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L_1-convergence and strong summability of Hankel transforms

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Abstract. In this paper we analyse L_1-convergence and strong summability of Hankel transforms.

1. Introduction

As usual, we define the Hankel transform $h_\mu f$ of a measurable function f on $(0,\infty)$ by

$$h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(x) x^{2\mu+1} dx,$$

where J_μ denotes the Bessel function of the first kind and order μ. We assume here that $\mu > -1/2$.

In this paper we study pointwise convergence and strong summability of Hankel transforms.

We consider, for every $1 \le p \le \infty$, the space $L_{p,\mu}$ that consists of all those complex valued and measurable functions f on $(0,\infty)$ such that

$$\|f\|_{p,\mu} = \left\{ \int_0^\infty |f(x)|^p x^{2\mu+1} dx \right\}^{1/p} < \infty, \quad \text{when } 1 \le p < \infty,$$

and

$$\|f\|_\infty = \operatorname{ess} \sup_{x \in (0,\infty)} |f(x)| < \infty, \qquad \text{when } p = \infty.$$

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$L_{p,\mu}$ is endowed with the topology associated to $\|\cdot\|_{p,\mu}$, $1 \le p \le \infty$.

C.S. Herz [12] established that the Hankel transform h_μ defines a bounded operator from $L_{p,\mu}$ into $L_{p',\mu}$, provided that $1 \le p \le 2$. Here p' denotes the conjugate of p, that is, $p' = \frac{p}{p-1}$.

The partial Hankel integral $S_T(f,\mu;\cdot)$ is defined by

$$S_T(f,\mu;x) = \int_0^T (xy)^{-\mu} J_\mu(xy) y^{2\mu+1} h_\mu(f)(y)dy, \quad x, T \in (0,\infty),$$

for every $f \in L_{p,\mu}$, $1 \le p \le 2$. The definition of the operator S_T can be extended to $L_{p,\mu}$, when $2 < p < \frac{4(\mu+1)}{2\mu+1}$, by using the Hankel convolution (see [4]). Numerous authors (see [6], [14] and [15], amongst others) have investigated the pointwise convergence of $S_T(f,\mu;x)$ to $f(x)$, as $T \to \infty$. In [1] and [2] we give necessary and sufficient conditions in order that

$$\lim_{T\to\infty} S_T(f,\mu;x) = f(x), \quad \text{a.e. } x \in (0,\infty). \tag{1}$$

In particular, we prove that if $x^{-\mu-1/2}f$ and $x^{-\mu-1/2}h_\mu(f)$ are in $L_{1,\mu}$ then (1) holds ([2, Theorem 3.1]).

The first objective on this paper is to recover f from $h_\mu(f)$ by means of L_1-convergence. As Corollary 2.1 shows, in general $S_T(f,\mu;\cdot) \notin L_{1,\mu}$, $T \in (0,\infty)$, when $f \in L_{1,\mu}$. Hence it makes no sense to think about the convergence of $S_T(f,\mu;\cdot)$, as $T \to \infty$, in $L_{1,\mu}$. In Section 2 we obtain necessary and sufficient conditions on f in order that the following

$$\lim_{T\to\infty} R_T(f,\mu;\cdot) = f$$

holds, when the limit is understood in $L_{1,\mu}$ and where

$$R_T(f,\mu;x) = -\int_0^T (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+2} \frac{d}{dy}(h_\mu(f)(y))dy,$$
$$T, x \in (0,\infty).$$

Note that, according to [19, §5.1 (7)], if $f \in L_{1,\mu} \cap L_{1,\mu+1}$ then $R_T(f,\mu;x) = S_T(f,\mu+1;x)$, $x, T \in (0,\infty)$.

I.I. Hirschman [13], D.T. Haimo [11] and F.M. Cholewinski [5] investigated the convolution operation for the Hankel transformation. Let

f, g be measurable functions on $(0,\infty)$. The Hankel convolution $f\#g$ of f and g is defined by

$$(f\#g)(x) = \int_0^\infty f(y)(\tau_x g)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy, \qquad \text{a.e. } x \in (0,\infty),$$

where the Hankel translation τ_x, $x \in (0,\infty)$, is defined through

$$(\tau_x g)(y) = \int_0^\infty g(z) D_\mu(x,y,z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dz, \quad \text{a.e. } y \in (0,\infty),$$

and being

$$D(x,y,z) = \frac{2^{3\mu-1}\Gamma(\mu+1)^2}{\Gamma(\mu+1/2)\sqrt{\pi}} (xyz)^{-2\mu} \Delta(x,y,z)^{2\mu-1}, \quad x,y,z \in (0,\infty),$$

where $\Delta(x,y,z)$ represents the area of the triangle having sides with lengths x, y and z, when such a triangle exists, and $\Delta(x,y,z) = 0$, otherwise.

We established in [4] that the definition of the partial Hankel integral S_T can be written through the Hankel convolution as follows

$$S_T(f,\mu;\cdot) = f\#\varphi_T,$$

for every $T \in (0,\infty)$, $1 \le p \le 2$ and $f \in L_{p,\mu}$, where $\varphi_T(x) = T^{2\mu+2}(xT)^{-\mu-1} J_{\mu+1}(xT)$, $x, T \in (0,\infty)$.

Motivated by the paper of D.V. Giang and F. Móricz [8] in Section 3 we analyse the strong summability of the Hankel transforms.

Let $q > 0$. We say that the Hankel transform of $f \in L_{1,\mu}$ is strongly summable of exponent q in $x \in (0,\infty)$ when

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |S_\nu(f,\mu;x) - f(x)|^q d\nu = 0. \tag{2}$$

By using Hölder's inequality we conclude that if (2) holds then it also holds when q is replaced by $r \in (0,q)$. Moreover, if $x \in (0,\infty)$ and $S_T(f,\mu;x) \longrightarrow f(x)$, as $T \to \infty$, then (2) holds for every $q > 0$. Hence strong summability is weaker than pointwise convergence.

Throughout this paper C will always denote a positive constant not necessarily the same in each ocurrence.

2. L_1-convergence of Hankel transforms

In this section, inspired in the paper of D.V. Giang and F. Móricz [10], we give conditions that allow to recover a function $f \in L_{1,\mu}$ from $h_\mu(f)$ by means of convergence in the space $L_{1,\mu}$.

Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous in $(0,\infty)$. According to [19, §5.1 (6)] a partial integration leads to

$$\begin{aligned} S_T(f,\mu;x) &= \int_0^T (xy)^{-\mu} J_\mu(xy) y^{2\mu+1} h_\mu(f)(y)dy \\ &= x^{-2\mu-2}\left\{ (xy)^{\mu+1} J_{\mu+1}(xy) h_\mu(f)(y) \Big]_0^T \right. \\ &\qquad \left. - \int_0^T (xy)^{\mu+1} J_{\mu+1}(xy) \frac{d}{dy}(h_\mu(f)(y))dy \right\} \end{aligned}$$

for every $x,T \in (0,\infty)$. Since $f \in L_{1,\mu}$, $h_\mu(f)$ is a bounded function on $(0,\infty)$. Hence, since $z^{-\nu}J_\nu(z)$ is bounded on $(0,\infty)$, we have

$$\lim_{y\to 0^+} (xy)^{\mu+1} J_{\mu+1}(xy) h_\mu(f)(y) = 0, \quad x \in (0,\infty).$$

Then, it follows that

$$S_T(f,\mu;x) = x^{-2\mu-2}(xT)^{\mu+1} J_{\mu+1}(xT) h_\mu(f)(T) + R_T(f,\mu;x), \quad x,T \in (0,\infty), \tag{3}$$

where $R_T(f,\mu;x) = -\int_0^T (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+2} \frac{d}{dy}(h_\mu(f)(y))dy$, $x,T \in (0,\infty)$. Note that if $f \in L_{1,\mu} \cap L_{1,\mu+1}$ then $h_\mu(f)$ is absolutely continuous on $(0,\infty)$ and according to [19, §5.1 (7)], $R_T(f,\mu;x) = S_T(f,\mu+1;x)$, $x,T \in (0,\infty)$.

We will obtain necessary and sufficient conditions in order that the following

$$\lim_{T\to\infty} R_T(f,\mu;\cdot) = f,$$

holds, in the sense of convergence in $L_{1,\mu}$.

Previously we need to establish some results.

Lemma 2.1. *Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous on $(0,\infty)$. Then*

$$R_T(f,\mu;x) - \sigma_T(f,\mu;x) = \frac{\lambda^2}{\lambda^2-1}(\sigma_{\lambda T}(f,\mu;x) - \sigma_T(f,\mu;x))$$
$$-\frac{2}{(\lambda^2-1)T^2}\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)y^{2\mu+3}h_\mu(f)(y)dy + \tau_T(f,\mu,\lambda;x),$$
$$x,T \in (0,\infty) \text{ and } \lambda > 1,$$

where $\sigma_T(f,\mu;\cdot)$, $T \in (0,\infty)$, denotes the Bochner–Riesz mean of f, that is,

$$\sigma_T(f,\mu;x) = \int_0^T (xy)^{-\mu}J_\mu(xy)\left(1-\left(\frac{y}{T}\right)^2\right)h_\mu(f)(y)y^{2\mu+1}dy,$$
$$x,T \in (0,\infty),$$

and

$$\tau_T(f,\mu,\lambda;x) = \frac{\lambda^2}{\lambda^2-1}\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)\left(1-\left(\frac{y}{\lambda T}\right)^2\right)y^{2\mu+2}$$
$$\times \frac{d}{dy}(h_\mu(f)(y))dy,$$

for every $x,T \in (0,\infty)$ and $\lambda > 1$.

Proof. Let $x,T \in (0,\infty)$ and $\lambda > 1$. According to [2, Lemma 2.2] and by (3) we can write

$$R_T(f,\mu;x) - \sigma_T(f,\mu;x) = S_T(f,\mu;x) - \sigma_T(f,\mu;x)$$
$$- x^{-2\mu-2}(xT)^{\mu+1}J_{\mu+1}(xT)h_\mu(f)(T)$$
$$= \frac{\lambda^2}{\lambda^2-1}[\sigma_{\lambda T}(f,\mu;x) - \sigma_T(f,\mu;x)]$$
$$- \frac{\lambda^2}{\lambda^2-1}\int_T^{\lambda T} y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\left(1-\left(\frac{y}{\lambda T}\right)^2\right)h_\mu(f)(y)dy$$
$$- x^{-2\mu-2}(xT)^{\mu+1}J_{\mu+1}(xT)h_\mu(f)(T).$$

Now a partial integration in the last integral allows to conclude the desired equality. □

The following lemma is analogous to the one presented in [9, Lemma 2].

Lemma 2.2. *Let $p \in (3/2, 2]$ and $-1/2 < \mu < p-2$. Assume that $f \in L_{p,\mu}$. Then*

$$\int_0^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y)dy \right| x^{2\mu+1}dx$$
$$\leq Cb(\lambda) \left\{ \frac{1}{T^{2\mu+2}(\lambda^{2\mu+2}-1)} \int_T^{\lambda T} |f(y)|^p y^{2\mu+1}dy \right\}^{1/p},$$
$$T \in (0,\infty) \text{ and } \lambda > 1,$$

where b is a continuous function on $(1,\infty)$ and $b(\lambda) \longrightarrow 0$, as $\lambda \to 1^+$. Here the constant C only depends on p and μ.

Proof. Let $T \in (0,\infty)$ and $\lambda > 1$. Firstly we split the integral in the left side of (4) as follows

$$\int_0^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y)dy \right| x^{2\mu+1}dx = I_1 + I_2,$$

where

$$I_1 = \int_0^{1/T} \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y)dy \right| x^{2\mu+1}dx$$

and

$$I_2 = \int_{1/T}^\infty \left| \frac{1}{T^2} \int_T^{\lambda T} (xy)^{-\mu-1} J_{\mu+1}(xy) y^{2\mu+3} f(y)dy \right| x^{2\mu+1}dx.$$

We now analyse I_1. By taking into account that $z^{-\eta}J_\eta(z)$, $\eta \geq -1/2$, is bounded on $(0,\infty)$ we can write that

$$I_1 \leq C\lambda^2 \int_0^{1/T} x^{2\mu+1} \int_T^{\lambda T} |f(y)| y^{2\mu+1}dydx = C\frac{\lambda^2}{T^{2\mu+2}} \int_T^{\lambda T} |f(y)|y^{2\mu+1}dy$$
$$\leq C\lambda^2 (\lambda^{2\mu+2}-1)^{1/p'} \left\{ \frac{1}{T^{2\mu+2}} \int_T^{\lambda T} |f(y)|^p y^{2\mu+1}dy \right\}^{1/p}.$$

On the other hand, by virtue of Hausdorff–Young's inequality for the Hankel transform ([12, Theorem 3]), we have

$$
\begin{aligned}
I_2 &= \frac{1}{T^2}\int_{1/T}^{\infty}\left|\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)y^{2\mu+3}f(y)dy\right|x^{-2}x^{2\mu+3}dx\\
&\le \frac{1}{T^2}\left\{\int_{1/T}^{\infty}x^{2\mu+3-2p}dx\right\}^{1/p}\\
&\quad\times\left\{\int_{1/T}^{\infty}x^{2\mu+3}\left|\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)y^{2\mu+3}f(y)dy\right|^{p'}dx\right\}^{1/p'}\\
&\le CT^{-2(\mu+2)/p}\left\{\int_T^{\lambda T}|f(y)|^p y^{2\mu+3}dy\right\}^{1/p}\\
&\le C\lambda^{2/p}\left\{\frac{1}{T^{2\mu+2}}\int_T^{\lambda T}|f(y)|^p y^{2\mu+1}dy\right\}^{1/p},
\end{aligned}
$$

provided that $\mu < p-2$.

Hence, we conclude that

$$
\begin{aligned}
&\int_0^{\infty}\left|\frac{1}{T^2}\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)y^{2\mu+3}f(y)dy\right|x^{2\mu+1}dx\\
&\qquad\le Cb(\lambda)\left\{\frac{1}{T^{2\mu+2}(\lambda^{2\mu+2}-1)}\int_T^{\lambda T}|f(y)|^p y^{2\mu+1}dy\right\}^{1/p},
\end{aligned}
$$

where $b(\lambda)=\lambda^2(\lambda^{2\mu+2}-1)+\lambda^{2/p}(\lambda^{2\mu+2}-1)^{1/p}$. □

We now characterize the convergence of $R_T(f,\mu;\cdot)$ to f, as $T\to\infty$, in $L_{1,\mu}$.

Proposition 2.1. *Let $-1/2<\mu<0$. Assume that $f\in L_{1,\mu}$, $x^{-\mu-1/2}f\in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous on $(0,\infty)$. Then*

$$R_T(f,\mu;\cdot)\longrightarrow f,\quad \text{as } T\to\infty,$$

in $L_{1,\mu}$ if and only if

$$\lim_{\lambda\to 1^+}\limsup_{T\to\infty}\int_0^{\infty}|\tau_T(f,\mu,\lambda;x)|x^{2\mu+1}dx=0. \tag{5}$$

Here $\tau_T(f,\mu,\lambda;x)$ is defined as in Lemma 2.1.

Proof. According to Lemma 2.1 we can write

$$\left|\int_0^\infty |R_T(f,\mu;x)-\sigma_T(f,\mu;x)|x^{2\mu+1}dx-\int_0^\infty |\tau_T(f,\mu,\lambda;x)|x^{2\mu+1}dx\right|$$
$$\leq \frac{\lambda^2}{\lambda^2-1}\int_0^\infty |\sigma_{\lambda T}(f,\mu;x)-\sigma_T(f,\mu;x)|x^{2\mu+1}dx$$
$$+\frac{2}{(\lambda^2-1)T^2}\int_0^\infty \left|\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)y^{2\mu+3}h_\mu(f)(y)dy\right|x^{2\mu+1}dx,$$
$$T\in(0,\infty) \text{ and } \lambda>1.$$

It is well-known that $\sigma_T(f,\mu;\cdot)\longrightarrow f$, as $T\to\infty$, in $L_{1,\mu}$, provided that $\mu\in(-1/2,1/2)$ ([7, p. 38]). Hence, for every $\lambda>1$,

$$\int_0^\infty |\sigma_{\lambda T}(f,\mu;x)-\sigma_T(f,\mu;x)|x^{2\mu+1}dx\longrightarrow 0, \quad \text{as } T\to\infty.$$

Moreover, since $f\in L_{1,\mu}$, $h_\mu(f)$ is bounded on $(0,\infty)$ and then $h_\mu(f)\in L_{p,\mu}(a,b)$, for every $0<a<b<\infty$ and $1\leq p<\infty$. Hence, from Lemma 2.2 we deduce, for $-1/2<\mu<0$,

$$\int_0^\infty \left|\frac{1}{T^2}\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)y^{2\mu+3}h_\mu(f)(y)dy\right|x^{2\mu+1}dx$$
$$\leq Cb(\lambda)\left\{\frac{1}{T^{2\mu+2}(\lambda^{2\mu+2}-1)}\int_T^{\lambda T}|h_\mu(f)(y)|^2y^{2\mu+1}dy\right\}^{1/2} \tag{6}$$
$$\leq Cb(\lambda)\sup_{y\geq T}|h_\mu(f)(y)|, \quad T\in(0,\infty) \text{ and } \lambda>1.$$

Here b is continuous on $(1,\infty)$ and $b(\lambda)\longrightarrow 0$, as $\lambda\to 1^+$ (see Lemma 2.2).

According now to Riemann–Lebesgue Lemma for Hankel transform [18, p. 457] from (6) we infer that

$$\int_0^\infty\left|\frac{1}{T^2}\int_T^{\lambda T}(xy)^{-\mu-1}J_{\mu+1}(xy)y^{2\mu+3}h_\mu(f)(y)dy\right|x^{2\mu+1}dx\to 0, \quad \text{as } T\to\infty,$$

uniformly when $1 < \lambda \leq a$, for every $a > 1$.

Thus we conclude that

$$\lim_{T\to\infty} \int_0^\infty |R_T(f,\mu;x) - \sigma_T(f,\mu;x)| x^{2\mu+1} dx = 0 \tag{7}$$

if and only if

$$\lim_{\lambda\to 1} \limsup_{T\to\infty} \int_0^\infty |\tau_T(f,\mu,\lambda;x)| x^{2\mu+1} dx = 0.$$

To finish the proof it is sufficient to take into account that, since $\sigma_T(f,\mu;\cdot) \longrightarrow f$, as $T \to \infty$, in $L_{1,\mu}$, (7) is equivalent to $R_T(f,\mu;\cdot) \longrightarrow f$, as $T \to \infty$, in $L_{1,\mu}$. $\square$

A consequence of Proposition 2.1 is the following one.

Corollary 2.1. *Let $-1/2 < \mu < 0$. Assume that $f \in L_{1,\mu}$, $x^{-\mu-1/2} f \in L_{1,\mu}$, that $h_\mu(f)$ is absolutely continuous on $(0,\infty)$ and that (5) holds. If $T \in (0,\infty)$ and $h_\mu(f)(T) \neq 0$, then $S_T(f,\mu;\cdot) \notin L_{1,\mu}$.*

Proof. It is well-known that [18, p. 199]

$$\sqrt{t} J_\mu(t) = \cos(t+\alpha) + O\left(\frac{1}{t}\right), \quad \text{as } t \to \infty,$$

for a certain $\alpha \in \mathbb{R}$. Hence, $t^{-\mu-1} J_{\mu+1}(t) \notin L_{1,\mu}$, when $-1/2 < \mu < 0$. By (3) the result follows. $\square$

According to Proposition 2.1 we can find sufficient conditions in order that $R_T(f,\mu;\cdot) \longrightarrow f$, as $T \to \infty$, in $L_{1,\mu}$.

Proposition 2.2. *Let $-1/2 < \mu < p-2$, with $p \in (3/2, 2]$. Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is absolutely continuous on $(0,\infty)$. If we have*

$$\lim_{\lambda\to 1^+} \limsup_{T\to\infty} \int_T^{\lambda T} y^{p-1} \left| \frac{d}{dy} h_\mu(f)(y) \right|^p dy < \infty, \tag{8}$$

then (5) is satisfied.

Proof. Choose $r \in (3/2, p)$ such that $-1/2 < \mu < r-2$. From Lemma 2.2 it infers that

$$\int_0^\infty |\tau_T(f,\mu,\lambda;x)| x^{2\mu+1} dx \le C \frac{\lambda^2}{\lambda^2-1} T^2 b(\lambda)$$

$$\times \left\{ \frac{1}{T^{2\mu+2}(\lambda^{2\mu+2}-1)} \int_T^{\lambda T} \left(1 - \left(\frac{y}{\lambda T}\right)^2\right)^r \left| \left(\frac{1}{y}\frac{d}{dy}\right) h_\mu(f)(y) \right|^r y^{2\mu+1} dy \right\}^{\frac{1}{r}},$$

$$T \in (0,\infty) \text{ and } \lambda > 1,$$

where $b(\lambda) = \lambda^2(\lambda^{2\mu+2}-1) + \lambda^{2/r}(\lambda^{2\mu+2}-1)^{1/r}$, $\lambda > 1$.

Hence we conclude after straightforward manipulations that, for every $T \in (0,\infty)$ and $\lambda > 1$,

$$\begin{aligned} &\int_0^\infty |\tau_T(f,\mu,\lambda;x)| x^{2\mu+1} dx \\ &\le C \frac{b(\lambda)}{(\lambda^{2\mu+2}-1)^{1/r}} \left\{ \int_T^{\lambda T} y^{r-1} \left| \frac{d}{dy} h_\mu(f)(y) \right|^r dy \right\}^{1/r}. \end{aligned} \tag{9}$$

By using Hölder's inequality it is not hard to see that (8) and (9) imply (5) holds. □

An immediate consequence of Proposition 2.2 is the following.

Corollary 2.2. *Let $-1/2 < \mu < 0$. Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)$ is ablutely continuous on $(0,\infty)$. If $y\frac{d}{dy}h_\mu(f)(y) = O(1)$, as $y \to \infty$, then (5) holds.*

3. Strong summability of Hankel transforms

In this section we study the strong summability of Hankel transforms.

Firstly we define the h_μ-Lebesgue points for a function $f \in L_{p,\mu}$ as follows. Let $1 \le p < \infty$ and $f \in L_{p,\mu}$. We will say that $x \in (0,\infty)$ is a h_μ-Lebesgue point for f of order p (to simplify $x \in HL^\mu(f,p)$) if and only if

$$\lim_{t \to 0^+} \frac{1}{t^{2\mu+2}} \int_0^t |(\tau_y f)(x) - f(x)|^p y^{2\mu+1} dy = 0.$$

It is clear that $HL^\mu(f,p)$ contains to $HL^\mu(f,q)$ provided that $1 \leq p < q < \infty$ and $f \in L_{p,\mu} \cap L_{q,\mu}$.

We now prove that $HL^\mu(f,p)$ is full in $(0,\infty)$.

Proposition 3.1. *Let $1 \leq p < \infty$ and $f \in L_{p,\mu}$. Then the Lebesgue measure of the set $(0,\infty) \setminus HL^\mu(f,p)$ is zero.*

Proof. Let $t > 0$. Define the set

$$A_t = \Big\{ x \in (0,\infty) : \limsup_{\epsilon \to 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y f)(x) - f(x)|^p y^{2\mu+1} dy > t \Big\}.$$

The proof will be finish when we prove that $\gamma(A_t) = 0$, where $d\gamma = x^{2\mu+1}dx$.

Let $\epsilon > 0$. We can write $f = g + h$ where g is a smooth function having compact support in $(0,\infty)$ and $\|h\|_{p,\mu} < \epsilon$. By proceeding as in the proof of [3, Proposition 2.1] we can obtain

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y g)(x) - g(x)|^p y^{2\mu+1} dy = 0, \quad x \in (0,\infty).$$

Hence by invoking Jensen's inequality and [13, §2 (2)] we obtain

$$\begin{aligned}
&\limsup_{\epsilon \to 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y f)(x) - f(x)|^p y^{2\mu+1} dy \\
&\qquad \leq \limsup_{\epsilon \to 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon |(\tau_y h)(x) - h(x)|^p y^{2\mu+1} dy \\
&\qquad \leq C \left(\limsup_{\epsilon \to 0^+} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon (\tau_y |h|^p)(x) y^{2\mu+1} dy + |h(x)|^p \right) \\
&\qquad \leq C(M(|h|^p)(x) + |h(x)|^p), \quad x \in (0,\infty).
\end{aligned} \tag{10}$$

Here, by M we denote the maximal function introduced by K. Stempak [16] defined by

$$M(F)(x) = \sup_{\epsilon > 0} \frac{1}{\epsilon^{2\mu+2}} \int_0^\epsilon \tau_x(|F|)(y) y^{2\mu+1} dy, \quad x \in (0,\infty),$$

when F is a measurable function on $(0,\infty)$.

Then we deduce from (10) that

$$A_t \subset \left\{ x \in (0,\infty) : M(|h|^p)(x) > \frac{t}{2C} \right\} \cup \left\{ x \in (0,\infty) : |h(x)|^p > \frac{t}{2C} \right\}.$$

Moreover, by invoking [17, (3)] we obtain

$$\gamma\left(\left\{x\in(0,\infty): M(|h|^p)(x)>\frac{t}{2C}\right\}\right)\le\frac{C}{t}\|h\|_{p,\mu}^p\le\frac{C}{t}\,\epsilon^p.$$

Also, it is clear that

$$\gamma\left(\left\{x\in(0,\infty): |h(x)|^p>\frac{t}{2C}\right\}\right)\le\frac{C}{t}\|h\|_{p,\mu}^p\le\frac{C}{t}\,\epsilon^p.$$

Hence, $\gamma(A_t)\le\frac{C}{t}\epsilon^p$. By letting $\epsilon\to0^+$ we can obtain the desired result. □

Our result about strong summability of Hankel transforms is the following.

Proposition 3.2. *Let $-1/2<\mu<0$. Let f be in $L_{1,\mu}\cap L_{p,\mu}$, for some $\mu+2<p<\infty$, and let x be in $HL^\mu(f,q)$, for some $q>0$. Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|S_\nu(f,\mu;x)-f(x)|^q d\nu=0. \tag{11}$$

Hence (11) *holds for almost every $x\in(0,\infty)$.*

Proof. Without loss of generality we can assume that $\mu+2<p<2$ and that $q=p'$ (see [8, Remark 4]).

Let $T\in(0,\infty)$. By invoking [4, p. 3] we can write, for every $\nu\in(0,T)$,

$$S_\nu(f,\mu;x)-f(x)=I_1(\nu,T)+I_2(\nu,T)$$

where

$$I_1(\nu,T)=\int_0^{1/T}[(\tau_xf)(y)-f(x)]\varphi_\nu(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy$$

and

$$I_2(\nu,T)=\int_{1/T}^{\infty}[(\tau_xf)(y)-f(x)]\varphi_\nu(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy.$$

Here we have taken into account that $\int_0^\infty\varphi_\nu(y)y^{2\mu+1}dy=2^\mu\Gamma(\mu+1)$. Recall that $\varphi_\nu(y)=\nu^{2\mu+2}(y\nu)^{-\mu-1}J_{\mu+1}(y\nu)$, $\nu,y\in(0,\infty)$.

Since the function $z^{-\eta}J_\eta(z)$, $\eta \geq -1/2$, is bounded on $(0,\infty)$, it infers that

$$\begin{aligned}&\left\{\frac{1}{T}\int_0^T |I_1(\nu,T)|^q d\nu\right\}^{1/q}\\ &\leq C\left\{\frac{1}{T}\int_0^T \left[\nu^{2\mu+2}\int_0^{1/T}|(\tau_x f)(y)-f(x)|\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy\right]^q d\nu\right\}^{1/q}\\ &\leq C\int_0^{1/T}|(\tau_x f)(y)-f(x)|\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy\left\{\frac{1}{T}\int_0^T \nu^{2(\mu+1)q}d\nu\right\}^{1/q}\\ &= CT^{2\mu+2}\int_0^{1/T}|(\tau_x f)(y)-f(x)|\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy.\end{aligned}$$

Hence, since $HL^\mu(f,q)$ is contained in $HL^\mu(f,1)$ it concludes that

$$\left\{\frac{1}{T}\int_0^T |I_1(\nu,T)|^q d\nu\right\}^{1/q} \longrightarrow 0, \quad \text{as } T\to\infty.$$

On the other hand, according to [12, Theorem 3] we obtain

$$\begin{aligned}&\left\{\frac{1}{T}\int_0^T |I_2(\nu,T)|^q d\nu\right\}^{1/q} \leq C\Bigg\{\frac{1}{T}\int_0^T\Bigg|\int_{1/T}^\infty[(\tau_x f)(y)\\ &\quad -f(x)]\nu^{2\mu+2}(y\nu)^{-\mu-1}J_{\mu+1}(y\nu)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy\Bigg|^q d\nu\Bigg\}^{1/q}\\ &\leq CT^{2[(\mu+1)q-\mu-2]/q}\left\{\int_{1/T}^\infty\left|\frac{(\tau_x f)(y)-f(x)}{y^2}\right|^p y^{2\mu+3}dy\right\}^{1/p}.\end{aligned}$$

We now define the function g through

$$g(t)=\int_0^t |(\tau_x f)(y)-f(x)|^p y^{2\mu+1}dy, \quad t\in(0,\infty).$$

Then by partial integration it obtains

$$\begin{aligned}
&\int_{1/T}^{\infty} \left| \frac{(\tau_x f)(y) - f(x)}{y^2} \right|^p y^{2\mu+3} dy = \int_{1/T}^{\infty} \frac{g'(y)}{y^{2(p-1)}} dy \\
&= \frac{g(y)}{y^{2(p-1)}} \Big]_{1/T}^{\infty} + 2(p-1) \int_{1/T}^{\infty} \frac{g(y)}{y^{2p-1}} dy \\
&= \lim_{y\to\infty} \frac{1}{y^{2(p-1)}} \int_0^y |(\tau_x f)(s) - f(x)|^p s^{2\mu+1} ds \\
&\quad - T^{2(p-1)} \int_0^{1/T} |(\tau_x f)(s) - f(x)|^p s^{2\mu+1} ds \\
&\quad + 2(p-1) \int_{1/T}^{\infty} \frac{g(y)}{y^{2p-1}} dy.
\end{aligned}$$

Hence since τ_x is a contractive operator from $L_{p,\mu}$ into itself ([17, p. 16]) and since, under our assumption, $HL^\mu(f,q)$ is contained in $HL^\mu(f,p)$, we can conclude that

$$\left\{ \frac{1}{T} \int_0^T |I_2(\nu,T)|^q d\nu \right\}^{1/q} \longrightarrow 0, \quad \text{as } T \to \infty.$$

Thus the proof is finished. □

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