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A note on the influence of minimal subgroups on the structure of finite groups

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Abstract. This paper studies the influence of the subgroups of prime order or order 4 of the focal subgroups on the structure of a finite group. It is proved that a group G belongs to a saturated formation containing the supersoluble groups if there exists a normal subgroup N of G such that G/N lies in the formation and the subgroups of prime order or order 4 of the focal subgroups of the Sylow subgroups of N are normal in the corresponding normalizers of the Sylow subgroups.

In this paper it is understood that all groups are finite.

Recall that if G is a group and P is a Sylow p-subgroup of G for a prime number p, then the subgroup $P \cap G'$ is called the focal subgroup of P with respect to G.

The main object of the present article is to study the influence of the subgroups of prime order or order 4 of the focal subgroups on the structure of the groups. It is a part of a project which studies the influence of the minimal subgroups on the structure of the groups (see Introduction in [1].)

Our main result is the following:

Theorem A. *Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{U}$ of supersoluble groups. Let N be a normal subgroup of a group G such that G/N belongs to $\mathcal{F}$. If for every Sylow subgroup P of N, every subgroup of prime order or order 4 of $P \cap G'$ is normal in $N_G(P)$, then G belongs to $\mathcal{F}$.*

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Recall that a formation is a class of groups which is closed under epimorphic images and subdirect products. A formation $\mathcal{F}$ is saturated if $G/\Phi(G) \in \mathcal{F}$ implies that G belongs to $\mathcal{F}$.

The proof of our main result depends heavily on the following lemmas which are of independent interest.

Lemma 1 [3; Theorem 1]**.** *Let P be a Sylow p-subgroup of a group G. If every subgroup of prime order or order 4 of $P \cap G'$ is contained in the center of $N_G(P)$, then G is p-nilpotent.*

Lemma 2. *Let G be a group of odd order. If for each Sylow subgroup P of G, every subgroup of prime order of $P \cap G'$ is normal in $N_G(P)$, then G has a Sylow tower of supersoluble type.*

Proof. We use induction on $|G|$. Let q be the smallest prime dividing $|G|$ and let Q be a Sylow q-subgroup of G. If $Q \cap G' = 1$, then G is q-nilpotent by Lemma 1. Suppose that $Q \cap G' \neq 1$ and let x be an element of order q in $Q \cap G'$. Then, by hypothesis, $\langle x \rangle$ is a normal subgroup of $T = N_G(Q)$. So $T/C_T(\langle x \rangle)$ is isomorphic to a subgroup of $\mathrm{Aut}(\langle x \rangle)$ which is of order $q-1$. Since q is the smallest prime dividing $|T|$, it follows that $\langle x \rangle \leq Z(T)$. Consequently every subgroup of prime order of $Q \cap G'$ is contained in $Z(N_G(Q))$. By Lemma 1, G is q-nilpotent. Let K be a normal Hall q'-subgroup of G. Then it is clear that K satisfies the hypotheses of the lemma. By induction, K has a Sylow tower of supersoluble type and so does G. The proof of the lemma is now complete.

The next lemma analyzes the case $p = 2$.

Lemma 3. *Let P be a Sylow 2-subgroup of a group G. If every subgroup of order 2 and 4 of $P \cap G'$ is normal in $N_G(P)$, then G is 2-nilpotent.*

Proof. Assume the result is false and choose for G a group of smallest order. Then G is not 2-nilpotent and so G has a subgroup K such that K is not 2-nilpotent but every proper subgroup of K is 2-nilpotent. According to a result due to Schmidt ([4; 9.1.9]), K has a normal Sylow 2-subgroup K_2 and $K = K_2K_p$ for a Sylow p-subgroup K_p of K, $p \neq 2$. Moreover K_2 is of exponent 2 or 4 and $K_2 = [K_2, K_p]$. Without loss of generality we can assume that K_2 is contained in P. So K_2 is really contained in $P \cap G'$. Notice that every subgroup of K_2 is normal in $N_G(P)$. Therefore $\Omega_1(K_2)$ is centralized by $N_G(P)$. Denote by $T = N_G(\Omega_1(K_2))$.

Then $< K, N_G(P) >$ is a subgroup of T. It is clear that the hypotheses of the lemma hold in T. So if T were a proper subgroup of G, we would have that T would be 2-nilpotent. So K would be 2-nilpotent, a contradiction. Therefore $T = G$ and so $1 \neq \Omega_1(K_2)$ is a normal subgroup of G. Since $N_G(P)$ is contained in $C_G(\Omega_1(K_2))$ and $C_G(\Omega_1(K_2))$ is a normal subgroup of G, it follows that $G = C_G(\Omega_1(K_2))$ and $\Omega_1(K_2)$ is really contained in $Z(G)$. In particular, $K_2 \neq \Omega_1(K_2)$ because K_p does not centralize K_2. This means that K_2 is of exponent 4. Let $\bar{G} = G/\Omega_1(K_2)$ and denote with bars the images in $\bar{G}$. Then $\bar{K}_2$ is of exponent 2 and $\bar{K}$ is a minimal non-2-nilpotent group. Let $\bar{a} \in \bar{K}_2$. Then $\bar{a} = a\Omega_1(K_2)$ for some $a \in K_2$ and $o(a) = 4$. By hypothesis, $\langle a \rangle$ is a normal subgroup of $N_G(P)$. So $\langle \bar{a} \rangle$ is a normal subgroup of $N_{\bar{G}}(\bar{P})$ and $N_{\bar{G}}(\bar{P})$ centralizes $\langle \bar{a} \rangle$. In particular, $\bar{P}$ centralizes $\bar{K}_2$.

Suppose that $N_G(P)$ is a proper subgroup of G. Then $N_G(P)$ is 2-nilpotent by minimality of G. This implies that $N_{\bar{G}}(\bar{P})$ is 2-nilpotent. Denote $\bar{A} = N_{\bar{G}}(\bar{K}_2)$. Then $\bar{A} = C_{\bar{G}}(\bar{K}_2)N_{\bar{A}}(\bar{P})$ since $\bar{P}$ is a Sylow 2-subgroup of $C_{\bar{G}}(\bar{K}_2)$. Since $N_{\bar{A}}(\bar{P})$ is 2-nilpotent, it follows that $N_{\bar{A}}(\bar{P}) = \bar{P} \times \bar{B}$, for a Hall $2'$-subgroup $\bar{B}$ of $N_{\bar{A}}(\bar{P})$. Therefore $\bar{A} = C_{\bar{G}}(\bar{K}_2)$ and $\bar{K}$ is contained in $C_{\bar{G}}(\bar{K}_2)$, a contradiction. Consequently P is a normal subgroup of G. This means that $\bar{G}$ centralizes $\bar{K}_2$, final contradiction.

Corollary 1. *If for every Sylow subgroup P of a group G, every subgroup of prime order or order 4 of $P \cap G'$ is normal in $N_G(P)$, then G has a Sylow tower of supersoluble type.*

Lemma 4 [1; Lemma 2]**.** *Let $\mathcal{F}$ be a saturated formation. Assume that G is a group such that G does not belong to $\mathcal{F}$ and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = MF(G)$. Then $G^{\mathcal{F}}$ is a p-group for some prime p, $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G.*

Here $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G, that is, the intersection of all normal subgroups N of G such that $G/N \in \mathcal{F}$.

Proof of Theorem A. Assume that the result is false and let G be a counterexample of minimal order. Among the normal subgroups H of G satisfying the hypotheses of the theorem, we choose N with $|N|$ minimal. By the above corollary, N has a Sylow tower of supersoluble type. So if p is the largest prime dividing $|N|$ and P is a Sylow p-subgroup of N, we have that P is a normal subgroup of G. Denote with bars the images in

$\bar{G} = G/P$. Then $\bar{G}$ has a normal subgroup $\bar{N}$ such that $\bar{G}/\bar{N}$ belongs to $\mathcal{F}$. Now if $\bar{Q}$ is a Sylow q-subgroup of $\bar{N}$, then $p \neq q$ and there exists a Sylow q-subgroup Q of G such that $\bar{Q} = QP/P$. Moreover $\bar{G}' = G'P/P$. Let $\bar{a}$ be an element of order q or 4 in $\bar{Q} \cap \bar{G}'$. Then $\bar{a} = aP$ for some element $a \in Q \cap G'$. By hypothesis, $\langle a \rangle$ is normal in $N_G(Q)$. So $\langle \bar{a} \rangle$ is a normal subgroup of $N_{\bar{G}}(\bar{Q})$. Therefore $\bar{G}$ satisfies the hypotheses of the theorem. The minimal choice of G yields $\bar{G} \in \mathcal{F}$ and by minimality of N it follows that $N = P$. This implies that every subgroup of prime order or order 4 of $P \cap G'$ is normal in G.

Assume that G does not belong to $\mathcal{F}$. Then $1 \neq G^{\mathcal{F}}$ is contained in $P \cap G'$ and so $G^{\mathcal{F}}$ is a p-group. By [2; Theorem 3.5], G has a maximal subgroup M such that $G = MF'(G)$, where $F'(G) = \operatorname{Soc}(G \bmod \Phi(G))$ and $G/\operatorname{Core}_G(M)$ does not belong to $\mathcal{F}$. Then $G = MG^{\mathcal{F}}$ and $G = MF(G)$ because $G^{\mathcal{F}}$ is a p-group. It is clear that M satisfies the hypotheses of the theorem for its normal subgroup $M \cap P$. So the minimal choice of G yields $M \in \mathcal{F}$.

By Lemma 4, $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. In both cases, we have that every subgroup of prime order or order 4 of $G^{\mathcal{F}}$ is normal in G. This implies that $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group of prime order. Since $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is G-isomorphic to $\operatorname{Soc}(G/\operatorname{Core}_G(M))$, it follows that $G/\operatorname{Core}_G(M)$ is supersoluble, a contradiction.

References

[1] M. Asaad, A. Ballester-Bolinches and M. C. Pedraza-Aguilera, A note on minimal subgroups of finite groups, *Comm. in Algebra* **24** no. 8 (1996), 2771–2776.

[2] A. Ballester-Bolinches, $\mathcal{H}$-normalizers and local definitions of saturated formations of finite groups, *Israel J. Math.* **67** (1989), 312–326.

[3] A. Ballester-Bolinches and Guo Xiuyun, Some results on p-nilpotence and solubility of finite groups, (*preprint*).

[4] D. J. S. Robinson, A course in the Theory of Groups, *Springer-Verlag*, *Berlin, Heidelberg, New York*, 1982.

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