b(r)/2}) \\
&\qquad + \frac{2}{r^k} l_1(A \cap S_l(p,r/2)_{a(r)/2}, B \cap S_l(p,r/2)_{b(r)/2}).
\end{aligned} \tag{26}$$

From here, from (20) and (24), and from the theorems on the compatibility of the tangency relations of sets of the classes $A^*_{p,k}$ (see Theorem 2 and Theorem 3 of the paper [4] we have

$$\text{(27)}\qquad \frac{1}{r^k}(l_1+l_2)(A\cap S_{l_1+l_2}(p,r)_{a(r)}, B\cap S_{l_1+l_2}(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0.$$

From the fact that $A, B \in D_p(E, l)$ for $l \in \overline{F}_f$, it follows that there exists a number $\tau > 0$ such that $A \cap S_l(p, r) \neq \emptyset$ and $B \cap S_l(p, r) \neq \emptyset$ for $r \in (0, \tau)$. Because $S_{l_1+l_2}(p, r) = S_l(p, r/2)$ and $r/2 \in (0, \tau/2) \subset (0, \tau)$, then $A \cap S_{l_1+l_2}(p, r) \neq \emptyset$ and $B \cap S_{l_1+l_2}(p, r) \neq \emptyset$, and so $A, B \in D_p(E, l_1+l_2)$. Hence it follows that the pair of sets (A, B) is (a, b)-clustered at the point p of the space (E, l_1+l_2). From here and from (27) it follows that $(A, B) \in T_{l_1+l_2}(a, b, k, p)$.

Now we assume that $(A, B) \in T_{l_1+l_2}(a, b, k, p)$ for $A, B \in A^*_{p,k} \cap D_p(E, l)$. Hence we have

$$\frac{1}{r^k}(l_1+l_2)(A\cap S_{l_1+l_2}(p,r)_{a(r)}, B\cap S_{l_1+l_2}(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0.$$

From here and from (11) it follows that

$$\text{(28)}\qquad \frac{1}{r^k}l_1(A\cap S_{l_1+l_2}(p,r)_{a(r)}, B\cap S_{l_1+l_2}(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0.$$

Hence and from Lemma 1 we get

$$\frac{1}{r^k}l_1(A\cap S_l(p,r/2)_{a(r)/2}, B\cap S_l(p,r/2)_{b(r)/2}) \xrightarrow[r\to 0^+]{} 0.$$

From here and from the fact that $l_1 \in \overline{F}_f$

$$\text{(29)}\qquad \frac{1}{r^k}f(\rho(A\cap S_l(p,r/2)_{a(r)/2}, B\cap S_l(p,r/2)_{b(r)/2})) \xrightarrow[r\to 0^+]{} 0.$$

Because for arbitrary sets $A, B \in E_0$

$$\begin{aligned} 0 &\le \rho(A\cap S_l(p,r/2)_{a(r)}, B\cap S_l(p,r/2)_{b(r)}) \\ &\le \rho(A\cap S_l(p,r/2)_{a(r)/2}, B\cap S_l(p,r/2)_{b(r)/2}), \end{aligned}$$

from here and from (29) it follows that

$$\frac{1}{r^k} f(\rho(A \cap S_l(p,r/2)_{a(r)}, B \cap S_l(p,r/2)_{b(r)})) \xrightarrow[r \to 0^+]{} 0. \tag{30}$$

Hence, from the properties of the function f and from the Theorem 3 of the paper [4] we get

$$\frac{1}{r^k} f\big(d_\rho((A \cap S_l(p,r/2)_{a(r)}) \cup (B \cap S_l(p,r/2)_{b(r)}))\big) \xrightarrow[r \to 0^+]{} 0. \tag{31}$$

Because for $A, B \in E_0$

$$\begin{aligned} 0 &\le f\big(d_\rho((A \cap S_l(p,r/2)_{a(r/2)}) \cup (B \cap S_l(p,r/2)_{b(r/2)}))\big) \\ &\le f\big(d_\rho((A \cap S_l(p,r/2)_{a(r)}) \cup (B \cap S_l(p,r/2)_{b(r)}))\big), \end{aligned}$$

from here and from (31) it follows that

$$\frac{1}{r^k} f\big(d_\rho((A \cap S_l(p,r/2)_{a(r/2)}) \cup (B \cap S_l(p,r/2)_{b(r/2)}))\big) \xrightarrow[r \to 0^+]{} 0. \tag{32}$$

Hence and from the fact that $l_1 \in \overline{F}_f$ we obtain

$$\frac{1}{r^k} l_1(A \cap S_l(p,r/2)_{a(r/2)}, B \cap S_l(p,r/2)_{b(r/2)}) \xrightarrow[r \to 0^+]{} 0,$$

i.e.

$$\frac{1}{t^k} l_1(A \cap S_l(p,t)_{a(t)}, B \cap S_l(p,t)_{b(t)}) \xrightarrow[t \to 0^+]{} 0. \tag{33}$$

From the assumption $A, B \in D_p(E,l)$ for $l \in \overline{F}_f$ and from (23) it follows that $A, B \in D_p(E,l_1)$ for $l_1 \in \overline{F}_f$. Hence we get that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E,l_1). From here and from the condition (33) it follows that $(A,B) \in T_{l_1}(a,b,k,p)$, and this means that $(A,B) \in T_{l_1}(a,b,k,p) \cup T_{l_2}(a,b,k,p)$. This ends the proof of the theorem. □

Let $A, B \in E_0$ and $l_1, l_2, \dots, l_n \in \overline{F}_f$; from the definition

$$\begin{gathered} (A,B) \in \bigcup_{i=1}^{n} T_{l_i}(a,b,k,p) \iff (A,B) \in T_{l_j}(a,b,k,p) \\ \text{for an } j \in \{1,2,\dots,n\}. \end{gathered} \tag{34}$$

Hence and from Theorem 1 we get

Corollary 1. *If the non-decreasing functions a, b fulfil the condition* (22) *and $l_1, l_2, \dots, l_n \in \overline{F}_f$, then $(A,B) \in \bigcup_{i=1}^{n} T_{l_i}(a,b,k,p)$ if and only if $(A,B) \in T_{l_1+\dots+l_n}(a,b,k,p)$, for $A, B \in A^*_{p,k} \cap D_p(E,l)$, where $l \in \overline{F}_f$.*

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