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Homomorphisms from $\mathbb{C}^*$ into $GL_n(\mathbb{C})$

By MATJAŽ OMLADIČ (Ljubljana), HEYDAR RADJAVI (Halifax)
and PETER ŠEMRL (Ljubljana)

Abstract. We determine all homomorphisms of the multiplicative group $\mathbb{C}^*$ into $GL_n(\mathbb{C})$. As a consequence we get a complete description of homomorphisms of $GL_n(\mathbb{C})$ into $GL_m(\mathbb{C})$, $m < n$. We also obtain the general form of multiplicative maps $g : \mathbb{C} \to M_n(\mathbb{C})$.

1. Introduction

Let F be any field. The theory of endomorphisms of the group $GL_n(F)$ is highly non-trivial, but well known. Our work is motivated by two results that can be proved in an elementary way. Motivated by a problem concerning homogeneous linear geometric objects of type $(m, n, 1)$ M. Kucharzewski and A. Zajtz [4] determined all homomorphisms of $GL_n(\mathbb{R})$ into $GL_m(\mathbb{R})$. Here, $m \leq n$. If $m < n$ then every homomorphism $\phi : GL_n(\mathbb{R}) \to GL_m(\mathbb{R})$ is of the form

$$\phi(A) = g(\det A) \tag{1}$$

where $g : \mathbb{R}^* \to GL_m(\mathbb{R})$ is a group homomorphism. For $m = n$, ϕ either has the form (1) with g being multiplicative, or

$$\phi(A) = f(\det A) T A T^{-1}$$

or

$$\phi(A) = f(\det A) T \left(A^{-1}\right)^{tr} T^{-1}$$

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where T is an invertible real $n \times n$ matrix and f is an endomorphism of the multiplicative group $\mathbb{R}^*$. The general form of such functions f is well known [1]. So, in order to understand the structure of homomorphisms from $GL_n(\mathbb{R})$ into $GL_m(\mathbb{R})$ completely, one has to solve the functional equation $g(ts) = g(t)g(s)$, where g maps $\mathbb{R}^*$ into $GL_m(\mathbb{R})$. The special case $m = 3$ has been completely solved by M. Kuczma and A. Zajtz [5]. Even in this low dimensional case the description of the general form of such maps is quite complicated. For the solution of this problem under the additional measurability hypothesis see [6] and [8].

Djoković [3] observed that the case $m < n$ can be extended to division rings of characteristic $\neq 2$. More precisely, he proved that if K is a division ring, char $K \neq 2$, $m < n$, and $\phi : GL_n(K) \to GL_m(K)$ is a homomorphism, then there exists a homomorphism $g : K^*/C \to GL_m(K)$ such that $\phi(A) = g(\det A)$. Here, C denotes the commutator subgroup of K^* and $\det A$ is Dieudonne's determinant.

The above results motivate the study of homomorphisms from a multiplicative group of nonzero elements of a given field to the general linear group over this field. As the real 3 by 3 case shows [5] this problem might be very difficult in general.

When studying the multiplicative Cauchy functional equation $g(ts) = g(t)g(s)$ for real functions a possible approach is to transform this equation into the logarithmic Cauchy equation $f(ts) = f(t) + f(s)$. Let $\mathcal{A}$ be any finite-dimensional algebra over the field F. Comparing the functional equations $g(ts) = g(t)g(s)$ and $f(ts) = f(t) + f(s)$ for mappings $f, g : F^* \to \mathcal{A}$ we notice that the second one is much easier to solve. Namely, let $\{a_1, \dots, a_m\}$ be a basis of $\mathcal{A}$. Then for every $t \in F^*$ there exist uniquely determined $k_1(t), \dots, k_m(t) \in F$ such that

$$f(t) = k_1(t)a_1 + \cdots + k_m(t)a_m.$$

Obviously, the functions $k_i : F^* \to F$, $i = 1, \dots, m$, satisfy the logarithmic Cauchy functional equation. So, it is enough to know the structure of logarithmic functions on the field in order to get the general form of logarithmic mappings $f : F^* \to \mathcal{A}$. Therefore, one possible approach to our problem would be to reduce it to the problem of describing the set of all logarithmic functions from F^* into the set of all $n \times n$ matrices over F. We will show that this approach works in the complex case.

2. The complex case

For the proof of our main result we will need the following simple lemma. We denote by $M_n(\mathbb{C})$ the algebra of all $n \times n$ complex matrices.

Lemma 2.1. *Let m be a positive integer and let $\phi : \mathbb{C}^* \to M_m(\mathbb{C})$ be a mapping satisfying the functional equation*

$$\phi(\lambda\mu) = \phi(\lambda) + \phi(\mu), \quad \lambda, \mu \in \mathbb{C}^*.$$

Assume also that $\phi(\lambda)$ is nilpotent for every nonzero λ and that $\phi(\lambda)$ and $\phi(\mu)$ commute for all $\lambda, \mu \in \mathbb{C}^*$. Then there exist pairwise commutative $m \times m$ nilpotents $N_1, \dots, N_l$, and additive functions $h_1, \dots, h_l : \mathbb{C} \to \mathbb{C}$ satisfying $h_r(2\pi i) = 0$, $r = 1, \dots, l$, such that

$$\phi(\lambda) = h_1(\log \lambda)N_1 + \cdots + h_l(\log \lambda)N_l$$

for every nonzero λ.

Remark. Note that $h_r(\log \lambda)$, $1 \le r \le l$, is well defined because of the requirement $h_r(2\pi i) = 0$.

Proof. Let $\{N_1, \dots, N_l\}$ be a maximal linearly independent subset of $\{\phi(\lambda) : \lambda \in \mathbb{C}^*\}$. Then for every nonzero complex number λ there exist uniquely determined complex numbers $q_1(\lambda), \dots, q_l(\lambda)$ such that

$$\phi(\lambda) = q_1(\lambda)N_1 + \ldots + q_l(\lambda)N_l.$$

Obviously,

$$q_r(\lambda\mu) = q_r(\lambda) + q_r(\mu)$$

for every $\lambda, \mu \in \mathbb{C}^*$ and every r, $1 \le r \le l$. If we define $h_r : \mathbb{C} \to \mathbb{C}$ by

$$h_r(\lambda) = q_r(e^{\lambda}), \quad \lambda \in \mathbb{C},$$

then clearly each h_r, $1 \le r \le l$, is additive with $h_r(2\pi i) = 0$. Hence,

$$\phi(\lambda) = h_1(\log \lambda)N_1 + \ldots + h_l(\log \lambda)N_l.$$

This completes the proof. $\square$

Theorem 2.2. *Let n be a positive integer and let $g : \mathbb{C}^* \to GL_n(\mathbb{C})$ be a map satisfying $g(\lambda\mu) = g(\lambda)g(\mu)$ for all nonzero complex numbers λ and μ. Then there exist an integer $k \geq 1$ and an invertible $n \times n$ matrix T such that*

$$(2) \quad g(\lambda) = T \begin{bmatrix} f_1(\lambda)e^{\phi_1(\lambda)} & 0 & \dots & 0 \\ 0 & f_2(\lambda)e^{\phi_2(\lambda)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_k(\lambda)e^{\phi_k(\lambda)} \end{bmatrix} T^{-1}, \quad \lambda \in \mathbb{C}^*.$$

Here, $\phi_p : \mathbb{C}^* \to M_{n_p}$, $1 \leq p \leq k$, is a mapping as in Lemma 2.1*, $n = n_1 + \ldots + n_k$, and $f_p : \mathbb{C}^* \to \mathbb{C}^*$ is a multiplicative function, $1 \leq p \leq k$.*

Proof. First we will prove by induction on n that there exist an integer $k \geq 1$ and an invertible $n \times n$ matrix T such that

$$g(\lambda) = T \begin{bmatrix} \rho_1(\lambda) & 0 & \dots & 0 \\ 0 & \rho_2(\lambda) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_k(\lambda) \end{bmatrix} T^{-1}, \quad \lambda \in \mathbb{C}^*,$$

where for every $\lambda \in \mathbb{C}^*$ and every integer p, $1 \leq p \leq k$, $\rho_p(\lambda)$ is an invertible $n_p \times n_p$ matrix having only one eigenvalue. In the case that $n = 1$ there is nothing to prove. So, assume that $n > 1$ and that our statement holds true for all positive integers smaller than n. If every $g(\lambda)$, $\lambda \in \mathbb{C}^*$, has only one eigenvalue, then we are done. So, let λ_0 be a nonzero complex number such that $g(\lambda_0)$ has at least two eigenvalues. Applying similarity, if necessary, we can assume with no loss of generality that

$$g(\lambda_0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

with A and B having no common eigenvalue. Now, as $g(\lambda)$ commutes with $g(\lambda_0)$ for every nonzero λ, we have

$$g(\lambda) = \begin{bmatrix} \tau_1(\lambda) & 0 \\ 0 & \tau_2(\lambda) \end{bmatrix}$$

for every $\lambda \in \mathbb{C}^*$. Applying the induction hypothesis on mappings τ_1 and τ_2 we conclude the induction proof of our first step.

So, from now on we can restrict ourselves to the case where multiplicative $\rho : \mathbb{C}^* \to GL_n(\mathbb{C})$ maps every nonzero λ into an invertible matrix with one eigenvalue. So, $\rho(\lambda) = f(\lambda)I + N(\lambda)$, where $f : \mathbb{C}^* \to \mathbb{C}^*$ is multiplicative and $N(\lambda)$ is nilpotent for every nonzero λ. A mapping $\psi : \mathbb{C}^* \to GL_n(\mathbb{C})$ defined by

$$\psi(\lambda) = (f(\lambda))^{-1}\rho(\lambda), \quad \lambda \in \mathbb{C},$$

is multiplicative. Moreover, for every $\lambda \in \mathbb{C}^*$ the only eigenvalue of $\psi(\lambda)$ is 1. So, $\psi(\lambda) = I + M(\lambda)$ with $M(\lambda)$ nilpotent. Therefore, we can define a new mapping $\phi : \mathbb{C}^* \to M_n(\mathbb{C})$ by

$$\phi(\lambda) = \log \psi(\lambda) = M(\lambda) - \frac{1}{2}M(\lambda)^2 + \ldots + \frac{(-1)^n}{n-1}M(\lambda)^{n-1}.$$

Obviously, $\phi(\lambda)$ is nilpotent for every nonzero complex number λ. Moreover, $\phi(\lambda)$ and $\phi(\mu)$ commute for all $\lambda, \mu \in \mathbb{C}^*$. It is easy to see that

$$\phi(\lambda\mu) = \phi(\lambda) + \phi(\mu), \quad \lambda, \mu \in \mathbb{C}^*.$$

Applying the fact that

$$\rho(\lambda) = f(\lambda)\psi(\lambda) = f(\lambda)\exp(\phi(\lambda))$$

we complete the proof. $\square$

As an application of Theorem 2.2 we will characterize homomorphisms of the multiplicative semigroup $\mathbb{C}$ into $M_n(\mathbb{C})$.

Theorem 2.3. *Let n be a positive integer and let $g : \mathbb{C} \to M_n(\mathbb{C})$ be a nonzero map satisfying $g(\lambda\mu) = g(\lambda)g(\mu)$ for all complex numbers λ and μ. Then there exist an integer $k \geq 0$, an integer $s \geq 0$, and an invertible $n \times n$ matrix T such that*

$$g(\lambda) = T \begin{bmatrix} I_s & 0 & 0 & \ldots & 0 & 0 \\ 0 & f_1(\lambda)e^{\phi_1(\lambda)} & 0 & \ldots & 0 & 0 \\ 0 & 0 & f_2(\lambda)e^{\phi_2(\lambda)} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & f_k(\lambda)e^{\phi_k(\lambda)} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} T^{-1},$$

$\lambda \in \mathbb{C}^*$, *and*

$$g(0) = T \begin{bmatrix} I_s & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} T^{-1}.$$

Here, $\phi_p : \mathbb{C}^* \to M_{n_p}$, $1 \le p \le k$, *is a mapping as in Lemma 2.1,* $n \ge s + n_1 + \cdots + n_k > 0$, I_s *is an* $s \times s$ *identity matrix, and* $f_p : \mathbb{C}^* \to \mathbb{C}^*$ *is a multiplicative function,* $1 \le p \le k$.

Proof. Obviously, $g(1)$ and $g(0)$ are idempotents satisfying $g(1)g(0) = g(0)g(1) = g(0)$ and $g(1) \neq 0$. Applying similarity, if necessary, we can assume that

$$g(1) = \begin{bmatrix} I_s & 0 & 0 \\ 0 & I_j & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$g(0) = \begin{bmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $s, j \ge 0$. Of course, in the cases $n = s + j$, $s = 0$, or $j = 0$ some columns and rows are absent. From $g(0) = g(0)g(\lambda) = g(\lambda)g(0)$ and $g(1)g(\lambda) = g(\lambda)g(1) = g(\lambda)$ we get that

$$g(\lambda) = \begin{bmatrix} I_s & 0 & 0 \\ 0 & g_1(\lambda) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda \in \mathbb{C}^*,$$

with $g_1 : \mathbb{C}^* \to M_j(\mathbb{C})$ being multiplicative. Using $g(\lambda)g(\lambda^{-1}) = g(1)$ we see that $g_1(\lambda)$ is invertible for every nonzero λ. Applying Theorem 2.2 we complete the proof. $\square$

3. Remarks

1. Applying the result of Djoković [3] together with our Theorem 2.2 we get a complete understanding of the structure of homomorphisms from

$GL_n(\mathbb{C})$ into $GL_m(\mathbb{C})$, $m < n$. The general form of additive functions $h : \mathbb{C} \to \mathbb{C}$ satisfying $h(2\pi i) = 0$ is well known [1]. So, to get the complete description of the set of homomorphisms from $GL_n(\mathbb{C})$ into $GL_m(\mathbb{C})$, $m < n$, we have to characterize all multiplicative functions $f : \mathbb{C}^* \to \mathbb{C}^*$. Although we believe that the structure of such functions is known we were not able to find the description of their general form in the literature. Therefore we will briefly explain the structure of multiplicative functions on $\mathbb{C}^*$.

In what follows $\arg\lambda \in [0, 2\pi)$ denotes the argument of a nonzero complex number λ. Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a multiplicative function. For every $\lambda \in \mathbb{C}^*$ we have

$$f(\lambda) = f(|\lambda|) f\left(\frac{\lambda}{|\lambda|}\right),$$

and so, it is enough to describe the general form of multiplicative functions $f_1 : (0, \infty) \to \mathbb{C}^*$ and $f_2 : S^1 \to \mathbb{C}^*$ in order to understand the structure of multiplicative functions on $\mathbb{C}^*$. Here, S^1 denotes the set of all complex numbers of modulus one. If $f_1 : (0, \infty) \to \mathbb{C}^*$ is a multiplicative function then $g_1 : (0, \infty) \to (0, \infty)$ defined by $g_1(t) = |f_1(t)|$ is multiplicative while $g_2 : \mathbb{R} \to [0, 2\pi)$ defined by $g_2(t) = \arg f_1(e^t)$ is additive modulo $2\pi\boldsymbol{Z}$, that is, $g_2(t+s) - g_2(t) - g_2(s) \in 2\pi\boldsymbol{Z}$. Here, $\boldsymbol{Z}$ denotes the set of all integers. The structure of multiplicative functions on the set of positive real numbers is known [1]. The general form of real functions additive modulo $2\pi\boldsymbol{Z}$ was obtained by van der Corput [2] and Vietoris [7]. Similarly, if $f_2 : S^1 \to \mathbb{C}^*$ is multiplicative then $h_1 : \mathbb{R} \to (0, \infty)$ defined by $h_1(t) = |f_2(e^{it})|$ is a periodic function satisfying the Cauchy exponential functional equation $h_1(t+s) = h_1(t)h_1(s)$. The general form of such functions is known [1]. The function $h_2(t) = \arg f_2(e^{it})$, $t \in \mathbb{R}$, is a periodic function additive modulo $2\pi\boldsymbol{Z}$. Van der Corput and Vietoris obtained the general form of real functions additive modulo $2\pi\boldsymbol{Z}$. A slight modification of their arguments gives a general form of real peridoic functions that are additive modulo $2\pi\boldsymbol{Z}$. So, it is possible to obtain the general form of multiplicative functions defined on $\mathbb{C}^*$. As the formulation of the result is rather long we will omit it.

2. It would be nice to have an analogue of Theorem 2.2 for some other fields. We believe that the real case is especially interesting. If we can solve the functional equation $g(ts) = g(t)g(s)$ for functions $g : \mathbb{R}^* \to GL_m(\mathbb{R})$, then we get the complete description of homomorphisms from $GL_n(\mathbb{R})$ into $GL_m(\mathbb{R})$ for $m \le n$.

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MATJAŽ OMLADIČ
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LJUBLJANA
JADRANSKA 19
1000 LJUBLJANA
SLOVENIA

E-mail: matjaz.omladic@fmf.uni-lj.si

HEYDAR RADJAVI
DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
CANADA B3H 4H8

E-mail: radjavi@mscs.dal.ca

PETER ŠEMRL
INSTITUTE OF MATHEMATICS, PHYSICS, AND MECHANICS
JADRANSKA 19
1000 LJUBLJANA
SLOVENIA

E-mail: peter.semrl@imfm.uni-lj.si

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