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On (0; 1) Pál-type interpolation with boundary conditions

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Abstract. Let the set of the knots

$$-1 = x_n < x_n^* < x_{n-1} < x_{n-1}^* < \cdots < x_1 < x_1^* < x_0 = 1 \quad (n \geq 2)$$

be given on the interval $[-1, 1]$. Find a polynomial $Q_m(x)$ of minimal degree satisfying the (0;1) interpolation properties

$$\begin{aligned} Q_m(x_i) &= y_i \qquad (i = 1, \dots, n-1), \\ Q'_m(x_i^*) &= y'_i \qquad (i = 1, \dots, n), \end{aligned}$$

with the boundary conditions

$$\begin{aligned} Q_m^{(j)}(1) &= \alpha_j \qquad (j = 0, \dots, k), \\ Q_m^{(j)}(-1) &= \beta_j \qquad (j = 0, \dots, l), \end{aligned}$$

where y_i, y'_i, α_j, β_j are given real numbers, and k, l are fixed non-negative integers.

In this paper the existence and uniqueness of the polynomial $Q_m(x)$ is proved if the inner nodal points $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ are the roots of the Jacobi polynomials $P_{n-1}^{(k+1,l)}(x)$ and $P_n^{(k,l-1)}(x)$, respectively. Explicit formulae for the fundamental polynomials of interpolation are given. Convergence and approximation theorems are also proved.

Recently many authors investigated the Pál-type interpolation, in which the nodal points $\{x_i\}_{i=0}^{n}$ are the roots of $\omega_{n+1}(x)$ and $\{x_i^*\}_{i=1}^{n}$ are the roots of $\omega'_{n+1}(x)$. The polynomial $Q_m(x)$ will be a modified Pál-type interpolational polynomial, because the knots $x_0, x_1, \dots, x_n$ are the roots of $\omega(x) = (1-x)^{k+1}(1+x)^l P_{n-1}^{(k+1,l)}(x)$, and the knots $x_0^*, x_1^*, \dots, x_n^*, x_{n+1}^*$, ($x_0^* = -1$, $x_{n+1}^* = 1$) are the roots of $\omega'(x)$.

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1. Introduction

In 1975 L. G. Pál [9] has introduced a modification of the Hermite–Fejér interpolation, in which the function values and the first derivatives are prescribed on two interscaled systems of nodal points $\{x_i\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^{n-1}$, that is

$$-\infty < x_1 < x_1^* < x_2 < \dots < x_{n-1} < x_{n-1}^* < x_n < +\infty, \tag{1.1}$$

where

$$\omega_n(x) = (x - x_1)\dots(x - x_n) \quad \text{and} \quad \omega_n'(x) = n(x - x_1^*)\dots(x - x_{n-1}^*).$$

He proved that for any given systems of real numbers

$$\{y_k\}_{k=1}^n \quad \text{and} \quad \{y_k'\}_{k=1}^{n-1},$$

there exists a polynomial $Q_{2n-1}(x)$ of minimal degree $(2n-1)$ satisfying the following interpolational properties

$$\begin{aligned} Q_{2n-1}(x_k) &= y_k \qquad (k = 1, \dots, n), \\ Q_{2n-1}'(x_k^*) &= y_k' \qquad (k = 1, \dots, n-1). \end{aligned} \tag{1.2}$$

This interpolational polynomial is not uniquely determined, hence for the uniqueness an additional condition is recommended. Introducing the additional condition $Q_{2n-1}(x_0) = 0$ at an additional knot $x_0 \neq x_k$ $(k = 1, \dots, n)$ Pál proved the uniqueness and gave an explicit formula for it.

Following Pál's idea many authors investigated this kind of interpolation and they called it *Pál-type interpolation*. In 1992 Xie [17] presented a new explicit formula of Pál-type interpolation on the interval $[-1, 1]$ with the additional knot x_n^*, where x_n^* is equal to one of the nodal points x_k $(k = 1, \dots, n)$. Earlier, in 1985 Eneduanya [1] investigated the special case when

$$\omega_n(x) = -n(n-1)\int_{-1}^x P_{n-1}(t)\,dt = (1 - x^2)P_{n-1}'(x), \tag{1.3}$$

where $P_n(x)$ is the Legendre polynomial of degree n with the usual normalization $P_n(1) = 1$. For the uniqueness Eneduanya used also the additional nodal point $x_n^* = -1$. Szili [15] investigated the Pál-type interpolation on the roots of the Hermite-polynomials with the additional point $x_0 = 0$. Both Eneduanya and Szili gave explicit formulae and proved approximation theorems. Joó and Szabó [6] gave a common generalization of the classical Fejér interpolation [2] and Pál interpolation. Szili [14] studied the inverse Pál interpolational problem on the roots of the integrated Legendre polynomials. Recently Joó and Pál ([4], [5]) investigated the lacunary $(0;0,1)$ interpolation on the roots of Jacobi polynomials and their derivatives, respectively. Sebestyén ([10], [11]) studied the same problem on the roots of Hermite polynomials and gave a completion of the Pál-type $(0;0,1)$ lacunary interpolation.

In this paper the following problem is investigated:
Let the set of the knots

$$(1.4)\quad -1 = x_n < x_n^* < x_{n-1} < x_{n-1}^* < \cdots < x_1 < x_1^* < x_0 = 1 \quad (n \geq 1)$$

be given on the interval $[-1,1]$. Find a polynomial $Q_m(x)$ of minimal degree satisfying the $(0;1)$ interpolation properties

$$(1.5)\qquad \begin{aligned} Q_m(x_i) &= y_i \qquad (i = 1, \ldots, n-1), \\ Q_m'(x_i^*) &= y_i' \qquad (i = 1, \ldots, n), \end{aligned}$$

with the boundary conditions

$$(1.6)\qquad \begin{aligned} Q_m^{(j)}(x_0) &= Q_m^{(j)}(1) = \alpha_j \qquad (j = 0, \ldots, k), \\ Q_m^{(j)}(x_n) &= Q_m^{(j)}(-1) = \beta_j \qquad (j = 0, \ldots, l), \end{aligned}$$

where y_i, y_i', α_j, β_j are given real numbers, and k, l are fixed non-negative integers.

As the polynomial $Q_m(x)$ satisfies $2n + k + l + 1$ conditions due to (1.5) and (1.6), so the expected minimal degree is $m = 2n + k + l$.

In Section 2 we give explicit formulae for the fundamental polynomials of interpolation and prove an existence and uniqueness theorem. In Section 2 we give an estimate for $|f(x) - Q_m(x)|$ on $[-1,1]$, if $f \in C^r[-1,1]$ and the knots (1.4) are the roots of appropriate ultraspherical polynomials. As the Legendre polynomial $P_n(x)$ $(P_n(1) = 1)$ is an ultraspherical polynomial with the parameter $\alpha = 0$, the interpolation on the knots (1.3) investigated by Eneduanya is a special case of our interpolational procedure with $k=0$, $l=1$. ENEDUANYA [1] proved that for $f\in C^r[-1,1]$, $x\in[-1,1]$

$$|f(x)-Q_{2n+1}(f;x)|=O(1)n^{-r+\frac{3}{2}}\log n\,\omega\left(f^{(r)};\frac{1}{n}\right),$$

where $O(1)$ is independent of n and x. XIE [16] improved this result: for $f\in C^r[-1,1]$, $x\in[-1,1]$

$$|f(x)-Q_{2n+1}(f;x)|=O(1)n^{-r+1}\omega\left(f^{(r)};\frac{1}{n}\right),$$

which implies the uniform convergence if $f(x)$ is continuously differentiable on $[-1,1]$. Now we can prove the uniform convergence of the interpolational procedure on $[-1,1]$ if $f\in C^{k+1}[-1,1]$ for $k\geqq 1$.

2. The existence and uniqueness

In what follows, we will use the notations: for fixed integers $k\geqq 0$, $l>0$ let

$$\omega_n^*(x)=P_n^{(k,l-1)}(x)\quad\text{and}\quad\omega_{n-1}(x)={\omega_n^*}'(x),\tag{2.1}$$

where $P_n^{(k,l-1)}(x)$ is the Jacobi polynomial of degree n with the normalization $P_n^{(k,l-1)}(1)=\binom{n+k}{n}$, and let

$$\varrho(x)=(1-x)^{k+1}(1+x)^l.\tag{2.2}$$

It is known that (see [13])

$$P_n^{(k,l-1)'}(x)=\frac{n+k+l}{2}P_{n-1}^{(k+1,l)}(x)\tag{2.3}$$

and $P_n^{(k,l-1)}(x)$ satisfies the differential equation

$$\begin{aligned}(1-x^2)P_n^{(k,l-1)''}(x)+\left[l-k-1-(k+l+1)x\right]P_n^{(k,l-1)'}(x)\\ +n(n+k+l)P_n^{(k,l-1)}(x)=0.\end{aligned}\tag{2.4}$$

Let the set of the knots be given by

$$-1=x_n<x_n^*<x_{n-1}<x_{n-1}^*<\dots<x_1<x_1^*<x_0=1\quad(n\ge 1),\tag{2.5}$$

where $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ are the roots of $\omega_{n-1}(x)$ and $\omega_n^*(x)$, respectively.

By the differential equation (2.4) we get

$$\big[\varrho(x)\omega_{n-1}(x)\big]' = -n(n+k+l)(1-x)^k(1+x)^{l-1}\omega_n^*(x)$$

and for any function $q(x)$, differentiable at x_i^*

$$\big[\varrho(x)\omega_{n-1}(x)q(x)\big]'_{x=x_i^*} = \varrho(x_i^*)\omega_{n-1}(x_i^*)q'(x_i^*). \tag{2.6}$$

We will denote by $l_j(x)$ and $l_j^*(x)$ the fundamental polynomials of Lagrange interpolation on the knots $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$, respectively, that is

$$l_j(x) = \frac{\omega_{n-1}(x)}{\omega'_{n-1}(x_j)(x-x_j)}, \quad \text{and} \quad l_j^*(x) = \frac{\omega_n^*(x)}{\omega_n^{*\prime}(x_j^*)(x-x_j^*)}. \tag{2.7}$$

Lemma 2.1. *On the knots in* (2.5) *the fundamental polynomials of interpolation are*

$$A_j(x) = \frac{\varrho(x)}{\varrho(x_j)(1+x_j)\omega_n^*(x_j)} \times \Big[(1+x)\omega_n^*(x)l_j(x) - \omega_{n-1}(x)\int_{-1}^{x}(1+t)l_j(t)\,dt\Big] \tag{2.8}$$

$(j = 1, \ldots, n-1)$;

$$B_j(x) = \frac{\varrho(x)\omega_{n-1}(x)}{\varrho(x_j^*)\omega_{n-1}(x_j^*)}\int_{-1}^{x} l_j^*(t)\,dt \qquad (j = 1, \ldots, n); \tag{2.9}$$

$$C_j(x) = (1-x)^j(1+x)^{l+1}\omega_{n-1}(x)\omega_n^*(x)p_j(x) + \varrho(x)\omega_{n-1}(x)\int_{-1}^{x}\frac{-(1+t)\omega_{n-1}(t)p_j(t) + q_j(t)\omega_n^*(t)}{(1-t)^{k+1-j}}\,dt \tag{2.10}$$

$(j = 0, \ldots, k)$, *where* $p_j(x)$ *and* $q_j(x)$ *are uniquely determined polynomials of degree* $\leqq k-j$;

$$D_j(x) = (1-x)^{k+1}(1+x)^j\omega_{n-1}(x)\omega_n^*(x)\tilde{p}_j(x) + \varrho(x)\omega_{n-1}(x)\int_{-1}^{x}\frac{-\omega_{n-1}(t)\tilde{p}_j(t) + \tilde{q}_j(t)\omega_n^*(t)}{(1+t)^{l-j}}\,dt \tag{2.11}$$

$(j=0,\dots,l)$, where $\tilde p_j(x)$ and $\tilde q_j(x)$ are uniquely determined polynomials of degree $\leqq l-j$ and $l-j-1$, respectively. The polynomials $A_j(x)$, $B_j(x)$, $C_j(x)$ and $D_j(x)$ are of degree at most $2n+k+l$.

Proof. By the definition of the functions $A_j(x)$, $B_j(x)$, $C_j(x)$ and $D_j(x)$ it is clear that they are polynomials of degree $2n+k+l$.

Using $l_j(x_i)=\delta_{i,j}$ and $l_j^*(x_i^*)=\delta_{i,j}$ and (2.6), it is easy to verify that

$$A_j(x_i)=\delta_{j,i} \quad (i=1,\dots,n-1), \qquad A_j'(x_i^*)=0 \qquad (i=1,\dots,n),$$
$$A_j^{(s)}(1)=0 \qquad (s=0,\dots,k), \qquad A_j^{(s)}(-1)=0 \quad (s=0,\dots,l),$$

for $j=1,\dots,n-1$;

$$B_j(x_i)=0 \quad (i=1,\dots,n-1), \qquad B_j'(x_i^*)=\delta_{i,j} \quad (i=1,\dots,n),$$
$$B_j^{(s)}(1)=0 \quad (s=0,\dots,k), \qquad B_j^{(s)}(-1)=0 \quad (s=0,\dots,l),$$

for $j=1,\dots,n$;

$$C_j(x_i)=0 \qquad (i=1,\dots,n-1), \qquad C_j'(x_i^*)=0 \qquad (i=1,\dots,n),$$
$$C_j^{(s)}(1)=\delta_{j,s} \quad (s=0,\dots,j-1), \qquad C_j^{(s)}(-1)=0 \quad (s=0,\dots,l),$$

for $j=0,\dots,k$. Now let us write the polynomial $p_j(x)$ from (2.10) in the form

$$p_j(x)=a_0^{(j)}+a_1^{(j)}(1-x)+\dots+a_{k-j}^{(j)}(1-x)^{k-j}.$$

From the equations

$$\begin{cases} C_j^{(j)}(1)=1 \\ C_j^{(s)}(1)=0 \quad (s=j+1,\dots,k) \end{cases}$$

the coefficients of $p_j(x)$ can be determined easily and uniquely. The integrand in (2.10) is a polynomial, if for $s=0,\dots,k-j$

$$\frac{d^s}{dx^s}\Big[(1+x)\omega_{n-1}(x)p_j(x)-q_j(x)\omega_n^*(x)\Big]_{x=1}=0.$$

As the coefficients of the polynomial $p_j(x)$ have already been determined, the unknown coefficients of the polynomial $q_j(x)$ can also be determined uniquely from these equations.

In a similar way it can be shown that for $j=0,\dots,l$

$$D_j(x_i)=0 \quad (i=1,\dots,n-1), \qquad D'_j(x_i^*)=0 \qquad (i=1,\dots,n),$$
$$D_j^{(s)}(1)=0 \quad (s=0,\dots,k), \qquad D_j^{(s)}(-1)=\delta_{j,s} \quad (s=0,\dots,l).$$

□

Theorem 1 (Existence and uniqueness). *If* $\{y_i\}_{i=1}^{n-1}$, $\{y'_i\}_{i=1}^{n}$, $\{\alpha_j\}_{j=0}^{k}$, $\{\beta_j\}_{j=0}^{l}$ *are given real numbers,* $k\geq 0$, $l>0$ *are arbitrary fixed integers, then on the nodal points* (2.5) *there exists a unique polynomial* $Q_m(x)$ *of degree at most* $m=2n+k+l$ *satisfying the equations* (1.5) *and* (1.6).

The polynomial $Q_m(x)$ *can be written in the form*

$$Q_m(x)=\sum_{i=1}^{n-1}y_iA_i(x)+\sum_{i=1}^{n}y'_iB_i(x)+\sum_{j=0}^{k}\alpha_jC_j(x)+\sum_{j=0}^{l}\beta_jD_j(x), \tag{2.12}$$

where $A_i(x)$, $B_i(x)$, $C_j(x)$ *and* $D_j(x)$ *are defined in* Lemma 2.1.

Proof. By Lemma 2.1 it is clear that the polynomial (2.12) satisfies the conditions of the theorem, which proves the existence of interpolational polynomial $Q_m(x)$.

For the uniqueness we assume that there is another polynomial $Q_m^*(x)$ of degree $\leqq m$ which also satisfies the equations (1.5) and (1.6). Then the polynomial

$$R_m(x)=Q_m(x)-Q_m^*(x)$$

satisfies the equations $R_m(x_i)=0$ $(i=1,\dots,n-1)$ and

$$R_m^{(s)}(1)=0 \quad (s=0,\dots,k), \qquad R_m^{(s)}(-1)=0 \quad (s=0,\dots,l-1),$$

so it can be written in the form

$$R_m(x)=\varrho(x)\omega_{n-1}(x)g_n(x),$$

where $g_n(x)$ is a polynomial of degree at most n. Furthermore, from the equations $R'_m(x_i^*)=0$ $(i=1,\dots,n)$ we get by (2.6)

$$R'_m(x_i^*)=\varrho(x_i^*)\omega_{n-1}(x_i^*)g'_n(x_i^*)=0,$$

that is $g_n'(x_i^*) = 0$ for $i = 1, \dots, n$. It is possible only in the case $g_n'(x) \equiv 0$, that is $g_n(x) \equiv c$ constant, hence

$$R_m(x) = c\varrho(x)\omega_n(x).$$

But also the equation $R_m^{(l)}(-1) = 0$ is to be satisfied, so the constant $c = 0$, that is $R_m(x) \equiv 0$, which proves the uniqueness. □

3. The convergence

In this section we will prove the convergence of the interpolational procedure, if $k = l - 1 \geqq 0$, that is the knots $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ are the roots of the ultraspherical polynomials $P_{n-1}^{(k+1)}(x)$ and $P_n^{(k)}(x)$, respectively.

We will need the following results and estimates on the ultraspherical polynomials $P_n^{(\alpha)}(x) = P_n^{(\alpha,\alpha)}(x)$ $(\alpha > -1,\ n \geqq 1)$ (see [13]):

$$P_n^{(\alpha)}(x) = (-1)^n P_n^{(\alpha)}(-x); \tag{3.1}$$

from (2.4)

$$(1-x^2)P_n^{(\alpha)''}(x) - 2(\alpha+1)xP_n^{(\alpha)'}(x) + n(n+2\alpha+1)P_n^{(\alpha)}(x) = 0; \tag{3.2}$$

and

$$|P_n^{(\alpha)}(x)| = O\big(n^\alpha\big) \qquad x \in [-1,1], \tag{3.3}$$

$$(1-x^2)^{\frac{\alpha}{2}+\frac{1}{4}}|P_n^{(\alpha)}(x)| = O\left(\frac{1}{\sqrt{n}}\right) \qquad x \in [-1,1], \tag{3.4}$$

where $O(n)$ is independent of x.

If $\xi_1, \dots, \xi_n$ are the roots of $P_n^{(\alpha)}(x)$ then we have the asymptotical relations

$$1 - \xi_j^2 \sim \begin{cases} \dfrac{j^2}{n^2} & (\xi_j \geqq 0) \\ \dfrac{(n-j)^2}{n^2} & (\xi_j < 0) \end{cases} \tag{3.5}$$

$$|P_n^{(\alpha)'}(\xi_j)| \sim \begin{cases} \dfrac{n^{\alpha+2}}{j^{\alpha+\frac{3}{2}}} & (\xi_j \geqq 0) \\ \dfrac{n^{\alpha+2}}{(n-j)^{\alpha+\frac{3}{2}}} & (\xi_j < 0) \end{cases} \tag{3.6}$$

where $a_n \sim b_n$ means that $|a_n| = O(b_n)$ and $|b_n| = O(a_n)$. If $\ell_j(x)$ denotes the fundamental polynomial of Lagrange interpolation on the knots $\xi_1, \dots, \xi_n$ which corresponds to the knot ξ_j, then (see [13], [7])

$$(3.7)\quad \ell_j(x) = \frac{P_n^{(\alpha)}(x)}{P_n^{(\alpha)'}(\xi_j)(x-\xi_j)} = W_n^{(\alpha)} \cdot \frac{1}{(1-\xi_j^2)[P_n^{(\alpha)'}(\xi_j)]^2} \sum_{\nu=0}^{n-1} \frac{1}{h_\nu^{(\alpha)}} P_\nu^{(\alpha)}(\xi_j) P_\nu^{(\alpha)}(x)$$

where

$$(3.8)\qquad W_n^{(\alpha)} = 2^{2\alpha} \frac{\Gamma^2(n+\alpha+1)}{\Gamma(n+1)\Gamma(n+2\alpha+1)} \sim C_1$$

$$(3.9)\quad h_\nu^{(\alpha)} = \frac{2^{2\alpha+1}}{2\nu+2\alpha+1} \frac{\Gamma^2(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\nu+2\alpha+1)} \begin{cases} \sim \dfrac{1}{\nu} & (\nu > 0) \\ = C_2 & (\nu = 0) \end{cases}$$

where the constants C_1, C_2 depend only on α.

Lemma 3.1. *For the ultraspherical polynomial* $P_n^{(\alpha)}(x)$ $(\alpha > -1,$ $n \geqq 1)$ *on the interval* $[-1, 1]$

$$(3.10)\qquad (1-x^2)^{\frac{\alpha}{2}+\frac14} \left| \int_{-1}^x P_n^{(\alpha)}(t)\,dt \right| = O\big(n^{\alpha-2} + n^{-\frac32}\big),$$

$$(3.11)\qquad (1-x^2)^{\frac{\alpha}{2}+\frac14} \left| \int_{-1}^x (1+t) P_n^{(\alpha+1)}(t)\,dt \right| = O\big(n^{-\frac32} + n^{\alpha-3}\big)$$

where $O(n)$ *is independent of* x.

Proof. Integrating the differential equation (3.2) we get

$$\int_{-1}^x P_n^{(\alpha)}(t)\,dt = \frac{1}{n(n+2\alpha+1)+2\alpha} \times \Big[2\alpha x P_n^{(\alpha)}(x) - (1-x^2) P_n^{(\alpha)'}(x) + 2\alpha P_n^{(\alpha)}(-1)\Big],$$

and applying the estimates (3.3), (3.4) and

$$P_n^{(\alpha)'}(x) = \frac{n+2\alpha+1}{2} P_{n-1}^{(\alpha+1)}(x)$$

we get (3.10).

Integrating the second integral by parts we get

$$(1-x^2)^{\frac{\alpha}{2}+\frac14}\int_{-1}^{x}(1+t)P_n^{(\alpha+1)}(t)\,dt$$
$$=\frac{2}{n+2\alpha+2}\Big[(1+x)(1-x^2)^{\frac{\alpha}{2}+\frac14}P_{n+1}^{(\alpha)}(x)-(1-x^2)^{\frac{\alpha}{2}+\frac14}\int_{-1}^{x}P_{n+1}^{(\alpha)}(t)\,dt\Big].$$

Now applying (3.4) and (3.10) we get the estimate (3.11). □

Lemma 3.2. *If $k=l-1\geqq 1$, $n\geqq 2$ the Lebesgue function of the first kind fundamental polynomials* (2.8) *satisfies*

$$\sum_{j=1}^{n-1}(1-x_j^2)|A_j(x)|=\begin{cases}O\big(n\log n\big) & (k=1)\\ O\big(n^{k-\frac12}\big) & (k\geqq 2)\end{cases}\tag{3.12}$$

for all $x\in[-1,1]$ where $O(n)$ is independent of x.

Proof. If $k=l-1$, substituting (2.1), (2.2) and (2.3) into (2.8) we get for $j=1,\dots,n-1$

$$\begin{aligned}A_j(x)&=\frac{(1-x^2)^{k+1}(1+x)P_n^{(k)}(x)}{(1-x_j^2)^{k+1}(1+x_j)P_n^{(k)}(x_j)}l_j(x)\\&\quad-\frac{(n+2k+1)(1-x^2)^{k+1}P_{n-1}^{(k+1)}(x)}{2(1-x_j^2)^{k+1}(1+x_j)P_n^{(k)}(x_j)}\int_{-1}^{x}(1+t)l_j(t)\,dt\\&=A_{j,1}(x)+A_{j,2}(x)\end{aligned}$$

From (3.2) with $\alpha=k$ and by (2.3)

$$P_n^{(k)}(x_j)=-\frac{1-x_j^2}{2n}P_{n-1}^{(k+1)'}(x_j),$$

hence applying (3.7) with $\alpha=k+1$ we get

$$(1-x_j^2)A_{j,1}(x)=-2n\frac{(1-x_j)(1+x)(1-x^2)^{\frac{k}{2}+\frac14}P_n^{(k)}(x)}{(1-x_j^2)^{\frac{3k}{2}+\frac{15}{4}}\big[P_{n-1}^{(k+1)'}(x_j)\big]^3}W_{n-1}^{(k+1)}$$
$$\times\sum_{\nu=0}^{n-2}\frac{1}{h_\nu^{(k+1)}}(1-x_j^2)^{\frac{k}{2}+\frac34}P_\nu^{(k+1)}(x_j)(1-x^2)^{\frac{k}{2}+\frac34}P_\nu^{(k+1)}(x),\tag{3.13}$$

and in a similar way

$$(3.14)\quad \begin{aligned}(1-x_j^2)A_{j,2}(x)=&\frac{n(n+2k+1)(1-x_j)(1-x^2)^{\frac{k}{2}+\frac{3}{4}}P_{n-1}^{(k+1)}(x)}{(1-x_j^2)^{\frac{3k}{2}+\frac{15}{4}}\left[P_{n-1}^{(k+1)'}(x_j)\right]^3}W_{n-1}^{(k+1)}\\ &\times\sum_{\nu=0}^{n-2}\frac{1}{h_\nu^{(k+1)}}(1-x_j^2)^{\frac{k}{2}+\frac{3}{4}}P_\nu^{(k+1)}(x_j)(1-x^2)^{\frac{k}{2}+\frac{1}{4}}\\ &\times\int_{-1}^{x}(1+t)P_\nu^{(k+1)}(t)\,dt\end{aligned}$$

where we applied again (3.7) with $\alpha = k+1$. Using (3.5) and (3.6) with $P_{n-1}^{(k+1)}(x)$ we get

$$(3.15)\qquad \frac{1}{(1-x_j^2)^{\frac{3k}{2}+\frac{15}{4}}|P_{n-1}^{(k+1)'}(x_j)|^3}=O\big(n^{-\frac{3}{2}}\big)\qquad (n\geqq 2)$$

hence for $k\geqq 1$ applying (3.4), (3.11) and (3.15)

$$(1-x_j^2)|A_{j,1}(x)|=O(1)n\,\frac{1}{\sqrt{n}}\,n^{-\frac{3}{2}}\left(C_1+\sum_{\nu=1}^{n-2}\nu\frac{1}{\sqrt{\nu}}\frac{1}{\sqrt{\nu}}\right)=O(1)$$

and

$$\begin{aligned}(1-x_j^2)|A_{j,2}(x)|&=O(1)n^2\,\frac{1}{\sqrt{n}}n^{-\frac{3}{2}}\left(C_2+\sum_{\nu=1}^{n-2}\nu\,\frac{1}{\sqrt{\nu}}(\nu^{-\frac{3}{2}}+\nu^{k-3})\right)\\ &=\begin{cases}O(\log n) & (k=1)\\ O\big(n^{k-\frac{3}{2}}\big) & (k\geqq 2)\end{cases}\end{aligned}$$

where the constants C_1, C_2 are independent of x, n. Taking the sum for j we get the statement of the lemma. □

Lemma 3.3. *If $k=l-1\geqq 1$, $n\geqq 2$, the Lebesgue function of the second kind fundamental polynomials* (2.9) *satisfies for all $x\in[-1,1]$*

$$(3.16)\qquad \sum_{j=1}^{n}|B_j(x)|=O\big(n^{k-\frac{1}{2}}\big)\qquad (k\geqq 1)$$

where $O(n)$ is independent of x.

Proof. For $k=l-1$ using (2.3) and (3.7) we can write (2.9) in the following form

$$\begin{aligned} B_j(x) &= \frac{(1-x^2)^{k+1}\omega_{n-1}(x)}{(1-x_j^{*2})^{k+1}\omega_{n-1}(x_j^*)}\int_{-1}^{x} l_j^*(t)\,dt \\ &= \frac{n+2k+1}{2}\,\frac{(1-x^2)^{\frac{k}{2}+\frac{3}{4}}P_{n-1}^{(k+1)}(x)}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}\big[P_n^{(k)'}(x_j^*)\big]^3}\cdot W_n^{(k)} \\ &\quad\times\sum_{\nu=0}^{n-1}\frac{1}{h_\nu^{(k)}}(1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}}P_\nu^{(k)}(x_j^*)(1-x^2)^{\frac{k}{2}+\frac{1}{4}}\int_{-1}^{x}P_\nu^{(k)}(t)\,dt. \end{aligned}$$

Using (3.5) and (3.6) with $P_n^{(k)}(x)$ we get

$$\frac{1}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_n^{(k)'}(x_j^*)|^3}=O\big(n^{-\frac{3}{2}}\big),$$

and applying (3.4), (3.8), (3.9) and Lemma 3.1 in a similar way as in Lemma 3.2 we get the estimates

$$|B_j(x)|=O\big(n^{k-\frac{3}{2}}\big)\qquad (k\geqq 1).$$

Finally, taking the sum for $j=1,\dots,n$ our statement is proved. □

Theorem 2. *Let $k\geqq 0$ be a fixed integer, $m=2n+2k+1$, and let the knots $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ be the roots of the ultraspherical polynomials $P_{n-1}^{(k+1)}(x)$ and $P_n^{(k)}(x)$, respectively. If $f\in C^r[-1,1]$ ($r\geqq k+1$, $n\geqq 2r-k+2$), then the interpolational polynomial*

$$\begin{aligned} Q_m(x;f) &= \sum_{i=1}^{n-1} f(x_i)A_i(x)+\sum_{i=1}^{n} f'(x_i^*)B_i(x) \\ &\quad+\sum_{j=0}^{k} f^{(j)}(1)C_j(x)+\sum_{j=0}^{k+1} f^{(j)}(-1)D_j(x) \end{aligned} \tag{3.17}$$

with the fundamental polynomials given in Lemma 2.1 *satisfies*

$$|f(x)-Q_m(x;f)|=\omega(f^{(r)};\frac{1}{n})\begin{cases} O\big(n^{-r+1}\big) & (k=0)\\ O\big(n^{k-r+\frac{1}{2}}\big) & (k\geqq 1)\end{cases}$$

for $x\in[-1,1]$.

PROOF. For $k=0$ we refer to the proof of the theorem of XIE [16] and we prove the statement for $k \geqq 1$. Let $f \in C^r[-1,1]$, then by the theorem of GOPENGAUZ [3] for every $m \geqq 4r+5$ there exists a polynomial $p_m(x)$ of degree at most m such that for $j=0,\dots,r$

$$|f^{(j)}(x)-p_m^{(j)}(x)| \leqq M_{r,j}\left(\frac{\sqrt{1-x^2}}{m}\right)^{r-j}\omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{m}\right),$$

where $\omega\big(f^{(r)};\cdot\big)$ denotes the modulus of continuity of the function $f^{(r)}(x)$ and the constants $M_{r,j}$ depend only on r and j. Moreover,

$$f^{(j)}(\pm 1)=p_m^{(j)}(\pm 1) \qquad (j=0,\dots,r).$$

Hence for $x \in [-1,1]$

$$\begin{aligned}
|f(x)-Q_m(x;f)| &\leqq |f(x)-p_m(x)|+|p_m(x)-Q_m(x;f)| \leqq |f(x)-p_m(x)| \\
&\quad+\sum_{i=1}^{n-1}|f(x_i)-p_m(x_i)|\,|A_i(x)|+\sum_{i=1}^{n}|f'(x_i^*)-p_m'(x_i^*)|\,|B_i(x)| \\
&\leqq M_{r,0}\frac{1}{n^r}\omega\left(f^{(r)};\frac{1}{n}\right)\left(1+\sum_{i=1}^{n-1}(1-x_i^2)|A_i(x)|\right) \\
&\quad+M_{r,1}\frac{1}{n^{r-1}}\omega\left(f^{(r)};\frac{1}{n}\right)\sum_{i=1}^{n}|B_i(x)|
\end{aligned}$$

Applying (3.12) and (3.16) we get

$$|f(x)-Q_m(x;f)|=O(1)\frac{1}{n^r}\big(1+\alpha_n+n^{k-1-\frac{1}{2}}\big)\omega\left(f^{(r)};\frac{1}{n}\right)$$

where $\alpha_n = n\log n$ or $n^{k-\frac{1}{2}}$, according to $k=1$ and $k \geqq 2$, which completes the proof for $k \geqq 1$. □

As a corollary of Theorem 2 we can state the following convergence theorem:

Theorem 3. *Let $k \geqq 0$ be a fixed integer, $m=2n+2k+1$, $n \geqq k+4$, and let the knots $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ be the roots of the ultraspherical polynomials $P_{n-1}^{(k+1)}(x)$ and $P_n^{(k)}(x)$, respectively. If $f \in C^{k+1}[-1,1]$, then $Q_m(x;f)$ described in* (3.17) *uniformly converges to $f(x)$ on $[-1,1]$ as $n \to \infty$.*

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