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Monomial selections of set-valued functions

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Abstract. We prove a general theorem on monomial selections of set-valued functions, thus solving the problems posed by Z. Páles [6]. As a corollary we obtain new stability results of the monomial functional equation.

1. Introduction

Let S be a semigroup and let X be a vector space. A function $a_n : S^n \to X$ is called n-additive if it is additive in each variable. We say that $f : S \to X$ is a monomial function of degree n if there exists an n-additive function $a_n : S^n \to X$ such that

$$f(x) = a_n^*(x) := a_n(x, \dots, x) \quad \text{for } x \in S.$$

Our aim in this paper is to investigate the existance of monomial selections for set-valued functions. For the convenience of the reader we will quote a few results of Z. Djoković from [3] which show some interesting properties of monomial functions. However, to do so, we need the notion of shift and difference operators. The set of all mappings from S to X we denote by X^S. We define the shift operator

$$E_u : X^S \to X^S \quad (u \in S)$$

as follows: the image of $f \in X^S$ under E_u is the mapping defined by

$$(E_u f)(x) := f(u + x) \quad \text{for } x \in S.$$

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The identity $1 : X^S \to X^S$ maps each $f \in X^S$ onto itself. The difference operator Δ_u is defined by

$$\Delta_u := E_u - 1.$$

One can easily notice that if S is abelian semigroup then operator E is commutative, and in particular we have

$$E_u\Delta_u = \Delta_u E_u, \qquad \Delta_u\Delta_v = \Delta_v\Delta_u.$$

Theorem D1 (Lemma 2 from [3]). *If $a_n : S^n \to X$ is symmetric and n-addditive, then*

$$a_n(u_1, \dots , u_n) = \frac{1}{n!}\Delta_{u_1} \dots \Delta_{u_n} a_n^*,$$

where $a_n(u_1, \dots , u_n)$ denotes the constant mapping which maps each $x \in S$ into $a_n(u_1, \dots , u_n) \in X$.

Theorem D2 (Corollary 2 from [3]). *A function $g : S \to X$ is a mononial of degree n iff the equation*

$$\frac{1}{n!}\Delta_u^n f(x) = g(u) \quad \text{for } x, u \in S \tag{1}$$

has a solution $f \in X^S$.

Theorem D3 (Corollary 3 from [3]). *A necessary and sufficient condition that $f : S \to X$ is monomial of degree n is that*

$$\frac{1}{n!}\Delta_u^n f(x) = f(u) \quad \text{for } x, u \in S.$$

M. Albert and J. Baker proved (Theorem 7 from [1]) that equation (1) is stable in the Hyers–Ulam sense. Z. Páles in his paper [6] asked a question concerning the selection of set-valued analogue of equation (1). A generalization of the problems he posed may be stated in following way:

Problem P. *Let X be a locally convex topological vector space and let $G : S \to \mathrm{cc}(X)$, where $\mathrm{cc}(X)$ denotes the family of nonempty compact convex subsets of X. Assume that there exists a function $f : S \to X$ such that*

$$\Delta_u^n f(x) \in G(u) \quad \text{for } x, u \in S.$$

Does there exists a monomial of degree n selection of G?

We will show in Theorem 1 that the solution to Problem P is positive. However, to obtain these results we first need to introduce in the following section the notion of invariant mean for functions with images in compact sets.

At the end of this section we would like to mention that the solution to the Problem P was obtained independently by R. Badora, R. Ger and Zs. Páles in [2].

2. Invariant means

Let X be a locally convex topological vector space. By $\mathcal{L}(S, \mathrm{cc}(X))$ we denote the vector space of all functions $f : G \to X$ such that $\mathrm{cl}\,\mathrm{conv}\, f(S) \in \mathrm{cc}(X)$. One can easily notice that if X is finite dimensional that $\mathcal{L}(S, \mathrm{cc}(X))$ simply denotes the set of all bounded functions on S with values in X.

The following definition is a generalization of the notion of classical left invariant mean (see [5]). We would like also to remark that the idea of extending invariant means from the space of bounded real valued functions to functions with values in a vector space is due to L. Székelyhidi [7] and further improved by Z. Gajda [4].

Definition 1. Let G be a semigroup (not necessarily abelian). A linear function $m : \mathcal{L}(S, \mathrm{cc}(X)) \to X$ is a left invariant mean if

(i) $m(E_u f) = m(f)$ for $u \in S$, $f \in \mathcal{L}(S, \mathrm{cc}(X)$,

(ii) $m(f) \in \mathrm{cl}\,\mathrm{conv}(f(S))$ for $f \in \mathcal{L}(S, \mathrm{cc}(X))$.

(point (i) refers to the fact that m is left invariant and (ii) that m is a mean)

We say that the semigroup is left amenable if $\mathcal{L}(S, \mathrm{cc}(\mathbb{R}))$ admits left invariant mean.

For some basic properties of invariant means and amenable groups we refer the reader to [5]. We would only like to mention that every abelian semigroup is amenable. Our aim in this section is to prove that if G is left amenable then $\mathcal{L}(G, \mathrm{cc}(X)$ admits left invariant mean. This result, on which our analysis rests, can be derived from [8], however it is relatively short and therefore included here for completeness.

From now on we assume in this section that S is left amenable semigroup and that m is a fixed left invariant mean on $\mathcal{L}(S, cc(\mathbb{R}))$. By X^* we denote the space of all continuous linear functionals on X and by $\pi_i : \mathbb{R}^n \to \mathbb{R}$ the canonical projection onto the i-th coordinate. We define

$$m_{\mathbb{R}^n}(f) := (m(\pi_1(f)), \dots, m(\pi_n(f))) \in \mathbb{R}^n, \tag{2}$$

for $f \in \mathcal{L}(S, \mathrm{cc}(\mathbb{R}^n))$.

Proposition 1. *$m_{\mathbb{R}^n}$ is a left invariant mean on $\mathcal{L}(S, \mathrm{cc}(\mathbb{R}^n))$ such that*

$$\phi(m_{\mathbb{R}^n}(f)) = m(\phi(f)) \tag{3}$$

for $\phi \in (\mathbb{R}^n)^*$, $f \in \mathcal{L}(S, \mathrm{cc}(X))$.

Proof. As m is left invariant and linear operation obviously $m_{\mathbb{R}^n}$ is also left invariant and linear. We prove that (3) holds. By (2) it holds for $\phi = \pi_i$, however as $m_{\mathbb{R}^n}$ is linear and π_i forms a base of $(\mathbb{R}^n)^*$ (3) holds for all $\phi \in (\mathbb{R}^n)^*$.

So suppose, for contradiction, that $m_{\mathbb{R}^n}$ is not a mean, that is that $m_{\mathbb{R}^n}$ does not satisfy condition (ii) from Definition 1. Then there exists an $f \in \mathcal{L}(S, \mathrm{cc}(\mathbb{R}^n))$ such that

$$m_{\mathbb{R}^n}(f) \notin \mathrm{cl}\,\mathrm{conv}(f(S)) \in \mathrm{cc}(\mathbb{R}^n),$$

so we can find $\phi \in (\mathbb{R}^n)^*$ such that

$$\phi(m_{\mathbb{R}^n}(f)) \notin \phi(\mathrm{cl}\,\mathrm{conv}(f(S))) = \mathrm{cl}\,\mathrm{conv}(\phi f)(S).$$

However, by (3) $m(\phi f) = \phi(m_{\mathbb{R}^n}(f))$, so we have obtained a contradiction with the fact that m is a mean on $\mathcal{L}(S, \mathrm{cc}(\mathbb{R}))$. □

Theorem 1. *There exists a left invariant mean m_X on $\mathcal{L}(S, \mathrm{cc}(X))$ such that*

$$\phi(m_X(f)) = m(\phi f) \quad \text{for } \phi \in X^*.$$

Proof. Let $\phi_1, \dots, \phi_n \in X^*$ be arbitrary, and let $f \in \mathcal{L}(S, \mathrm{cc}(X))$. Let $\Phi = (\phi_1, \dots, \phi_n) : X \to \mathbb{R}^n$. Let $K := \mathrm{cl}\,\mathrm{conv}(f(S))$. As K is compact, $\mathrm{im}(\Phi \circ f) \subset \Phi(K) \in \mathrm{cc}(\mathbb{R}^n)$, so $m_{\mathbb{R}^n}(\Phi \circ f) \subset \Phi(K)$ which means that there exists an $x \in K$ such that

$$(m(\phi_1 \circ f), \dots, m(\phi_n \circ f)) = (\phi_1(x), \dots, \phi_n(x)).$$

This implies that

$$K \cap (\phi_1)^{-1}(m(\phi_1 \circ f)) \cap \ldots \cap (\phi_n)^{-1}(m(\phi_n \circ f)) \neq \emptyset,$$

and therefore the family

$$\{K \cap \phi^{-1}(m(\phi \circ f))\}_{\phi \in X^*}$$

is centered. As $\phi^{-1}(m(\phi \circ f))$ is closed in X and K is compact we obtain that this family has nonempty intersection, which means that there exists $x_f \in V$ such that

$$\phi(x_f) = m(\phi \circ f) \quad \text{for } \phi \in X^*. \tag{4}$$

We put $m_X(f) := x_f$. As X is locally convex X^* separates points in X so $m_X(f)$ is uniquely defined. By the construction we know that $m_X(f) \in K$, so m_X is a mean.

Making use of (4), the uniqueness of m_X, and the fact that m is linear and left invariant, one can check that m_X is linear and left translation invariant. □

3. Monomial selections

In the following part of the paper we assume that S is abelian semigroup, X is locally convex topological vector space, and m is a left invariant mean on X (which exists by Theorem 2 as every abelian semigroup is amenable).

Before showing the main theorem we will have to prove first some introductory results.

Lemma 1. *Let $k, n \in \mathbb{N}$ and let $u \in S$ be arbitrary. Then*

$$\Delta_{ku}^n = \left(\sum_{i=0}^{k-1} E_{iu} \right)^n \Delta_u^n.$$

Proof. Clearly

$$\begin{aligned} \Delta_{ku} &= E_{ku} - 1 = E_u^k - 1 = (E_u - 1)(E_u^{k-1} + \ldots + E_u^1 + E_u^0) \\ &= (E_u - 1)(E_{(k-1)u} + \ldots + E_{1u} + E_{0u}) = \Delta_u \left(\sum_{i=0}^{k-1} E_{iu} \right). \end{aligned}$$

The commutativity of operators Δ and E makes the proof complete. □

For a function $g(x, u)$ of variables x and u by $m_x(g(x, u))$ we denote the mean of g with respect to the variable x.

Proposition 2. *Let $G : S \to \mathrm{cc}(X)$ and let $f : S \to X$ be such a function that*

$$\frac{1}{n!}\Delta_u^n f(x) \in G(u) \quad \text{for } u, x \in S. \tag{5}$$

We put

$$F(u) := m_x\left(\frac{1}{n!}\Delta_u^n f(x)\right).$$

Then F is a well defined function and

$$\Delta_u^n F(x) = \sum_{j=0}^{n}\binom{n}{j}(-1)^j(n-j)^n F(u). \tag{6}$$

Proof. By (5) and the fact that $G(u)$ is compact we obtain that F is well defined. We now show (6).

$$\Delta_u^n F(x) = (E_u - 1)^n F(x) = \sum_{i=0}^{n}\binom{n}{i}(-1)^i E_u^{n-i} F(x)$$

$$= \sum_{i=0}^{n}\binom{n}{i}(-1)^i E_{(n-i)u} F(x) = \sum_{i=0}^{n}\binom{n}{i}(-1)^i F(x + (n-i)u)$$

$$= \sum_{i=0}^{n}\binom{n}{i}(-1)^i m_z\left(\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(-1)^j f(z + (n-j)(x + (n-i)u))\right)$$

$$= m_z\left(\sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(-1)^j f(z + (n-j)(x + (n-i)u))\right)$$

$$= m_z\left(\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}(-1)^i\frac{1}{n!}\binom{n}{j}(-1)^j f(z + (n-j)(x + (n-i)u))\right)$$

$$= m_z\left(\sum_{j=0}^{n}\sum_{i=0}^{n}\binom{n}{j}(-1)^j\frac{1}{n!}\binom{n}{i}(-1)^i f(z+(n-j)x+(n-i)(n-j)u)\right)$$

$$= m_z\left(\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i f((z+(n-j)x)+(n-i)(n-j)u)\right)$$

$$= m_z \left(\sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{1}{n!} \Delta^n_{(n-j)u} f(z + (n-j)x) \right).$$

By (5)

$$\frac{1}{n!} \Delta^n_{(n-j)u} f(z) \in G((n-j)u) \quad \text{for } z \in S,$$

which implies that

$$\phi_j(z) := \frac{1}{n!} \Delta^n_{(n-j)u} f(z + (n-j)x) \in G((n-j)u) \quad \text{for } z \in S,$$

and therefore $\phi_j \in \mathcal{L}(S, \mathrm{cc}(X))$. So by the linearity of m we obtain that

$$m_z \left(\sum_{j=0}^{n} \binom{n}{j} (-1)^j \phi_j(z) \right) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j m_z(\phi_j(z)).$$

Thus we have proved that

$$\Delta^n_u F(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j m_z \left(\frac{1}{n!} \Delta^n_{(n-j)u} f(z + (n-j)x) \right).$$

However, as the mean is left translation invariant, this implies that

$$\Delta^n_u F(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j m_z \left(\frac{1}{n!} \Delta^n_{(n-j)u} f(z) \right).$$

Making now use of Lemma 1 we get

$$\begin{aligned} \Delta^n_u F(x) &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j m_z \left(\left(\sum_{k=0}^{n-j-1} E_{ku} \right)^n \frac{1}{n!} \Delta^n_u f(z) \right) \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^n m_z \left(\frac{1}{n!} \Delta^n_u f(z) \right) \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^n F(u). \qquad \square \end{aligned}$$

Lemma 2. *Let $n \in \mathbb{N}$. Then*

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^n = n!\,.$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the formula $f(x) = x^n$. One can easily notice that f is a monomial function of edgree n, and therefore by Theorem D1

$$\frac{1}{n!} \Delta_u^n f = f(u) \quad \text{for } u \in \mathbb{R}.$$

This means that we can apply Proposition 1 and get that the function

$$F(x) = m_z(\Delta_x^n f(z)) = m_z(f(x)) = f(x)$$

satisfies the equation

$$\Delta_u^n F(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^n F(x).$$

Hence

$$n! f(x) = \Delta_u^n f(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^n f(x).$$

Stating $x = 1$ we obtain the assertion of the lemma. $\square$

In the following theorem we characterize the set-valued functions which admit monomial selections. Notice that this theorem is a direct generalization of Theorem D2.

Theorem 2. *Let $G : S \to \mathrm{cc}(X)$. Then G admits a monomial selection of degree n if and only if there exists a function $f : S \to X$ such that*

$$\frac{1}{n!} \Delta_u^n f(x) \in G(u) \quad \textit{for } u, x \in S.$$

Proof. Suppose that G admits a monomial selection f od degree n. Then thanks to Theorem D1 we get

$$\frac{1}{n!} \Delta_u^n f(x) = f(x) \in G(u) \quad \text{for } u, x \in S.$$

For the other implication, suppose that there exists a function $f : S \to X$ such that

$$\frac{1}{n!}\Delta_u^n f(x) \in G(u) \quad \text{for } u, x \in S.$$

Then by Proposition 2 and Lemma 2 we obtain that there exists a function $F : S \to X$ such that $F(u) \in G(u)$ for $u \in S$ and

$$\frac{1}{n!}\Delta_u^n F(x) = F(u).$$

Then by Theorem D3 we obtain that F is a monomial function of degree n. $\square$

We would like to remark that thanks to Theorem 2 one can easily generalize Theorems 2 and 3 from [6] for monomial functions of degree n instead of additive (that is monomial of degree 1). However, in fact this generalization is trivial, as the proofs remain the same, and therefore we only mention it.

Corollary 1. *Let $\psi, \phi : S \to \mathbb{R}$ and let $f : S \to \mathbb{R}$ be such that*

$$\psi(u) \le \frac{1}{n!}\Delta_u^n f(x) \le \phi(u) \quad \textit{for } u, x \in S.$$

Then there exists a monomial function $F : S \to \mathbb{R}$ of degree n such that

$$\psi(u) \le F(u) \le \phi(u) \quad \textit{for } u \in S.$$

Proof. We define $G(u) := [\psi(u), \phi(u)]$ and make use of Theorem 2. $\square$

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