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Star-Menger and related spaces

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Abstract. In this paper we introduce and study some notions related to the classical concepts of being a Menger space or a Rothberger space.

1. Introduction and definitions

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X. Then the symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The symbol $\mathsf{S}_{\rm fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ (see [7], [12]).

We are going now to introduce new selection hypotheses similar to the previous ones. As usual, for a subset A of a space X and a collection $\mathcal{P}$ of subsets of X, $\mathrm{St}(A,\mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is the set $\cup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$; for $A = \{x\}$, $x \in X$, we write $\mathrm{St}(x,\mathcal{P})$ instead of $\mathrm{St}(\{x\},\mathcal{P})$. We assume that all spaces are Hausdorff.

1.1. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. Then:

(a) The symbol $\mathsf{S}_1^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{\mathrm{St}(U_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$;

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(b) The symbol $\mathsf{S}^*_{\rm fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\{{\rm St}(V,\mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$;

(c) By $\mathsf{U}^*_{\rm fin}(\mathcal{A},\mathcal{B})$ we denote the selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of members of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{{\rm St}(\cup\mathcal{V}_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$ or there is some $n \in \mathbb{N}$ such that ${\rm St}(\cup\mathcal{V}_n,\mathcal{U}_n) = X$.

1.2. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X and let $\mathcal{K}$ be a family of subsets of X. Then we say that X belongs to the class $\mathsf{SS}^*_{\mathcal{K}}(\mathcal{A},\mathcal{B})$ if X satisfies the following selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{{\rm St}(K_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all one-point [resp., finite, compact] subspaces of X we write $\mathsf{SS}^*_1(\mathcal{A},\mathcal{B})$ [resp., $\mathsf{SS}^*_{\rm fin}(\mathcal{A},\mathcal{B})$, $\mathsf{SS}^*_{\rm comp}(\mathcal{A},\mathcal{B})$] instead of $SS^*_{\mathcal{K}}(\mathcal{A},\mathcal{B})$.

1.3. Remark. The following games are naturally corresponded to the selection hypotheses introduced above.

(1) For $\mathsf{S}^*_1(\mathcal{A},\mathcal{B})$ we have the game $\mathsf{G}^*_1(\mathcal{A},\mathcal{B})$ played (on a space X) as follows: Two players, ONE and TWO, play an inning per positive integer. In the n-th inning ONE chooses an $\mathcal{U}_n \in \mathcal{A}$, to which TWO responds by choosing a $U_n \in \mathcal{U}_n$. The play $\mathcal{U}_1, U_1; \dots; \mathcal{U}_n, U_n; \dots$ is won by TWO if $\{{\rm St}(U_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$; otherwise, ONE wins;

(2) The game $\mathsf{G}^*_{\rm fin}(\mathcal{A},\mathcal{B})$ is played similarly, except that in the n-th inning TWO chooses a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. The play $\mathcal{U}_1, \mathcal{V}_1; \dots; \mathcal{U}_n, \mathcal{V}_n; \dots$ is won by TWO if $\bigcup_{n\in\mathbb{N}}\{{\rm St}(V,\mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$; otherwise, ONE wins;

(3) The game $\mathsf{S}G^*_1(\mathcal{A},\mathcal{B})$ is played in the following way: in the n-th inning ONE chooses some $\mathcal{U}_n \in \mathcal{A}$ and TWO responds by choosing a point $x_n \in X$. The play $\mathcal{U}_1, x_1; \dots; \mathcal{U}_n, x_n; \dots$ is won by TWO if $\{{\rm St}(x_n, A_n) : n \in \mathbb{N}\}$ belongs to $\mathcal{B}$; otherwise, ONE wins;

(4) The game $\mathsf{S}G^*_{\rm fin}(\mathcal{A},\mathcal{B})$ is played similarly, except that in the n-th inning TWO chooses a finite subset F_n of X. The play $\mathcal{U}_1, F_1; \dots; \mathcal{U}_n, F_n; \dots$ is won by TWO if $\{{\rm St}(F_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is a member of $\mathcal{B}$; otherwise, ONE wins;

(5) The game $\mathsf{SG}^*_{\mathrm{comp}}(\mathcal{A},\mathcal{B})$ is played as the previous game, but in the n-th inning TWO chooses a compact subset K_n of X. TWO wins if $\{\mathrm{St}(K_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, the play is won by ONE. $\square$

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of topologically significant open covers of a space X:

$\mathcal{O}$ – the collection of all open covers of X;

Ω – the collection of ω-covers of X. An open cover $\mathcal{U}$ of X is an *ω-cover* [4] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$;

Γ – the collection of γ-covers of X. An open cover $\mathcal{U}$ of X is a *γ-cover* [4] if it is infinite and for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

Recall that a space X is said to have the *Menger property* [9], [5], [6] (resp. the *Rothberger property* [11]) if the selection hypothesis $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ (resp. $\mathsf{S}_1(\mathcal{O},\mathcal{O})$) is true for X (see also [10], [7], [12]).

Following this terminology we introduce the following definition.

1.4. Definition. A space X is said to have: (1) the *star-Rothberger property*, (2) the *star-Menger property*, (3) the *strongly star-Rothberger property*, (4) the *strongly star-Menger property*, (5) *star-K-Menger property* if it satisfies the selection hypothesis: (1') $\mathsf{S}^*_1(\mathcal{O},\mathcal{O})$, (2') $\mathsf{S}^*_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, (3') $\mathsf{SS}^*_1(\mathcal{O},\mathcal{O})$, (4') $\mathsf{SS}^*_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, (5') $\mathsf{SS}^*_{\mathrm{comp}}(\mathcal{O},\mathcal{O})$.

2. Relations between star covering properties

We give first the following diagram which illustrates relationships between here defined properties and some other star covering properties, whose definitions can be found in [2]. Most of the implications follow almost directly from the definitions; we give a simple one in Proposition 2.1. Recall that a space X is said to be *strongly starcompact* [*strongly star-Lindelöf*, *star-L-Lindelöf*] if for every open cover $\mathcal{U}$ of X there is a finite [countable, Lindelöf] subset A of X such that $\mathrm{St}(A,\mathcal{U}) = X$. X is *star-compact* [*star-Lindelöf*] if for every open cover $\mathcal{U}$ of X there exists a finite [countable] $\mathcal{V} \subset \mathcal{U}$ such that $\mathrm{St}(\cup\mathcal{V},\mathcal{U}) = X$.

We will also give some examples and assertions in order to compare the properties (and their combinations) from this diagram and to clarify it.

Observe a simple fact that any property from the diagram is an invariant of contunuous mappings and is inherited by closed-and-open subspaces.

$$
\begin{tikzcd}
 & \text{star-Rothberger} \arrow[r, Rightarrow] & \text{star-Menger} \arrow[r, Rightarrow] & \text{star-Lindelöf} & \\
 & & \text{star-}K\text{-Menger} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \text{star-}L\text{-Lindelöf} \arrow[u, Rightarrow] & \\
 & \text{strongly star-Roth.} \arrow[r, Rightarrow] \arrow[uu, Rightarrow] & \text{strongly star-Men.} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \begin{array}{c}\text{strongly}\\ \text{star-Lind.}\end{array} \arrow[u, Rightarrow] & \text{separable} \arrow[l, Rightarrow] \\
 & \text{Rothberger} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \text{Menger} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \text{Lindelöf} \arrow[u, Rightarrow] & \\
\text{starcomp.} \arrow[uuuurr, dashed] & \text{strongly starcomp.} \arrow[l, Rightarrow] \arrow[uur] & \text{compact} \arrow[l, Rightarrow] \arrow[u, Rightarrow] & &
\end{tikzcd}
$$

Diagram 1

2.1. Proposition. *Every (strongly)star-Menger space X is (strongly) star-Lindelöf.*

Proof. Consider only the case when X is a star-Menger space. Let $\mathcal{U}$ be an open cover of X. Then, by definition, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}) = X$. Then $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a countable subfamily of $\mathcal{U}$ satisfying $\mathrm{St}(\cup\mathcal{V}, \mathcal{U}) = X$, i.e. X is a star-Lindelöf space. $\square$

2.2. (Matveev [8]) *Example.* Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω (i.e. the intersection of every two distinct elements of $\mathcal{A}$ is finite) and let $X = \omega \cup \mathcal{A}$ be the Mrówka–Isbell space constructed from $\mathcal{A}$ [3], [2]. Then:

(i) X is strongly star-Menger $\iff |\mathcal{A}| < \mathbf{d}$;

(ii) If $|\mathcal{A}| = \mathbf{c}$, then X is not star-Menger,

where $\mathbf{d}$ is the dominating number (see [2]).

2.3. Example. There is a strongly star-Menger space X which is not strongly starcompact.

Let $X = [0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$ be the Tychonoff plank. It is shown in [2] that X is not strongly starcompact. We prove that X is strongly star-Menger.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $\mathbb{N} = N_1 \cup N_2 \cup \dots$ be a partition of $\mathbb{N}$ into infinitely many finite pairwise disjoint subsets. Fix $n \in \mathbb{N}$. For each $k \in N_n$ there is an $\alpha_k < \omega_1$ such that the set $\{(\beta, k) : \alpha_k < \beta \leq \omega_1\}$is contained in some member U of $\mathcal{U}_n$, which means that for $x_k = (\omega_1, k)$ one has $\mathrm{St}(x_k, \mathcal{U}_n) \supset U$. Let $A_n = \{x_k : k \in N_n\}$, $\alpha_n = \sup\{\alpha_k : k \in N_n\}$ and $\alpha = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\alpha < \omega_1$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U}_n) \supset (\alpha, \omega_1] \times [0, \omega)$.

Further, the subspace $T = [0, \omega_1) \times \{\omega\}$ of X is homeomorphic to $[0, \omega_1)$ and consequently T is strongly star-Menger. Thus there is a sequence $(B_n : n \in \mathbb{N})$ of finite subsets of T such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(B_n, \mathcal{U}_n) \supset T$.

Finally, the subspace $K = [0, \alpha] \times [0, \omega]$ of X is compact and thus strongly star-Menger. There exists a sequence $(C_n : n \in \mathbb{N})$ of finite subsets of K so that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(C_n, \mathcal{U}_n) \supset K$.

For each $n \in \mathbb{N}$ put $F_n = A_n \cup B_n \cup C_n$. Then the sequence $(F_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is a strongly star-Menger space. $\square$

Recall that a space X is said to be *meta-compact* [*meta-Lindelöf*] if every open cover $\mathcal{U}$ of X has a point-finite [point-countable] open refinement $\mathcal{V}$ (i.e., every point of X belongs to at most finitely many [countably many] members of $\mathcal{V}$).

2.4. Theorem. *Every strongly star-Menger metacompact space is a Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of a strongly star-Menger metacompact space X. For every $n \in \mathbb{N}$ let $\mathcal{V}_n$ be a point-finite open refinement of $\mathcal{U}_n$. As X is strongly star-Menger, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{V}_n) = X$. Elements of each F_n belong to finitely many members $V_{n,1}, \dots, V_{n,k(n)}$ of $\mathcal{V}_n$; let $\mathcal{V}'_n = \{V_{n,1}, \dots, V_{n,k(n)}\}$. Then $\mathrm{St}(F_n, \mathcal{V}_n) = \bigcup \mathcal{V}'_n$, so that we have $\bigcup_{n\in\mathbb{N}} \cup\mathcal{V}'_n = X$. For every $V \in \mathcal{V}'_n$ choose a member U_V of $\mathcal{U}_n$ such that $V \subset U_V$. Then, for every n, $\mathcal{W}_n = \{U_V : V \in \mathcal{V}'_n\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n = X$, i.e. X is a Menger space. $\square$

2.5. Theorem. *Every strongly star-Menger meta-Lindelöf space X is Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V}$ be a point-countable refinement of $\mathcal{U}$. Since X is strongly star-Menger there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{V}) = X$. For every $n \in \mathbb{N}$ denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersect F_n. Since $\mathcal{V}$ is point-countable and F_n is finite, $\mathcal{W}_n$ is countable. So the collection $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a countable subfamily of $\mathcal{V}$ and is a cover of X. For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \subset U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$ and X is a Lindelöf space. □

It is known that in the class of Hausdorff spaces strongly starcompactness and countable compactness coincide, so that countable compact Hausdorff spaces are strongly star-Menger. From the previous theorem we obtain the next well known result [1]:

2.6. Corollary. *A countably compact meta-Lindelöf space is compact.*

2.7. Example. There is a strongly star-Menger space which is not Menger.

Let $X = [0, \omega_1)$ be the set of all countable ordinals with the order topology. Since X is a Hausdorff countably compact space, i.e. a strongly starcompact space, it is strongly star-Menger. On the other hand, X cannot have the Menger property because it is even not Lindelöf.

The following theorem gives an information when star-Menger spaces satisfy the Menger property.

2.8. Theorem. *For a paracompact* (*Hausdorff*) *space X the following are equivalent:*

(a) *X is a star-Menger space;*

(b) *X is a star-K-Menger space;*

(c) *X is a strongly star-Menger space;*

(d) *X is a Menger space.*

Proof. We have to prove only that (a) implies (d). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of a paracompact star-Menger space X. By the well known Stone characterization of paracompactness [3] for every

$n \in \mathbb{N}$ let $\mathcal{V}_n$ be an open star-refinement of $\mathcal{U}_n$. Since X is star-Menger there exists a sequence $\{\mathcal{W}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subfamily of $\mathcal{V}_n$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{W}_n, \mathcal{V}_n) = X$. For every $W \in \mathcal{W}_n$ let U_W be a member of $\mathcal{U}_n$ such that $\mathrm{St}(W, \mathcal{V}_n) \subset U_W$. Then $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$ is a finite subfamily of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \cup\mathcal{U}'_n = X$ which means that X is a Menger space. □

In a similar way we obtain

2.9. Theorem. *A paracompact space X is Rothberger iff it is star-Rothberger iff it is strongly star-Rothberger.*

From Theorem 2.5, Theorem 2.8 and the fact that regular Lindelöf spaces are paracompact we have

2.10. Corollary. *A regular strongly star-Menger meta-Lindelöf space is a Menger space.*

Let us observe that the following result is true without any separation axiom.

2.11. Theorem. *A paracompact space X is star-K-Menger if and only if it is a Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let for each $n \in \mathbb{N}$, $\mathcal{V}_n$ be an open locally finite refinement of $\mathcal{U}_n$. Since X is star-K-Menger, there exists a sequence $(K_n : n \in \mathbb{N})$ of compact subspaces of X satisfying $\bigcup_{n\in\mathbb{N}} \mathrm{St}(K_n, \mathcal{V}_n) = X$. The set $\mathcal{V}'_n$ of all members of $\mathcal{V}_n$ which meet K_n is finite because $\mathcal{V}_n$ is locally finite and $\bigcup \mathcal{V}'_n = \mathrm{St}(K_n, \mathcal{V}_n)$. Therefore, $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}'_n = X$. For every $V \in \mathcal{V}'_n$ pick a $U_V \in \mathcal{U}_n$ with $V \subset U_V$ and let $\mathcal{W}_n = \{U_V : V \in \mathcal{V}'_n\}$. Then the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ guarantees that X is a Menger space. □

It is easy to check that the star-Menger property is preserved by countable topological sums. However, the product of two star-Menger spaces need not be star-Menger as simple examples show. The same holds for strongly star-Menger spaces.

2.12. Example. The product of a strongly star-Menger space and a strongly star-Lindelöf space which is not strongly star-Menger.

The ordinal space $X = [0, \omega_1)$ is strongly star-Menger. Let Y be the set $[0, \omega_1]$ with the following topology: for every $\alpha < \omega_1$ the set $\{\alpha\}$ is open; a set containing ω_1 is open iff its complement in Y is countable. Then Y is a Lindelöf space, hence a strongly star-Lindelöf space. The space $X \times Y$ is not strongly star-Menger because it is not strongly star-Lindelöf as it was shown in [2; Ex. 3.3.3].

However, we have the following result.

2.13. Theorem. *If X is a star-Menger (star-Rothberger) space and Y is a compact space, then $X \times Y$ is a star-Menger (star-Rothberger) space.*

Proof. We shall prove the star-Menger case. Let $\{\mathcal{W}_n : n \in \mathbb{N}\}$ be a sequence of open covers of $X \times Y$; without loss of generality one can suppose that every $\mathcal{W}_n$ is a basic open cover of the form $\mathcal{U}_n \times \mathcal{V}_n$, $\mathcal{U}_n$ an open cover of X and $\mathcal{V}_n$ an open cover of Y. For a fixed $x \in X$, each $\mathcal{W}_n$ is an open cover for the compact subspace $\{x\} \times Y$ of $X \times Y$. Therefore, there exists a finite subfamily $\mathcal{U}_{n,x} \times \mathcal{V}_{n,x}$ of $\mathcal{W}_n$ such that $\cup(\mathcal{U}_{n,x} \times \mathcal{V}_{n,x}) \supset \{x\} \times Y$. Let $U_{n,x} = \cap\, \mathcal{U}_{n,x}$. Then $\mathcal{G}_n = \{U_{n,x} : x \in X\}$ is an open cover of X for every $n \in \mathbb{N}$. Since X has the star-Menger property there are finite $\mathcal{H}_n = \{U_{n,x_1}, \ldots, U_{n,x_{k(n)}}\} \subset \mathcal{G}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{H}_n, \mathcal{G}_n) = X$. Denote $\mathcal{W}'_n = (\mathcal{U}_{n,x_1} \times \mathcal{V}_{n,x_1}) \cup \cdots \cup (\mathcal{U}_{n,x_{k(n)}} \times \mathcal{V}_{n,x_{k(n)}})$. We have that for every $n \in \mathbb{N}$, $\mathcal{W}'_n$ is a finite subfamily of $\mathcal{W}_n$ and

$$\bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{W}'_n, \mathcal{W}_n) \supset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\cup\mathcal{H}_n, \mathcal{G}_n) \times Y = X \times Y. \qquad \square$$

Matveev observed in [8] that there is a consistent example of a strongly star-Menger space X whose product with a compact space Y is not strongly star-Menger. By Theorem 2.13 $X \times Y$ is a star-Menger space, so that we have a consistent example of a star-Menger space which is not strongly star-Menger.

We close this section by the following three questions.

2.14. Question. Characterize hereditarily (strongly) star-Menger [(strongly) star-Rothberger] spaces.

2.15. Question. Find out a space X such that all finite powers of X are (strongly) star-Menger [resp. (strongly) star-Rothberger] spaces but X^ω is not. Characterize spaces X such that X^ω (resp. every finite power of X) is (strongly) star-Menger [(strongly) star-Rothberger].

2.16. Question. Let $\mathcal{M}$ be the class of spaces X such that for every (strongly) star-Menger [(strongly) star-Rothberger] space Y the product $X \times Y$ is (strongly) star-Menger [(strongly) star-Rothberger]. Describe the class $\mathcal{M}$.

3. Other properties

In [7; Th. 1.1], it was shown that a Lindelöf space X satisfies $S_1(\Gamma,\Gamma)$ if and only if X satisfies $S_{\text{fin}}(\Gamma,\Gamma)$. Closely following the line of reasoning from the proof of that result we have:

3.1. Theorem. *For a Lindelöf space X we have $S_1^*(\Gamma,\Gamma)=S_{\text{fin}}^*(\Gamma,\Gamma)$.*

Proof. Clearly, $S_1^*(\Gamma,\Gamma)$ implies $S_{\text{fin}}^*(\Gamma,\Gamma)$. Let X satisfies $S_{\text{fin}}^*(\Gamma,\Gamma)$ and let $(\mathcal{U}_n : n\in\mathbb{N})$ be a sequence of γ-covers of X. Suppose that $\mathcal{U}_n=\{U_{n,1},U_{n,2},\dots\}$. We shall define a new sequence $(\mathcal{V}_n : n\in\mathbb{N})$ of γ-covers of X as follows:

$$\mathcal{V}_n=\{V_{n,1},V_{n,2},\dots\},\ \text{where } V_{n,k}=U_{1,k}\cap U_{2,k}\cap\cdots\cap U_{n,k}.$$

We see that $\mathcal{V}_1=\mathcal{U}_1$, $\mathcal{V}_i$ refines $\mathcal{U}_i$ for $i\geq 2$ and $V_{n,k}\subset V_{m,k}$ whenever $n\geq m$. Let us check that every $\mathcal{V}_n$ is a γ-cover for X. Let $x\in X$. For every $i=1,2,\dots,n$ there is some $m_i\in\mathbb{N}$ such that $x\in U_{i,k}$ for all $k>m_i$. If $m_0=\max\{m_1,m_2,\dots,m_n\}$, then $x\in V_{n,k}$ for all $k>m_0$.

Since X satisfies $S_{\text{fin}}^*(\Gamma,\Gamma)$ there exists a sequence $(\mathcal{W}_n : n\in\mathbb{N})$, each $\mathcal{W}_n$ a finite subset of $\mathcal{V}_n$, such that $\{\mathrm{St}(W,\mathcal{V}_n) : W\in\mathcal{W}_n,\ n\in\mathbb{N}\}$ is a γ-cover of X. Now, we use tha fact that every $\{\mathrm{St}(W,\mathcal{V}_n) : W\in\mathcal{W}_n\}$ is finite while $\{\mathrm{St}(W,\mathcal{V}_n) : W\in\mathcal{W}_n,\ n\in\mathbb{N}\}$ is infinite being a γ-cover.

Pick a member $V_{1,k_1}\in\mathcal{W}_1$. Then $X\setminus\mathrm{St}(V_{1,k_1},\mathcal{V}_1)\neq\emptyset$. Take now some $V_{2,k_2}\in\mathcal{W}_2$ such that $\mathrm{St}(V_{2,k_2},\mathcal{V}_2)\neq\mathrm{St}(V,\mathcal{V}_1)$ for all $V\in\mathcal{W}_1$; we can suppose this because of the fact mentioned above. Then $X\setminus(\mathrm{St}(V_{1,k_1},\mathcal{V}_1)\cup\mathrm{St}(V_{2,k_2},\mathcal{V}_2))\neq\emptyset$. We continue this procedure and obtain a sequnce

$$(V_{n,k_n} : n\in\mathbb{N}),\quad V_{n,k_n}\in\mathcal{W}_n$$

such that, by construction, $\{\mathrm{St}(V_{n,k_n},\mathcal{V}_n) : n\in\mathbb{N}\}$ is a γ-cover of X. It is understood, $\{\mathrm{St}(U_{n,k_n},\mathcal{U}_n) : n\in\mathbb{N}\}$ is a γ-cover of X witnessing membership of X to the class $S_1^*(\Gamma,\Gamma)$. □

We need now the following simple lemma taken from [7; L. 3.2].

3.2. Lemma. *If $\mathcal{U}$ is an ω-cover of a space X, then $\{U^2 : U\in\mathcal{U}\}$ is an ω-cover of X^2.*

3.3. Theorem. *If every finite power of a space X satisfies $\mathsf{S}_1^*(\Omega,\mathcal{O})$, then X satisfies $\mathsf{S}_1^*(\Omega,\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X. Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots \cup N_n \cup \dots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For every $i \in \mathbb{N}$ and every $j \in N_i$ let $\mathcal{V}_j = \{U^i : U \in \mathcal{U}_j\}$. According to Lemma 3.2, for every $i \in \mathbb{N}$, the sequence $(\mathcal{V}_j : j \in N_i)$ is a sequence of ω-covers of X^i. By assumption, for every $i \in \mathbb{N}$ one can choose a sequence $({U_j}^i : j \in N_i)$ so that for each j, $U_j \in \mathcal{U}_j$ and $\{\mathrm{St}({U_j}^i,\mathcal{V}_j) : j \in N_i\}$ is an open cover for X^i.

We shall prove that $\{\mathrm{St}(U_j,\mathcal{U}_j) : j \in \mathbb{N}\}$ is an ω-cover for X which witnesses that X satisfies $\mathsf{S}_1^*(\Omega,\Omega)$. Indeed, let $A = \{a_1, a_2, \dots, a_p\}$ be a finite subset of X. Then $(a_1, a_2, \dots, a_p) \in X^p$ so that there is some $k \in N_p$ such that $(a_1, a_2, \dots, a_p) \in \mathrm{St}({U_k}^p, \mathcal{V}_k)$; it is clear that $A \subset \mathrm{St}(U_k,\mathcal{U}_k)$. $\square$

In a similar way one may prove:

3.4. Theorem. *If every finite power of a space X is a star-Menger space, then X satisfies $\mathsf{S}_{\mathrm{fin}}^*(\Omega,\Omega)$.*

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