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Contact CR-submanifolds in Sasakian manifolds – a foliated approach

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Abstract. We study properties of contact CR-submanifolds of Sasakian manifolds. We consider a Sasakian manifold as a foliated manifold. Using the correspondence between foliated and transverse structure we reduce many results about geometrical objects in Sasakian manifolds to the corresponding results in Kähler manifolds.

In this short note we study properties of contact CR-submanifolds of Sasakian manifolds. For us a Sasakian manifold is a manifold foliated by a very particular transversely Kähler foliation. In fact the one-dimensional foliation of a Sasakian manifold generated by the characteristic vector field is a transversely Kähler isometric flow. We call this foliation the characteristic foliation of a Sasakian manifold. Using the correspondence between foliated and transverse structures, cf. [21, 22], we can reduce many theorems about geometrical objects in Sasakian manifolds to theorems about corresponding objects in Kähler manifolds. For the first time this approach has been used successfully by H. Reckziegel in [17] to study horizontal submanifolds. Although Reckziegel did not use the words "foliation" and "foliated" his approach is essentially a "foliated" one. The effectiveness of this method can be easily tested comparing results about the local structure of Sasakian manifolds with those about Kähler manifolds in two books of K. Yano and M. Kon, cf. [23. 24].

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1. Introduction

We recall some basic definitions on foliations and foliated objects from [21, 22].

Let $\mathcal{F}$ be a foliation on a manifold M. The foliation $\mathcal{F}$ is given by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ modelled on a manifold N_0, i.e.

i) $\{U_i\}$ is an open covering of M,
ii) $f_i : U_i \to N_0$ are submersions with connected fibres defining $\mathcal{F}$,
iii) g_{ij} are local diffeomorphisms of N_0 and $g_{ij} \circ f_j = f_i$ on $U_i \cap U_j$.

The image of the submersion f_i is an open subset of the manifold N_0. The disjoint union $N = \coprod f_i(U_i)$ (also a q-manifold) we call the transverse manifold of $\mathcal{F}$ associated to the cocycle $\mathcal{U}$ and the pseudogroup $\mathcal{H}$ generated by g_{ij} the holonomy pseudogroup (representative) on the transverse manifold N.

The foliated geometric structures, i.e. those which in local coordinates can be expressed in the transverse coordinates only, correspond bijectively to holonomy invariant ones on the transverse manifold, cf. [21, 22]. The precise definition presented in the above mentioned works involves natural bundles on the category of foliated manifolds and their foliated mappings. A foliated structure is a foliated section of such a natural bundle. In our case we need something much simpler and accessible. The normal bundle of a foliation and its tensor products are natural bundles on the category of the foliated manifolds. A foliated section S of the normal bundle $N(M, \mathcal{F})$ is a foliated geometric structure – a foliated vector field. It can be represented by a local vector field X which in an adapted chart $\Phi = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ has the form $\Sigma_i X_i(x, y)\partial/\partial x_i + \Sigma_\alpha X_\alpha(y)\partial/\partial y_\alpha$. Foliated vector fields are sections of the normal bundle which are given by infinitesimal automorphisms of the foliation. A k-form ω as a local section of a suitable vector bundle can be also a foliated geometric structure. In this case our condition transforms into the following local representation in an adapted chart:

$$\omega = \Sigma_{\alpha_1 < \cdots < \alpha_k} \omega_{\alpha_1, \ldots, \alpha_k}(y) dy_{\alpha_1} \wedge \ldots dy_{\alpha_k}.$$

Likewise a tensor field A of the type (1,1), when restricted to the normal bundle of the foliation (considered as the subbundle of the tangent bundle of the manifold) in an adapted chart can be represented as follows:

$$A = \Sigma_{\alpha_i, \alpha_j} a_{\alpha_i, \alpha_j}(y) dy_{\alpha_i} \otimes v_{\alpha_j}$$

where v_{α_j} are the sections of the normal bundle (thus in this case standard local vector fields) which correspond to the local vector fields $\partial/\partial y_{\alpha_j}$.

A transversely Kähler foliation is a foliation whose normal bundle has a foliated Kähler structure, i.e. both the metric and the complex structures are foliated structures. In view of the above mentioned theorem any transverse manifold of a transversely Kähler foliation is a Kähler manifold whose Kähler structure is $\mathcal{H}$-invariant.

For a given pseudogroup $\mathcal{H}$ of local diffeomorphisms of a q-manifold T, there are many foliations of various dimensions whose holonomy pseudogroups are equivalent, cf. [11, 22], to $\mathcal{H}$. Any property that depends only on the equivalence class of the holonomy pseudogroup is called a transverse property, i.e. all foliations whose holonomy groups are equivalent to the given pseudogroup have this property. Being Riemannian, transversely Kähler or transversely symplectic are transverse properties, cf. [7, 22]. Being minimalisable is also a transverse property, cf. [10].

Now let us recall the definition of a Sasakian manifold. Let M be a smooth manifold of dimension $2n+1$. The manifold M is called an almost contact metric manifold if there exist on M:

1. a non-vanishing vector field ξ and a 1-form η such that $\eta(\xi) = 1$;

2. a tensor field φ of type $(1,1)$ such that $\varphi^2 = -Id + \eta \otimes \xi$, this implies that $\varphi(\xi) = 0$ and $\eta \circ \varphi \equiv 0$;

3. a Riemannian metric g such that $g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X)\eta(Y)$.

An almost contact metric manifold is called Sasakian if, additionally, it satisfies the following condition, cf. [19] and [3, p. 73]:

4. $(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$, and hence, cf. [3, p. 74], $\nabla_X \xi = -\varphi(X)$, $d\eta(X,Y) = g(X, \varphi Y)$ and the $2n+1$-form $\eta \wedge d\eta^n$ does not vanish.

The last condition, $\nabla_X \xi = -\varphi(X)$, ensures that the vector field ξ is a Killing vector field for the metric g, cf. [3]. Therefore, this vector field defines a Riemannian foliation $\mathcal{F}$ of dimension 1 which is an isometric flow, cf. [5, 6]. The vector field ξ is called the characteristic vector field and $\mathcal{F}$ the characteristic foliation of the Sasakian manifold M. It is not difficult to verify that $L_\xi \varphi | \ker \eta \equiv 0$. Therefore the tensors g and φ induce foliated tensors in the normal bundle of the characteristic foliation which can be identified with the bundle $\ker \eta$. Let $\bar{g}$ and $\bar{J}$ be the corresponding tensors on the transverse manifold N (of the characteristic foliation). The Riemannian connection ∇ of M induces a transversely projectable connection in $\ker \eta$ which projects onto the Riemannian connection of $(N, \bar{g})$, cf. [15]. The condition (4) ensures that the almost complex structure $\bar{J}$ is integrable, thus $(N, \bar{J}, \bar{g})$ is a Hermitian manifold. The equality $d\eta(X,Y) = g(X, \varphi Y)$ means that the 2-form $d\eta$ is base-like. The corresponding 2-form Φ on the transverse manifold is its Kähler form, cf. [16]; and the submersions f_i are canonical fibrations, cf. [17]. The holonomy pseudogroup is a pseudogroup of Kähler transformations and $\mathcal{F}$ is transversely Kähler.

There are many transversely Kähler isometric flows which are not given by any Sasakian structure. Let Ψ be an isometric flow defining a transversely Kähler foliation. The transverse manifold N of this foliation admits a holonomy invariant Kähler structure $(\bar{g}, \bar{J})$. Let ξ be the vector field tangent to the flow Ψ, g the Riemannian metric for which the flow is isometric, and let Q be the orthogonal complement of ξ in the metric g. Then we put:

$$\eta : \eta(\xi) \equiv 1, \quad \eta|Q \equiv 0;$$

$$\varphi : \varphi(\xi) \equiv 0 \quad df_i(\varphi(X)) = \bar{J}(df_i(X)) \quad \text{for any } X \in TU_i.$$

One can easily check that the structure (g, φ, ξ, η) defined above satisfies the conditions 1)$-$3), i.e. it is an almost contact metric structure. The condition (4) is not a transverse one which can be deduced easily from the Boothby–Wang theorem, cf. [3, 4].

Let us take a compact Kähler manifold B and S^1-bundles over B. They are classified by integral 2-forms on B, cf. [13]. The manifold B is the space of leaves of any foliation defined by such an S^1-bundle. The holonomy pseudogroup of any such foliation is equivalent to the trivial pseudogroup on B (generated by the identity transformation of B). Therefore if “being Sasakian” were a transverse property, then any such a foliation would be given by a Sasakian structure. However, according to the Boothby–Wang theorem this is not true – the corresponding S^1-bundle should be classified by a very special 2-form: the Kähler form of some Kähler structure of B. For more properties of transversely Kähler foliations see [8].

First of all we are going to compare various curvature tensors of the manifolds (M, g, φ) and $(N, \bar{g}, \bar{J})$. For any $X \in T_x N_i$ and $y \in f_i^{-1}(x)$ denote by X^* the only vector of $\ker \eta_y$ such that $df_i(X^*) = X$. The considerations of [23, Chapter VI], yield the following relations:

1. $(\bar{J}X)^* = \varphi(X^*)$;
2. $g(X^*, Y^*) = \bar{g}(X, Y)$;
3. $(\bar{\nabla}_X Y)^* = \nabla_{X^*} Y^* + g(Y^*, \varphi X^*)\xi$, where ∇ and $\bar{\nabla}$ are the Levi–Civita connections of g and $\bar{g}$, respectively;
4. $(\bar{R}(X,Y)Z)^* = R(X^*, Y^*)Z^* + g(Z^*, \varphi Y^*)\varphi X^* - g(Z^*, \varphi X^*)\varphi Y^* - 2g(Y^*, \varphi X^*)\varphi Z^*$ where R and $\bar{R}$ are the curvature tensors of ∇ and $\bar{\nabla}$, respectively;
5. $\bar{S}(X,Y) = S(X^*, Y^*) + 2g(X^*, Y^*)$ where S and $\bar{S}$ are the Ricci curvature tensors of (M, g) and $(N, \bar{g})$, respectively;
6. $\bar{r} = r + 2n$ where r and $\bar{r}$ are the scalar curvatures of (M, g) and $(N, \bar{g})$, respectively;
7. $\bar{K}(X, \bar{J}X) = K(X^*, \varphi X^*) + 3$ where K and $\bar{K}$ are the sectional curvature tensors of (M, g) and $(N, \bar{g})$, respectively;

As an example we shall prove the following theorem, cf. [3]:

Theorem 1. *The φ-sectional curvature determines completely the sectional curvature of a Sasakian manifold.*

PROOF. It is well-known that for any plane tangent to the characteristic vector field the sectional curvature is equal to 1. – This is a consequence of the curvature formula: $R(X,Y)\xi = g(Y,\xi)X - g(X,\xi)Y$ which holds for any vector fields X and Y on the manifold M. Using the above mentioned formula the calculation of the sectional curvature of a plane which is neither tangent to ξ nor transverse (i.e. tangent to $\ker \eta$) involves the sectional curvature of the projection of the plane onto $\ker \eta$, and terms depending only on the basis of the plane. (We split any vector into a vector tangent to $\ker \eta$ and a vector parallel to ξ.) Therefore everything depends on the scalar curvature for transverse planes. The formula (4) establishes the relation between the sectional curvature in the transverse direction in a Sasakian manifold and the corresponding sectional curvature in the transverse manifold. The formula (7) gives the precise relation between φ-sectional curvature and the holomorphic sectional curvature of M and N, respectively. As N is a Kähler manifold, its holomorphic sectional curvature determines its sectional curvature, so the φ-sectional curvature determines the sectional curvature of a Sasakian manifold M. □

The formula (7) leads to the following proposition:

Proposition 1. *The characteristic foliation of a Sasakian space form $M(c)$ is a transversely Kähler isometric flow modelled on a Kähler space form $N(c-3)$.*

Now let us turn our attention to submanifolds. Let W be an $m+1$ dimensional submanifold of M tangent to ξ, i.e. for any $x \in W$, $\xi(x) \in T_xW$. For any point of this submanifold we can find a very special adapted chart at this point. The proof of this fact is a simple generalization of the Frobenius theorem.

Lemma 1. *Let x be a point of a submanifold W tangent to the characteristic vector field of a Sasakian manifold M. Then there exists an adapted chart $\psi : V \to R^{2n+1}$, $\psi = (\psi_1, \dots, \psi_{2n+1})$ at x such that the set $U = \{y \in V \mid \psi_{m+2}(y) = \cdots = \psi_{2n+1}(y) = 0\}$ is a connected component of $V \cap W$ containing x and $(\psi_1 \mid U, \ \dots, \ \psi_{m+1}|U) : U \to R^{m+1}$ is an adapted chart for the induced foliation of W.*

This lemma leads us to the following

Proposition 2. *Let W be a submanifold tangent to the characteristic foliation of a Sasakian manifold M. Then for any point x of W there exist neighbourhoods U and V of x in W and M, respectively, having the following properties:*

- i) *U is a connected component of $V \cap W$ containing x;*
- ii) *U is a foliated subset of V (for the characteristic foliation);*
- iii) *there exists a Riemannian submersion with connected fibres $f : V \to N_0$ onto a Kähler manifold N_0 defining the characteristic foliation;*
- iv) *there exists a submanifold $\bar{W}$ of N_0 such that $U = f^{-1}(\bar{W})$.*

Proof. Let U and V be neighbourhoods of the point x from Lemma 1. Then we define the submersion f as $p_{2n} \circ \psi : V \to R^{2n}$ where p_{2n} is the projection $(x_1, \dots, x_{2n+1}) \mapsto (x_2, \dots, x_{2n+1})$. On the set $\operatorname{im} f \subset R^{2n}$ the Sasakian structure of M induces a Kähler structure for which the submersion is a Riemannian submersion. Since the characteristic foliation restricted to V is defined by this submersion and the set U is saturated, there exists a submanifold $\bar{W}$ of N_0 satisfying the condition (iv). $\square$

2. Contact CR-submanifolds

First we recall the definition of a contact CR-submanifold, cf. [24].

Definition 1. Let W be a connected submanifold of a Sasakian manifold M tangent to the characteristic vector field. W is called a contact CR-submanifold of M if there exists a differentiable distribution D on W of constant dimension, $D : x \mapsto D_x \subset T_xW$, satisfying the following conditions:

i) D is invariant with respect to φ, i.e. for any $x \in W$, $\varphi D_x \subset D_x$;

ii) the complementary orthogonal distribution $D^\perp : x \mapsto D_x^\perp \subset T_xW$ is anti-invariant with respect to φ, i.e. for any $x \in W$, $\varphi D_x^\perp \subset T_xW^\perp$.

A contact CR-submanifold W is non-trivial if $\dim D = h > 0$ and $\dim D^\perp = q > 0$; cf. [24, p. 48].

The image by φ of the tangent bundle TW splits into two distributions $\varphi(D)$ and $\varphi(D^\perp)$, tangent and orthogonal to W, respectively:

$$\varphi(D_x) \subset T_xW \cap \varphi(T_xW) \quad \text{and} \quad \varphi(D_x^\perp) \subset T_xW^\perp \cap \varphi(T_xW).$$

One can easily check that

$$\varphi(D_x) = T_xW \cap \varphi(T_xW) = D_0 \quad \text{and} \quad \varphi(D_x^\perp) = T_xW^\perp \cap \varphi(T_xW).$$

Thus the distribution D_0 has constant dimension and $D = D_0$ or $D = D_0 \oplus T\mathcal{F}$, and $D^\perp = D_0^\perp \oplus T\mathcal{F}$ or $D_0^\perp$, respectively, where $D_0^\perp$ is the orthogonal complement of $D_0 \oplus T\mathcal{F}$. This means that the tangent bundle TW of W admits the following decomposition: $T\mathcal{F} \oplus D_0 \oplus D_0^\perp$. Moreover, the distributions D_0 and $D_0^\perp$ define the decomposition of the subbundle $\ker \eta|_{TW} = \operatorname{im} \varphi|_{TW}$. For the rest of the paper we assume that $D = D_0 \oplus T\mathcal{F}$.

The above description of the distributions D and $D^\perp$ coupled with the fact that the tensors g and φ induce foliated tensors on $\ker \eta$ yield the following (cf. [24])

Proposition 3. *Let W be a submanifold tangent to the characteristic vector field of a Sasakian manifold. Then W is a contact CR-submanifold iff the coresponding submanifolds in the transverse manifold are CR-submanifolds.*

Proof. As the problem is a local one, we can consider the characteristic fibration $f : M \to N$ and a submanifold $W = f^{-1}(\bar{W})$; in this case the statement is trivial. □

Having described in detail the distributions D and $D^\perp$ we turn our attention to their properties. The argument in the proof of Theorem III.3.1 of [24] ensures only that the distribution $D^\perp \oplus T\mathcal{F}$ is integrable. Thus we have the following version of Theorem III.3.1:

Theorem 2. *Let W be a contact CR-sub-manifold of a Sasakian manifold M. Then the distribution $D^\perp \oplus T\mathcal{F}$ is completely integrable and its integral submanifolds are anti-invariant submanifolds (tangent to the characteristic vector field).*

For the same reason we obtain the following version of Theorem III.3.2 of [24], where B is the second fundamental form of the submanifold W in M and $\nabla_X \xi = PX$:

Theorem 3. *Let W be a contact CR-submanifold of a Sasakian manifold M. Then the distribution D is integrable iff $B(X, PY) = B(Y, PX)$ for any $X, Y \in D$. Its integral submanifolds are invariant submanifolds of M.*

Remark. As the properties described by the above theorems are local, they can be derived from the corresponding theorems for CR-submanifolds of Kähler manifolds, compare Theorems IV.4.1 and IV.4.2 of [24].

Proposition 4. *Let W be a contact CR-submanifold tangent to the characteristic vector field of a Sasakian manifold M. If $g(B(X,Y), \varphi Z) = 0$ for any $X, Y \in D_0$, $Z \in D_0^\perp$ then any geodesic of W tangent to D_0 at one point remains tangent to D_0 at any point of its domain.*

Proof. Since the foliation $\mathcal{F}|W$ is a Riemannian foliation, a geodesic orthogonal to $\mathcal{F}$ at one point is orthogonal to $\mathcal{F}$ at any point of its domain, and it is a $D_0 \oplus D_0^\perp$-horizontal lift of the corresponding geodesic in the transverse manifold, cf. [18, 25, 15]. Let us consider a geodesic $\alpha : (a, b) \to W$ tangent to D_0 at 0 and the set $A = \{t \in (a, b) : \dot{\alpha}(t) \in D_0\}$.

The set A is closed and $0 \in A$. We shall show that it is also open. As the problem is local we can reduce our considerations to a foliated submanifold of a Sasakian manifold with the characteristic foliation given by a global submersion with connected fibres, i.e. the characteristic fibration $f : M \to N$ and $W = f^{-1}(\bar{W})$ where $\bar{W}$ is a CR-submanifold of the Kähler manifold N. Therefore $T\bar{W}$ admits a decomposition into orthogonal distributions $\bar{D}$ and $\bar{D}^\perp$ such that $D = f^{-1}(\bar{D})$ and $D_0 = \ker\eta \cap f^{-1}(\bar{D})$, $D_0^\perp = \ker\eta \cap f^{-1}(\bar{D}^\perp)$. Let B be the second fundamental form of the submanifold W in M and $\bar{B}$ be the second fundamental form of the submanifold $\bar{W}$ in N. Then $B(X^*, Y^*) = \bar{B}(X, Y)^*$, cf. [24], p. 101, where, for any vector X tangent to $\bar{W}$, X^* is its $\ker\eta$ $(D_0 \oplus D_0^\perp)$ – lift to M, and hence $\bar{g}(\bar{B}(X, Y), \bar{\varphi}Z) = 0$ for any $X, Y \in \bar{D}$ and $Z \in \bar{D}^\perp$. Then Proposition IV.4.2 of [24] ensures that $\bar{D}$ is a totally geodesic foliation of $\bar{W}$. Let $\bar{\alpha}$ be the geodesic in $\bar{W}$ corresponding to α. If α is tangent to D_0 at $t \in (a, b)$, then $\bar{\alpha}$ is tangent to $\bar{D}$ at this point. Since the foliation $\bar{D}$ is totally geodesic, $\bar{\alpha}$ must be contained in some leaf of $\bar{D}$. Hence α being the $D_0 \oplus D_0^\perp$-horizontal lift of $\bar{\alpha}$, it must be tangent to D_0. Therefore the set A is open, and thus $A = (a, b)$. □

Taking as a model Kähler manifolds we can introduce the following notions:

Definition 2. We say that a contact CR-submanifold W is:

i) D_0-totally geodesic iff $B(X, Y) = 0$ for any $X, Y \in D_0$;

ii) contact mixed foliate if $B(X, Y) = 0$ for any $X \in D$ and $Y \in D^\perp$, and $B(PX, Y) = B(X, PY)$ for any $X, Y \in D_0$.

It is not difficult to verify the following

Lemma 2. i) *W is D_0-totally geodesic iff $\bar{W}$ is $\bar{D}$-totally geodesic;* ii) *W is contact mixed foliate iff $\bar{W}$ is mixed foliate.*

Proposition 5. *Let W be a contact CR-submanifold tangent to the characteristic vector field of a Sasakian manifold M. If W is D_0-totally geodesic, then D is a foliation and any geodesic of W tangent to D_0 at one point remains tangent to D_0 at any point of its domain.*

Proof. It is a consequence of Lemma 2, Corollary IV.4.3 of [24] and of considerations similar to those of the second part of the proof of Proposition 4. □

Proposition 6. *If W is a contact mixed foliate non-trivial contact CR-submanifold of a Sasakian manifold space form $M(c)$, then $c \leq -3$.*

Proof. The transverse manifold of the characteristic foliation has constant holomorphic sectional curvature equal to $c+3$. The problem is local and Lemma 2 together with Proposition IV.4.3 of [24] ensures that $c+3 \leq 0$. Thus $c \leq -3$. □

Corollary 1. *Let W be a contact mixed foliate contact CR-submanifold of a Sasakian space form $M(c)$. If $c > -3$, then W is either an invariant submanifold or an anti-invariant submanifold of $M(c)$.*

This is a counterpart of Corollary IV.4.4 of [24]. Theorem IV.6.1 of [24] or [1] yield the following

Theorem 4. *Let W be a contact totally umbilical non-trivial contact CR-submanifold of a Sasakian manifold M. If $\dim D_0^{\perp} > 1$, then a geodesic orthogonal to ξ and tangent to W at one point has this property on an open subset of its domain.*

Proof. The corresponding submanifold $\bar{W}$ in the transverse manifold is totally umbilical. Since the characteristic foliation is Riemannian we have to show that the geodesic is tangent to W on an open subset of its domain. This property is a local one and therefore we can reduce our considerations to the canonical fibration. The geodesic is the $\ker\eta$-horizontal lift of a geodesic in N. Therefore it is sufficient to know that the submanifold $\bar{W}$ is totally geodesic. This is precisely the fact which Bejancu's theorem ensures. □

Theorem 5. *Let W be a totally geodesic contact CR-submanifold of a Sasakian manifold M. Then D and $D^{\perp} \oplus T\mathcal{F}$ are Riemannian foliations, and locally:*

i) *W is diffeomorphic to $R \times N_0 \times N_1$,*
ii) *the foliation D is given by the projection $R \times N_0 \times N_1 \to N_1 \subset N$,*
iii) *the foliation $D^{\perp} \oplus T\mathcal{F}$ is given by the projection $R \times N_0 \times N_1 \to N_0 \subset N$,*
iv) *the submanifold $\bar{W} \subset N$ is a Riemannian product of $N_0 \times N_1$, of a totally geodesic invariant submanifold N_0, and a totally geodesic anti-invariant submanifold N_1 of N.*

Proof. The problem is local and we can reduce our considerations to the case of canonical fibration. Therefore we can assume that $W =$

$f^{-1}(\bar{W})$ for some CR-submanifold $\bar{W}$ of the Kähler manifold N and that the submersion $f : M \to N$ is a Riemannian submersion. The orthogonal complement of $T\mathcal{F}$ on W is equal to $\ker \eta = D_0 \oplus D_0^\perp$. Therefore $D_0 = (df|W)^{-1}(\bar{D}) \cap \ker \eta$ and $D_0^\perp = (df|W)^{-1}(\bar{D}^\perp) \cap \ker \eta$ where $\bar{D}$ and $\bar{D}^\perp$ are invariant and antiinvariant distributions, respectively, of the CR-submanifold $\bar{W}$ of N.

Since $\bar{W}$ is totally geodesic, cf. [24], Prop. V.2.5, Theorem IV.6.2 of [24] assures that the submanifold $\bar{W}$ of N is a Riemannian product $N_0 \times N_1$ of a totally geodesic invariant submanifold N_0 and a totally geodesic anti-invariant submanifold N_1 of N. Therefore it remains to prove that the foliations D and $D^\perp \oplus T\mathcal{F}$ are Riemannian foliations of the submanifold W. The subbundle $D_0^\perp$ is the orthogonal complement of D, therefore the foliation D is Riemannian iff any geodesic of W which is tangent to $D_0^\perp$ at one point remains tangent to $D_0^\perp$ at any point of its domain, cf. [25, 15]. Likewise the subbundle D_0 is the orthogonal complement of $D^\perp \oplus T\mathcal{F}$, therefore the foliation $D^\perp \oplus T\mathcal{F}$ is Riemannian iff any geodesic of W which is tangent to D_0 at one point remains tangent to D_0 at any point of its domain.

Let us take a geodesic γ of W which is tangent to $D_0^\perp$ at one point x. Since f is a Riemannian submersion γ is a horizontal geodesic, i.e. tangent to $\ker \eta$. Its image $f\gamma$ is a geodesic in $\bar{W}$, cf. [12], which is tangent to $\bar{D}^\perp$ at one point. As both distributions, $\bar{D}$ and $\bar{D}^\perp$, are totally geodesic, the geodesic $f\gamma$ remains tangent to $\bar{D}$ throughout its domain. The $\ker \eta$-orthogonal lift γ' passing through the point x of $f\gamma$ is a geodesic in M and W which is tangent to $D_0^\perp$. Both geodesics, γ and γ', have the same tangent vector at the point x, therefore they must be equal.

Similar considerations are valid for the other distribution. □

Final remarks. 1. The same method can be applied to submanifolds transverse to the characteristic vector field of a Sasakian manifold.

2. This method is also applicable to the $\mathcal{S}$-structures of D. E. Blair, cf. [2].

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