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Perfect measures and equicontinuity

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Abstract. In this paper we give a characterization of the equicontinuous subsets of the space $M_p(X,E')$ of vector-valued perfect measures on a completely regular topological space X generalizing the result of KOUMOULLIS [2] for the scalar case.

Let X be a completely regular Hausdorff space, E a normed space over K, the field of real or complex numbers, and $C_b(X,E)$ the space of all continuous E-valued functions on X. As in [2] $M_p(X)$ denotes the space of all scalar-valued Baire perfect measures on X, which is the dual of the space $C_b(X)$ with the topology β_p. For a real valued continuous function f defined on X, $f^{\sim}$ is its continuous extension to βX. For $\mu \in (C_b(X), \|.\|'$, $\mu^{\sim} : C(\beta X) \longmapsto K$ is defined by $\mu^{\sim}(f^{\sim}) = \mu(f^{\sim}_{|X})$. A subset G in a completely regular Hausdorff space Y will be called *distinguished* if there exist a continuous mapping ϕ from Y onto a separable metric space such that $G = \phi^{-1}(\phi(G))$. The class of all distinguished subsets of $\beta(X)$ disjoint from X will be denoted by $D(\beta X)$. For D in $D(\beta X)$, the topology γ_D, on $C_b(X,E)$ is defined to be the one generated by the seminorms $\|.\|_g$, as g varies over the set $B_D(X)$ of all bounded scalar-valued functions on βX, vanishing at infinity and zero on D, $\|f\|_g = \sup_{x\in\beta(X)}(\|f(x)\|^{\sim}|g(x)|)$. As in [3, Theorem 2.4] it can be verified that γ_D is the finest locally convex topology agreeing with the topology of uniform convergence on compact subsets of $\beta(X\backslash D)$, on norm-bounded subsets of $C_b(X,E)$. The topology β_p on $C_b(X,E)$ is defined in [1] as $\beta_p = \bigwedge\{\gamma_D : D \in D\}$. A locally convex topology on $C_b(X,E)$ will be called *locally solid* if it has

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a 0-nbd base consisting of absolutely convex sets V, such that $f \in V$ and $\|g\| \leq \|f\|$ implies $g \in V$. It is proved in [1, Theorem 5] that the topology β_p on $C_b(X,E)$ is locally solid. For a duality $\langle F, G\rangle$, $A \subset F$ the polar of A is defined as $A^o = \{g \in G, |\langle a, g\rangle| \leq 1$, for every $a \in A\}$. We define $M_p(X,E')$ as $M_p(X,E') = \{\mu \in M_\sigma(X,E)$, $\mu_x \in M_p(X)$, for every $x \in E\}$, here μ_x is defined as $\mu_x(f) = \mu(f \otimes x)$. In [1, Theorem 8] it was proved that if $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_p)$ then the dual of $(C_b(X,E), \beta_p)$ is $M_p(X,E')$. In $C_b(X)$, the space of all bounded real-valued continuous functions defined on X, has also been defined the strict topology β_o which yield as a dual space a subspace of $M_p(X)$ which is the space $M_t(X)$ of tight Baire measures [3] and it is well known that $M_p(X) = M_t(X)$ when X is a separable metric space [4]. Also a subset H of $M_t(X)$ is β_o equicontinuous if H is norm bounded and for every $\epsilon > 0$ there is a compact subset K of X such that $|\mu|_*(X \backslash (K)) < \epsilon$ for every $\mu \in H$ [4].

The main purpose of this work is to extend to the vector case the characterization of equicontinuity of subsets of scalar perfect measures given by Koumoullis in [2].

Lemma. *Let X and Y be completely regular spaces, $\varphi : X \to Y$ a continuous map, E a normed vector space, $L \in E'$ and $e \in E$. We define the maps*

$$(C_b(Y), \beta_p) \xrightarrow{\psi_e} (C_b(Y,E), \beta_p) \xrightarrow{T_\varphi} (C_b(X,E), \beta_p) \xrightarrow{\Phi_L} (C_b(X), \beta_p)$$

by $\Psi_e(g) = g \otimes e$, $T_\varphi(h) = h \circ \varphi$ and $\Phi_L(h) = L \circ h$. Then we have the following:

1. *Ψ_e, T_φ and Φ_L are continuous.*
2. *If $L(e) = 1$, $\Phi_L \circ T_\varphi \circ \Psi_e = T^*_\varphi$ where T^*_φ is given by $T^*_\varphi(g) = g \circ \varphi$*
3. *If $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_p)$ and H is a β_p-equicontinuous subset of $M_p(X,E')$, then $\{|\mu| \circ \Phi_L : \mu \in H\}$ is an β_p -equicontinuous subset of $M_p(X,E')$ and $\left\{|\mu| \circ T^*_\varphi : \mu \in H\right\}$ is an β_p-equicontinuous subset of $M_p(Y)$*

Proof.

1. The continuity of T_φ is Theorem 3 in [1]. The proofs of the continuity of Ψ_e and Φ_L are just easy applications of Corollary 6 in [1].

2. For any $g \in C_b(Y)$ we have that:

$$\begin{aligned}\Phi_L \circ T_\varphi \circ \Psi(g) &= \Phi_L(T_\varphi(g \otimes e)) = \Phi_L((g \otimes e) \circ \varphi) \\ &= (L \circ (g \otimes e) \circ \varphi)) = g \circ \varphi = T^*_\varphi(g)\end{aligned}$$

therefore the equality follows.

3. From Theorem 1 in [1] we have that $|\mu| \in M_p(X)$ for every $\mu \in M_p(X, E')$. Thus the set $\{|\mu| \circ \Phi_L : \mu \in H\}$ is included in $M_p(X, E')$ since Φ_L is continuous. To prove the equicontinuity of this set we can assume without lost of generality that $\|L\| \leq 1$. Since H is β_p-equicontinuous, there exists an absolutely convex solid O-neighborhood W in $(C_b(X, E), \beta_p)$ such that $W \subset H^o$. Now for every $f \in W$ and $\mu \in H$,

$$\begin{aligned}|\mu| \circ \Phi_L(f) = |\mu(L \circ f) &= \sup\{|\mu(h)| : h \in C_b(X, E),\ \|h\| \leq |L \circ f|\} \\ &\leq \sup\{|\mu(h)| : h \in C_b(X, E),\ \|h\| \leq \|f\|\} \\ &\leq \sup\{|\mu(h)| : h \in W\} \leq 1,\end{aligned}$$

where the first inequality follows from $|L \circ f| \leq \|L\| \leq \|f\|$ and the second from the fact that W is solid. Therefore $\{|\mu| \circ \Phi_L : \mu \in H\}$ is β_p-equicontinuous. For the proof of the equicontinuity of the set $\left\{|\mu| \circ T^*_\varphi : \mu \in H\right\}$ we choose L and e so that $L(e) = 1$. As before we observe that this set is included in $M_p(Y)$. Since

$$T_\varphi \circ \Psi_e : (C_b(Y), \beta_p) \longrightarrow (C_b(X, E), \beta_p)$$

is a continuous operator (by (1)) and

$$\left\{|\mu| \circ T^*_\varphi : \mu \in H\right\} = \{|\mu| \circ \Phi_L \circ T_\varphi \circ \Psi_e : \mu \in H\}$$

(by (2)), the conclusion follows from the equicontinuity of $\{|\mu| \circ \Phi_L : \mu \in H\}$ □

Main Theorem

Theorem. *Suppose $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$. A subset H of $M_p(X, E')$ is β_p-equicontinuous if and only if: (a) H is norm bounded and (b) for any continuous function f from X onto a separable metric* *space Y and every $\epsilon > 0$ there is a compact subset K of Y such that $|\mu|(X\backslash f^{-1}(K)) < \epsilon$ for every $\mu \in H$.*

Proof. Let $\phi : X \longmapsto Y$ be a continuous map from X onto the separable metric space Y and H an equicontinuous subset of $M_p(X, E')$. Then by the Lemma, $\left\{|\mu| \circ T_\phi^* : \mu \in H\right\}$ is an β_o-equicontinuous subset of $M_t(Y)$, and by Theorem 7.15 [3] for any $\epsilon \geq 0$ there exists a compact subset $K \subseteq Y$ such that $|\mu|(X\backslash\phi^{-1}(K)) \leq \epsilon$ for every $\mu \in H$. Also, since $\beta_p \leq \| \ \|$, H is norm bounded.

On the other hand, if $H \subseteq M_p(X, E')$ satisfies (a) and (b). Put $\alpha_o = \sup\{|\mu| : \mu \in H\}$. To prove that H is β_p-equicontinuous is equivalent to prove that H is γ_D-equicontinuous for every $D \in D(\beta X)$. Also since γ_D is the finest locally convex topology agreeing with the topology of uniform convergence on compact subsets of $\beta X\backslash D$, on norm-bounded subsets of $C_b(X, E)$ it is enough to prove that for any $k > 0$, there exist a compact subset $K_D \subseteq \beta X\backslash D$ and some $\eta > 0$ such that the set

$$K = \left\{f \in C_b(X, E) : \|f\| \leq k, \ \|f\|^{\sim}_{|K_D} < \eta\right\}$$

is contained in the set

$$\{g \in C_b(X, E) : |\mu(g)| \leq 1, \ \mu \in H\}.$$

Take $\eta = 1/(2 + 2\alpha_o)$. Since D is a distinguished subset of βX disjoint from X there exist a continuous perfect map ρ from βX onto a compact metric space M such that $D = \rho^{-1}(\rho(D))$. Since $\rho(X)$ is a separable metric space it follows from (b) that there exists a compact subset K of $\rho(X)$ such that $|\mu|(X\backslash\rho^{-1}(K)) < 1/(2k + 1)$ for every $\mu \in H$. Take $K_D = \rho^{-1}(K)$. Then K_D is a compact subset of βX and is included in $\beta X\backslash D$ because $D = \rho^{-1}(\rho(D))$ and D is disjoint from X. Therefore we have that

$$|\mu|^{\sim}[(\beta X\backslash D)\backslash K_D] < 1/(2k + 1).$$

Then if $f \in K$ and $\mu \in H$ we have

$$\begin{aligned} |\mu|(f) &\leq \int \|f\| d|\mu| = \int \|f\|^{\sim} d|\mu|^{\sim} \\ &\leq \int_{K_D} \|f\|^{\sim} d|\mu|^{\sim} + \int_{(\beta X\backslash D)\backslash K_D} \|f\|^{\sim} d|\mu|^{\sim} \\ &\leq \alpha_o/(2 + 2\alpha_o) + k/(2k + 1) \leq 1 \end{aligned}$$

this proves that H is β_p-equicontinuous. □

Corollary. *Suppose $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_p)$. A subset H of $M_p(X,E')$ is β_p-equicontinuous if and only if the set $\{|\mu| : \mu \in H\}$ is a β_p-equicontinuous subset of $M_p(X)$*

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