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On homogeneous generalized functions

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0. Introduction

Let $\mathcal{V}$ be an abstract locally convex function space whose elements are defined on $\mathbf{R} = (-\infty, \infty)$ or $(0, \infty)$ and for which

$$\phi(\cdot) \mapsto \phi\left(\frac{\cdot}{y}\right), \qquad y > 0,$$

is a continuous mapping from $\mathcal{V}$ to $\mathcal{V}$. If an element g from $\mathcal{V}'$, where $\mathcal{V}'$ is the dual space of the space $\mathcal{V}$, satisfies the equation

$$g(y\cdot) = y^\alpha g(\cdot), \quad y > 0, \quad \alpha \in \mathbf{R}, \tag{1}$$

in the sense that $\langle g(yx), \phi(x)\rangle = y^\alpha \langle g(x), \phi(x)\rangle$ for all $\phi \in \mathcal{V}$, then g is called a homogeneous generalized function from $\mathcal{V}'$ of order α. If

$(*)$ "$\mathcal{D}(I)$ is dense in $\mathcal{V}$ and the inclusion $i : \mathcal{D}(I) \to \mathcal{V}$ is continuous",

where $\mathcal{D}(I)$ is the well-known Schwartz space, then the structure of homogeneous generalized functions in $\mathcal{V}'$ (and thus, of measurable and continuous homogeneous functions) is very well known. The aim of this paper is to give the solutions of (1) in $\mathcal{V}'$ when $(*)$ does not hold.

Homogeneous generalized functions appear naturally in investigation of the asymptotic behavior of generalized functions (see [3] and [6]). Note, in papers of G. Tröger ([4], [5]) and M. Charmichael and R. S. Pathak ([1]) homogeneous generalized functions have been used without checking if they exist and of which form they are (see Examples at the end of the paper). The purpose of our paper is the investigation of such elements on more general function spaces. We shall give a necessary and sufficient condition for a generalized function from $\mathcal{V}'$ to be homogeneous of order $\alpha \in \mathbf{R}$, and sufficient conditions for the space $\mathcal{V}'$ such that all generalized functions from $\mathcal{V}'$ which are homogeneous of order $\alpha \in \mathbf{R}$ are of the form $A_1 x_+^\alpha + A_2 x_-^\alpha$, $x \in \mathbf{R}$, or $A_1 x_+^\alpha$, $x \in (0, \infty)$ $(A_1, A_2 \in \mathbf{R})$. Thus, we deal with the problem of finding the domain and solving equation (1) in this domain with solutions which are expected. The simply formulated problem does not have so simple answer and this is the main interest of the paper.

Note that the standard method used for finding homogeneous geralized functions of order α in which the equation $xf' = \alpha f$ is transformed to $u' = 0$ by putting $f = x^\alpha u$ is not legitimate in the general case because the mapping $\varphi \mapsto x^\alpha \varphi$ is not defined in $\mathcal{V}$, in general.

1. Let I be one of the intervals $(-\infty, \infty)$, $(0, \infty)$. A space $\mathcal{W}_\beta(I)$ is defined as a subspace of $C^\infty(I \setminus \{0\})$ for which all the seminorms

$$(2) \qquad p_{n,\beta}(\phi) = \int\limits_I \Big| |x|^\beta x^n \phi^{(n)}(x) \Big| \, dx, \quad n \in \mathbf{N}_0,$$

where $\mathbf{N}$ is the set of natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, are finite. We identify functions which are different only at $x = 0$, so $p_{0,\beta}$ is a norm.

One can simply prove that for a given $\phi \in \mathcal{W}_\beta(I)$ the sequence $(\phi\kappa_n)_{n\in N}$, where κ_n is an even positive function such that $\kappa_n(x) = 0$ for $x \in [0, 1/2n] \cup [n+1, \infty)$ and $\kappa_n(x) = 1$ for $x \in [1/n, n]$, $n \in N$, converges to ϕ in $\mathcal{W}_\beta(I)$ as $n \to \infty$. This implies that the space $\mathcal{D}(I)$ is dense in $\mathcal{W}_\beta(I)$. Thus, all homogeneous generalized functions of order α in $\mathcal{W}_\beta(I)$ are of the form

$$(3) \qquad A_1 x_+^\alpha + A_2 x_-^\alpha \ \text{ resp. } \ A_1 x_+^\alpha \quad \text{if } I = \mathbf{R} \ \text{ resp. } I = (0, \infty),$$

where $x_\pm^\alpha = x^\alpha H(\pm x)$, $x \in \mathbf{R}$ and H is the characteristic function of $(0, \infty)$.

Let $\mathcal{U}_\beta(I)$, $\beta \in \mathbf{R}$, be a subspace of $\mathcal{W}_\beta(I)$ with a locally convex topology, such that the inclusion mapping $i : \mathcal{U}_\beta(I) \to \mathcal{W}_\beta(I)$ is continuous, as well as the following one

$$(4) \qquad \mathcal{U}_\beta(I) \to \mathcal{U}_\beta(I) : \phi(\cdot) \mapsto \phi(\cdot/y), \quad y \in (0, \infty).$$

If $f \in \mathcal{W}'_\beta(I)$ and it is homogeneous of order α then $f|_{\mathcal{U}_\beta(I)}$, the restriction of f on $\mathcal{U}_\beta(I)$, is of the form (3). Our aim is the characterization of $\mathcal{U}_\beta(I)$, which implies that all the homogeneous generalized functions from $\mathcal{U}'_\beta(I)$ are of the form (3). At first sight this characterization may appear to be artificial but it is naturally connected with the problem which we are studying.

Note that $\mathcal{D}(I)$ need not be dense in $\mathcal{U}_\beta(I)$. This is in fact the only interesting case. $\mathcal{U}_\beta(I)$ can contain $\mathcal{D}(I)$ but not as a dense subspace (Examples II, III). Also, $\mathcal{U}_\beta(I)$ cannot contain $\mathcal{D}(I)$ at all (Examples I, IV). If $\mathcal{U}_\beta(I)$ contains $\mathcal{D}(I)$ then clearly $\mathcal{U}_\beta(I)$ is dense in $\mathcal{W}_\beta(I)$. Even in that case we could not use this fact for the proof that any homogeneous element g on $\mathcal{U}_\beta(I)$ can be extended on $\mathcal{W}_\beta(I)$ i.e. $g \in \mathcal{W}'_\beta(I)$ and thus it is of the form (3).

For $\mathcal{U}_\beta(I)$ we shall assume: if $\phi \in \mathcal{U}_\beta(I)$ and $y_0 \in (0,\infty)$, then

$$\lim_{y\to y_0} \frac{\phi(\cdot/y) - \phi(\cdot/y_0)}{y - y_0} = \frac{d}{dy}\phi(\cdot/y)\mid_{y=y_0} \tag{5}$$

in the sence of convergence in $\mathcal{U}_\beta(I)$.

This condition implies the continuity of the mapping

$$(0,\infty) \to \mathcal{U}_\beta(I) : y \mapsto \frac{d}{dy}\phi(x/y) = -x\phi'(x/y)/y^2, \qquad x \in I \setminus \{0\}, \quad \phi \in \mathcal{U}_\beta(I). \tag{6}$$

Moreover, for every $l \geq 2$, $l \in \mathbf{N}$, the mapping

$$(0,\infty) \to \mathcal{U}_\beta(I) : y \mapsto \frac{d^l}{dy^l}\phi(x/y), \quad x \in I \setminus \{0\}, \phi \in \mathcal{U}_\beta(I), \tag{7}$$

is continuous. If $l = 2$ this follows from (6) and the fact that

$$-\frac{x}{y^2}\phi'(x/y) = \frac{1}{y}\psi(x/y), \quad \text{where } \psi(x) = -x\phi'(x) \in \mathcal{U}_\beta(I), \qquad x \in I \setminus \{0\}, \quad y > 0.$$

For $l > 2$ this can be proved by induction.

Let us note, beside the fact that the mapping $\varphi \mapsto x\varphi$ is not defined in $\mathcal{U}_\beta(I)$, that the mapping $\varphi \mapsto \varphi'$ too fails to be defined in $\mathcal{U}_\beta(I)$ in the general case.

Proposition 1. *A generalized function $f \in \mathcal{U}'_\beta(I)$ is homogeneous of order $\alpha \in R$ if and only if for each $\phi \in \mathcal{U}_\beta(I)$*

$$\langle f(x), (x\phi(x))' + \alpha\phi(x)\rangle = 0; \tag{8}$$

i.e. if and only if f fulfils the equation $xf' = \alpha f$ in $\mathcal{U}'_\beta(I)$.

Proof. The assumption (5) implies $x\phi' \in \mathcal{U}_\beta(I)$ when $\phi \in \mathcal{U}_\beta(I)$ and $x \in I \setminus \{0\}$. Since $(x\phi)' + \alpha\phi = x\phi' + (1+\alpha)\phi$, (8) makes sense when $f \in \mathcal{U}'_\beta(I)$ and $x \in I \setminus \{0\}$.

Let f be a homogeneous generalized function of order α and $\phi \in \mathcal{U}_\beta(I)$. We have

$$\langle f(yx), \phi(x)\rangle = y^\alpha \langle f(x), \phi(x)\rangle = \langle f(x), \phi(x/y)/y\rangle, \quad y > 0.$$

Because of (6) we can differentiate this equality with respect to y. In this way we get

$$\alpha y^{\alpha-1}\langle f(x), \phi(x)\rangle = \langle f(x), -y^{-2}\phi(x/y) - y^{-3}x\phi'(x/y)\rangle, \quad y > 0,$$

i.e.

$$\begin{aligned}\alpha y^\alpha\langle f(x),\phi(x)\rangle &= \langle f(x), -y^{-1}(\phi(x/y)+x\phi'(x/y)/y)\rangle = \\ = \langle f(yx), -(\phi(x)+x\phi'(x))\rangle &= y^\alpha\langle f(x), -(\phi(x)+x\phi'(x))\rangle = \\ &= y^\alpha\langle f(x), -(x\phi(x))'\rangle \qquad y>0.\end{aligned}$$

This implies

$$\alpha y^\alpha\langle f(x),\phi(x)\rangle = y^\alpha\langle f(x), -(x\phi(x))'\rangle, \quad y>0,$$

and in particular for $y=1$, $\langle f(x), (x\phi(x))'+\alpha\phi(x)\rangle = 0$.
Conversely, suppose that $\langle f(x), (x\phi(x))'+\alpha\phi(x)\rangle = 0$ holds for all $\phi \in \mathcal{U}_\beta(I)$. Let $\phi(\cdot/y)$, $y>0$, be substituted for $\phi(\cdot)$. We get

$$\langle(\alpha+1)f(x),\phi(x/y)\rangle = -\langle f(x), x\phi'(x/y)/y\rangle, \qquad y>0.$$

Since $-x\phi'(x/y)/y = y\frac{d}{dy}\phi(x/y),\ x\in I\setminus\{0\},\ y>0$, from (6) we obtain

$$(\alpha+1)\langle f(x),\phi(x/y)\rangle = y\langle f(x), \frac{d}{dy}\phi(x/y)\rangle, \qquad y>0.$$

Let us consider the function $y\mapsto F(y) = \langle f(x),\phi(x/y)\rangle, y>0$. It has a derivative at each $y\in(0,\infty)$ (moreover it belongs to $C^\infty(0,\infty)$), and satisfies

$$\begin{aligned} yF'(y) &= (\alpha+1)F(y), \quad y>0 \\ F(1) &= \langle f(x),\phi(x)\rangle. \end{aligned} \tag{9}$$

Therefore, $F(y) = \langle f(x),\phi(x)\rangle y^{\alpha+1}, \ \ y>0$. Thus,

$$\langle f(yx),\phi(x)\rangle = \langle f(x),\phi(x/y)\rangle/y = F(y)/y = y^\alpha\langle f(x),\phi(x)\rangle$$

which means that f is homogeneous of order α. □

2. In this section we assume $I=(-\infty,\infty)$.

Let $\mathcal{U}_\beta(I)$ have a dense subspace $\mathcal{Z}_\beta(I)$, $\beta\in\mathbf{R}$, with the following property:

If $\phi\in\mathcal{Z}_\beta(I)$ then for some ε, $0<\varepsilon<1$, $\delta>0$, $A>0$ and $c>0$, which depend on ϕ,

$$\begin{cases} |\phi(x)| & \le c/|x|^{1+\beta-\varepsilon}, \quad |x|<\delta, \quad x\ne 0 \\ |\phi(x)| & \le c/|x|^{1+\beta+\varepsilon}, \quad |x|>A. \end{cases} \tag{10}$$

Furthermore, let $\mathcal{Z}_\beta(I)$ contain functions ϕ_0 and ϕ_1 such that

$$\int_{-\infty}^{\infty} |t|^\beta \phi_0(t)dt = 1, \quad \int_{-\infty}^{0} |t|^\beta \phi_0(t)dt = 0,$$
$$\int_{-\infty}^{\infty} |t|^\beta \phi_1(t)dt = 1, \quad \int_{0}^{\infty} t^\beta \phi_1(t)dt = 0.$$

Remark. If for any element of $\mathcal{U}_\beta(I)$ (10) holds, then we can take for $\mathcal{Z}_\beta(I)$ any proper and dense subset of $\mathcal{U}_\beta(I)$ as well as the whole space $\mathcal{U}_\beta(I)$. In this paper we are not interested in finding the smallest dense subspace of $\mathcal{U}_\beta(I)$ for whose elements (10) holds because this needs more specific assumptions.

Proposition 2. (a) *If $\phi \in \mathcal{Z}_\beta(I)$ then there is $\theta \in \mathcal{W}_\beta(I)$ such that*

$$\phi(x) = \left(\int_0^\infty t^\beta \phi(t)dt\right)\phi_0(x) + \left(\int_{-\infty}^0 |t|^\beta \phi(t)dt\right)\phi_1(x) + (x\theta(x))' + \beta\theta(x), \quad x \in I \setminus \{0\}. \tag{11}$$

(b) *If (11) holds for any $\phi \in \mathcal{Z}_\beta(I)$ with $\theta \in \mathcal{U}_\beta(I)$, then a generalized function $f \in \mathcal{U}'_\beta(I)$ is homogeneous of order β if and only if it is of the form*

$$f(x) = A_1 x_+^\beta + A_2 x_-^\beta, \quad x \in I, \quad A_1, A_2 \in \mathbf{R}. \tag{12}$$

(c) *If elements of $\mathcal{Z}_\beta(I)$ are real analytic on $I \setminus \{0\}$ then θ in (11) is real analytic on $I \setminus \{0\}$, too.*

Proof. (a) Let $\phi \in \mathcal{Z}_\beta(I)$. Because of (10) we have that $c_0(\phi) = \int_0^\infty t^\beta \phi(t)dt$ and $c_1(\phi) = \int_{-\infty}^0 |t|^\beta \phi(t)dt$ are finite. By direct computation one can prove that the function

$$\theta(x)=\begin{cases} -\frac{1}{x^{\beta+1}}\left(\int\limits_x^\infty t^\beta\phi(t)dt-c_0(\phi)\int\limits_x^\infty t^\beta\phi_0(t)dt-\right.\\ \qquad\left.-c_1(\phi)\int\limits_x^\infty t^\beta\phi_1(t)dt\right), \quad x>0,\\ \frac{1}{x^{\beta+1}}\left(\int\limits_{-\infty}^x |t|^\beta\phi(t)dt-c_0(\phi)\int\limits_{-\infty}^x |t|^\beta\phi_0(t)dt-\right.\\ \qquad\left.-c_1(\phi)\int\limits_{-\infty}^x t^\beta\phi_1(t)dt\right) \quad x<0,\end{cases}$$

satisfies (11).

For $x<-A$, we have

$$|x|^\beta\frac{1}{|x|^{\beta+1}}\int\limits_{-\infty}^x \left||t|^\beta\phi(t)\right|dt\le\frac{c}{|x|}\int\limits_{-\infty}^x 1/|t|^{\quad 1+\varepsilon}dt=\frac{c}{\varepsilon|x|^{1+\varepsilon}},$$

and for $x>A$,

$$x^\beta\frac{1}{x^{\beta+1}}\int\limits_x^\infty t^\beta|\phi(t)|dt\le\frac{c}{\varepsilon x^{1+\varepsilon}}\ .$$

From above it follows that $\int\limits_{|x|\ge A}\left||x|^\beta\theta(x)\right|dx<\infty$.

Let us prove that for $n\in\mathbf{N}$, $\int\limits_{|x|\ge A}\left||x|^\beta x^n\theta^{(n)}(x)\right|dx<\infty$. Since θ fulfils (11), we have

$$x\theta'(x)=\phi(x)-c_0(\phi)\ \phi_0(x)-c_1(\phi)\ \phi_1(x)-(\beta+1)\theta(x),\quad x\in I\setminus\{0\},$$

By induction one can prove that, for $n\in\mathbf{N}$,

$$\begin{aligned}|x|^\beta x^n\theta^{(n)}(x)&=|x|^\beta x^{n-1}\phi^{(n-1)}(x)-c_0(\phi)|x|^\beta x^{n-1}\phi_0^{(n-1)}(x)-\\ -c_1(\phi)|x|^\beta x^{n-1}\phi_1^{(n-1)}(x)&-(\beta+n)|x|^\beta x^{n-1}\theta^{(n-1)}(x),\quad x\in I\setminus\{0\}\end{aligned}\tag{13}$$

Since $\phi,\phi_0,\phi_1\in\mathcal{W}_\beta$, from the definition of the space $\mathcal{W}_\beta(I)$, (13) and the inductive assumption that $\left||x|^\beta x^{n-1}\theta^{(n-1)}\right|$ is integrable on $|x|>A$, it follows that $\left||x|^\beta x^n\theta^{(n)}\right|$ is integrable on $|x|>A$.

To show that $\theta \in \mathcal{W}_\beta(I)$, we have to prove that for $n \in \mathbf{N}_0$, $x \mapsto \big||x|^\beta x^n \theta^{(n)}(x)\big|$ is integrable in a neighborhood of zero. Let $n = 0$. For $x > 0$ we have

$$\theta(x) = -\frac{1}{x^{\beta+1}}\Bigg(-\int\limits_0^x t^\beta \phi(t)dt + c_0(\phi) - c_0(\phi)\Bigg(-\int\limits_0^x t^\beta \phi_0(t)dt + 1\Bigg) - \\ - c_1(\phi)\Bigg(-\int\limits_0^x t^\beta \phi_1(t)dt\Bigg)\Bigg)$$

i.e.

$$\theta(x) = \frac{1}{x^{\beta+1}}\Bigg(\int\limits_0^x t^\beta \phi(t)dt - c_0(\phi)\int\limits_0^x t^\beta \phi_0(t)dt - c_1(\phi)\int\limits_0^x t^\beta \phi_1(t)dt\Bigg).$$

By (10) there exist $\varepsilon \in (0,1)$, $\delta > 0$, and $c > 0$, which depend on ϕ, ϕ_0, ϕ_1, such that

$$\big||x|^\beta \theta(x)\big| \le \frac{c(1 - c_0(\phi) - c_1(\phi))}{\varepsilon |x|^{1-\varepsilon}}, \quad 0 < x < \delta.$$

One can analogously prove the similar inequality for $-\delta < x < 0$. Hence $\big||x|^\beta \theta(x)\big|$ is integrable in a neighborhood of zero. Let $n \in \mathbf{N}$. Since $\phi, \phi_0, \phi_1 \in \mathcal{W}_\beta$, from the definition of the space $\mathcal{W}_\beta(I)$, (13) and the inductive assumption that $\big||x|^\beta x^{n-1}\theta^{(n-1)}(x)\big|$ is integrable in a neighborhood of zero, it follows that $\big||x|^\beta x^n \theta^{(n)}(x)\big|$ is integrable in a neighborhood of zero.

Note, that from the proof it follows that θ satisfies (11), as well.

(b) Assume that $\phi \in \mathcal{Z}_\beta(I)$ has representation (11) with $\theta \in \mathcal{U}_\beta(I)$. We have

$$\langle f, \phi\rangle = \langle f, c_0\phi_0\rangle + \langle f, c_1\phi_1\rangle + \langle f, (x\theta)' + \beta\theta\rangle.$$

Since f is a homogeneous generalized function of order β, by Proposition 1, it follows

$$\langle f, \phi\rangle = c_0\langle f, \phi_0\rangle + c_1\langle f, \phi_1\rangle = A_1\langle x_+^\beta, \phi\rangle + A_2\langle x_-^\beta, \phi\rangle.$$

(c) One has to show that for any $x \neq 0$ there are $C_x > 0$ and $k_x > 0$ such that $|\theta^{(n)}(x)| \le C_x {k_x}^n n!$, $n \in \mathbf{N}_0$. This can proved by induction by using the representation of θ from the first part of the proof. □

3. In this section we assume that $I = (0, \infty)$ and that $\mathcal{Z}_\beta(I)$ is a dense subspace of $\mathcal{U}_\beta(I)$ for which (10) holds on $(0, \infty)$. Clearly, $\mathcal{Z}_\beta(I)$ contains a function $\phi_0 = \phi_{0,\beta}$ such that $\int\limits_0^\infty |t|^\beta \phi_0(t)dt = 1$.

By the same arguments as in Section 2. we have the following

Proposition 3. (a) *If* $\phi \in \mathcal{Z}_\beta(I)$, *then there is* $\theta \in \mathcal{W}_\beta(I)$ *such that*

$$\phi(x) = \int_0^\infty |t|^\beta \phi(t)dt\, \phi_0(x) + (x\theta(x))' + \beta\theta(x), \quad x \in I \setminus \{0\} \tag{14}$$

(b) *Let (14) hold for any* $\phi \in \mathcal{Z}_\beta(I)$ *with* $\theta \in \mathcal{U}_\beta(I)$. *A generalized function* $f \in \mathcal{U}_\beta(I)$ *is homogeneous of order* β *if and only if it is of the form*

$$u(x) = A_1 x_+^\beta, \quad x \in I, \quad A_1 \in \mathbf{R}. \tag{15}$$

(c) *If elements of* $\mathcal{Z}_\beta(I)$ *are real analytic on* I *then* θ *in (14) is real analytic on* I, *too.*

4. Examples In the first there examples we take $\mathcal{Z}_\beta(I) = \mathcal{U}_\beta(I)$, while in the last one we have $\mathcal{Z}_\beta(I) \subset \mathcal{U}_\beta(I)$ and it is not possible to take $\mathcal{Z}_\beta(I) = \mathcal{U}_\beta(I)$.

Example I. The well-known space $\mathcal{Z}(a)$, $a > 0$, (see [3]) is defined as the space of entire functions ψ on $\mathbf{C}$ for which all the norms

$$\|\psi\|_p = \sup\left\{(1+|z|)^k \psi(z) e^{-a|\mathrm{Im} z|}, \quad k \le p, \quad z \in \mathbf{C}\right\}, \quad p \in \mathbf{N}_0,$$

are finite. Denote by $\mathcal{Z}_{\mathbf{R}}(a)$ the space of restrictions of functions from $\mathcal{Z}(a)$ on the real line supplied with the corresponding convergence structure from $\mathcal{Z}(a)$. Note that the mapping $\phi(\cdot) \to \phi(\cdot/k)$, $k > 0$ is not defined in this space. Let $\mathcal{Z} = \bigcup_{a>0} \mathcal{Z}(a)$ and $\mathcal{Z}_{\mathbf{R}}$ be the space of corresponding restrictions supplied with the convergence structure from $\mathcal{Z}$. Clearly, $\mathcal{Z}_{\mathbf{R}}$ does not contain smooth functions with compact supports. $\mathcal{Z}_{\mathbf{R}}$ fulfils all the conditions assumed for $\mathcal{U}_0(\mathbf{R})$ and $\mathcal{Z}_0(\mathbf{R})$. Let us prove that for any $\phi \in \mathcal{Z}_{\mathbf{R}} = \mathcal{U}_0(\mathbf{R}) = \mathcal{Z}_0(\mathbf{R})$ the corresponding function θ in (11) is also from $\mathcal{Z}_{\mathbf{R}}$. (11) implies that $x\theta$ is a restriction of an entire function because for $z \in \mathbf{C}$

$$z\theta(z) = \int_0^z \phi(\xi)d\xi - \int_0^\infty \phi(t)dt \int_0^z \phi_0(\xi)d\xi - \int_{-\infty}^0 \phi(t)dt \int_0^z \phi_1(\xi)d\xi \tag{17}$$

and on the real line we have

$$x\theta(x) = \int_0^x \phi(\xi)d\xi - \int_0^\infty \phi(t)dt \int_0^x \phi_0(\xi)d\xi - \int_{-\infty}^0 \phi(t)dt \int_0^x \phi_1(\xi)d\xi =$$

$$= \begin{cases} -\left(\int\limits_x^\infty \phi(t)dt - \int\limits_0^\infty \phi(t)dt \int\limits_x^\infty \phi_0(t)dt - \int\limits_{-\infty}^0 \phi(t)dt \int\limits_x^\infty \phi_1(t)dt\right), & x > 0 \\ 0 \qquad\qquad , & x = 0 \\ \int\limits_{-\infty}^x \phi(t)dt - \int\limits_0^\infty \phi(t)dt \int\limits_{-\infty}^x \phi_0(\xi)d\xi - \int\limits_{-\infty}^0 \phi(t)dt \int\limits_{-\infty}^x \phi_1(t)dt, & x < 0. \end{cases}$$

From (17) it follows that $\theta \in \mathcal{Z}$ and thus $\theta|_{\mathbf{R}} \in \mathcal{Z}_{\mathbf{R}}$.

From Proposition 2 we have that all the homogeneous generalized functions of order 0 from $\mathcal{Z}'_{\mathbf{R}}$ are of the form

$$A_1 H(x) + A_2 H(-x), \quad A_1, A_2 \in \mathbf{R}.$$

Note that the isomorphism of $\mathcal{Z}'$ and $\mathcal{Z}'_{\mathbf{R}}$, $f \to \tilde{f}$, is given by

$$\langle f, \phi \rangle = \langle \tilde{f}, \phi|_{\mathbf{R}} \rangle, \quad \phi \in \mathcal{Z}.$$

Example II. In [4], [5], the author investigated Zemanian's space $M_{a,b}$([7]). Recall, that for infinitely differentiable complex-valued functions ϕ on $(0,\infty)$ and $a, b \in \mathbf{R}$ the set of seminorms is defined by

$$\mu_{a,b,p}(\phi) = \sup_{t\in(0,\infty)} \left(t^{1-a+p}(1+t)^{a-b}|\phi^{(p)}(t)|\right), \quad p \in \mathbf{N}_0$$

and the test function space by

$$M_{a,b} = \{\phi \in C^\infty((0,\infty)) : \mu_{a,b,p}(\phi) < \infty, \quad \text{for all } \ p \in \mathbf{N}_0\} \ ,$$

which is equipped with the corresponding topology. The space $M_{a,b}$ fulfils all the conditions assumed for the spaces $\mathcal{U}_\beta((0,\infty))$ and $\mathcal{Z}_\beta((0,\infty))$, where $-b > \beta > -a$. $\mathcal{D}_\beta((0,\infty))$ is not dense in $M_{a,b}$ (see [7]). By using Proposition 3 we get that all homogeneous distributions of order β, $-b > \beta > -a$ are of the form (15).

Example III. The space $H_{c,\theta}(0,\infty)$, $c \in \mathbf{R}$, $\theta \geq 0$, investigated in [1] is defined as the space of smooth functions ϕ on $(0,\infty)$ for which all the seminorms

$$\gamma_k(\phi) = \sup_{t\in(0,\infty)} \left|t^{-c+k_e\theta t_\phi(k)}(t)\right|, \quad k \in \mathbf{N}_0,$$

are finite.

If $\theta > 0$ then the mapping (2) is not defined for arbitrary $k > 0$. So, instead of $H_{c,\theta}(0,\infty)$, $\theta > 0$, one has to consider the space $\bigcup_{\theta>0} H_{c,\theta}(0,\infty)$ equipped with the inductive topology. $\mathcal{D}(0,\infty)$ is not dense in this space. In the corresponding dual space homogeneous generalized functions of order $\beta > -(c+1)$ are of the form (14).

Example IV. Let $\mathcal{A}_{\alpha,\beta}$ be the space of real analytic functions ϕ on $\mathbf{R}\setminus\{0\}$, for which all the seminorms

$$\|\phi\|_p = \sup_{x\in\mathbf{R}\setminus\{0\}} (1+|x|)^\alpha |x|^\beta \left| x^p \phi^{(p)}(x)\right|, \quad p \in \mathbf{N}_0, \tag{16}$$

are finite, supplied with the corresponding topology. $\mathcal{A}_{\alpha,\beta}$ does not contain smooth functions with compact support and this space is not complete.

$\mathcal{A}_{\alpha,\beta}$ is a subspace of the space $\mathcal{B}_{\alpha,\beta}$ which we define to be the space of all smooth functions ϕ on $\mathbf{R}\setminus\{0\}$ for which all the seminorms (16) are finite.

Any functions φ from $\mathcal{B}_{\alpha,\beta}$ is of the form $\varphi = \varphi_1 + \varphi_2$, where

$$\varphi_1 = \begin{cases} \varphi|_{(0,\infty)} & , x \in (0,\infty) \\ 0 & , x \in (-\infty, 0) \end{cases} \quad \text{and} \quad \varphi_2 = \begin{cases} 0 & , x \in (-\infty,0) \\ \varphi|_{(-\infty,0)} & , x \in (0,\infty). \end{cases}$$

We have $\phi_1, \check{\phi}_2 \in M_{a,b}$, $(\check{\phi}(x) = \phi(-x))$, where $1-a=\alpha$, $a-b=\beta$. Using the fact that $M_{a,b}$ is a complete space ([7,p. 103]) we get that $\mathcal{B}_{\alpha,\beta}$ is complete. Note, that $\mathcal{D}(\mathbf{R}\setminus\{0\})$ is not dense in $\mathcal{B}_{\alpha,\beta}$. Let us prove that $\mathcal{A}_{\alpha,\beta}$ is not complete. Let $\phi \in \mathcal{D}((0,\infty))$ and

$$\phi_j(x) = \frac{j}{\pi^{1/2}} \int e^{-j^2(x-t)^2} \phi(t)dt\ , \quad j \in \mathbf{N}.$$

One can prove that this is a sequence from $\mathcal{A}_{\alpha,\beta}$ which converges to ϕ in $\mathcal{B}_{\alpha,\beta}$. Since ϕ does not belong to $\mathcal{A}_{\alpha,\beta}$ we get that $\mathcal{A}_{\alpha,\beta}$ is not complete.

The space $\mathcal{A}_{\alpha,\beta}$ fulfils all the conditions assumed for the spaces $\mathcal{U}_\beta(\mathbf{R})$ and $\mathcal{Z}_\beta(\mathbf{R})$, where $b-1<\beta<b+a-1$. $(\mathcal{U}_\beta(\mathbf{R}) = \mathcal{Z}_\beta(\mathbf{R}))$.

Let us consider the closure $\overline{\mathcal{A}}_{a,b}$ of $\mathcal{A}_{a,b}$ in $\mathcal{B}_{a,b}$. Take $\mathcal{U}_\beta(\mathbf{R})$ such that $\mathcal{A}_{a,b} \subset \mathcal{U}_\beta \subset \overline{\mathcal{A}}_{a,b}$ and $\mathcal{A}_{a,b} \neq \mathcal{U}_\beta \neq \overline{\mathcal{A}}_{a,b}$, and take $\mathcal{Z}_\beta(\mathbf{R}) = \mathcal{A}_{a,b}$, $b-1<\beta<b+a-1$. In this way we have constructed an example of spaces $\mathcal{U}_\beta(\mathbf{R})$ and $\mathcal{Z}_\beta(\mathbf{R})$ such that $\mathcal{Z}_\beta(\mathbf{R})$ is a proper and dense subset of $\mathcal{U}_\beta(\mathbf{R})$; moreover, we could not take for $\mathcal{Z}_\beta(\mathbf{R})$ the space $\mathcal{U}_\beta(I)$.

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