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A note on additive commutativity-preserving mappings

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Abstract. We characterize additive surjective commutativity-preserving mappings on M_n, $n \geq 2$.

The problem of characterizing linear transformations on M_n, the algebra of $n \times n$ complex matrices, that preserve some properties, has been considered in a number of papers. It turns out that this kind of mapping is often of the form

$$(1) \qquad X \mapsto \sigma AXA^{-1} + f(X)I \quad \text{or} \quad X \mapsto \sigma AX^{\text{tr}}A^{-1} + f(X)I,$$

where σ is a non-zero complex number, X^{tr} denotes the transpose of X, and f is a linear functional on M_n. It is natural to try to get similar results studying not linear but merely additive preservers. Omladič and Šemrl [10], [9] characterized additive spectrum-preserving mappings and additive mappings preserving operators of rank one. We say that ϕ *preserves commutativity* if $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$ (briefly $A \leftrightarrow B$), and it *preserves commutativity in both directions* if also $\phi(A) \leftrightarrow \phi(B)$ implies $A \leftrightarrow B$. Bijective additive mappings preserving commutativity on more general algebras have been described by Brešar, Miers, Banning and Mathieu [4], [5], [2]. This note is a continuation of the work of the present author [11], where we obtained the general form of an additive surjective mapping on M_n, $n \geq 3$, that preserves commutativity in both directions. The methods we use here are different, and we

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replace the assumption of preserving commutativity in both directions by the weaker one, preserving commutativity in one direction only. Moreover, the characterization of such mappings on M_2 is included. For $n \geq 3$ we obtain as a result the mappings of the form

$$X \mapsto \sigma T(X) + p(X)I \tag{2}$$

where $\sigma \neq 0$ is a complex constant, p a complex valued additive mapping on M_n, and $T : M_n \to M_n$ is defined either by $[x_{ij}] \mapsto A\,[f\,(x_{ij})]\,A^{-1}$, or $[x_{ij}] \mapsto A\,[f\,(x_{ij})]^{\mathrm{tr}}\,A^{-1}$ for some invertible matrix A and a ring automorphism f on $\mathbb{C}$. The mapping $\lambda \mapsto \overline{\lambda}$ of a complex number to its conjugate is a nontrivial continuous ring automorphism of $\mathbb{C}$. Moreover, there exist nowhere continuous ring automorphisms of $\mathbb{C}$ [1]. It is not surprising that the result for $n = 2$ differs essentially from that for $n \geq 3$. Even in the linear case the mappings of the form (1) are not the only ones that arise as bijective commutativity preservers on M_2 [13]. Any mapping of the form (2) can be regarded as a compositum of a linear bijective commutativity-preserving mapping and a ring automorphism $[x_{ij}] \mapsto [f\,(x_{ij})]$, additively perturbed by a mapping $X \mapsto p\,(X)\,I$. The same holds true in the two dimensional case. The set of all bijective linear mappings $\phi : M_2 \to M_2$ satisfying $\phi\,(I) = \lambda I$, for some $\lambda \neq 0$, is equal to the set of all bijective linear mappings on M_2 that preserve commutativity. This is a straightforward consequence of the fact that the commutant X' (the set of all matrices from M_n commuting with X) of any non-scalar matrix $X \in M_2$ is only two dimensional, i.e.:

$$X' = \{\alpha X + \beta I,\ \alpha, \beta \in \mathbb{C}\}. \tag{3}$$

Before giving the proofs we introduce some notation: $[A, B] = AB - BA$, $E_{ij} = [\delta_{ij}]$, where δ_{ij} is the Kronecker symbol. The mapping $\phi : M_n \to M_n$ is called f-quasilinear, for some $f : \mathbb{C} \to \mathbb{C}$, if it is additive, and if the relation $\phi(\alpha X) = f(\alpha)\phi(X)$ holds for all complex numbers α and $X \in M_n$.

Theorem. *Let ϕ be an additive surjective commutativity-preserving mapping on M_n, $n \geq 2$.*

If $n \geq 3$ then there exists a ring automorphism $f : \mathbb{C} \to \mathbb{C}$, a non-zero complex constant σ, an invertible matrix A and an additive function $p : M_n \to \mathbb{C}$ such that ϕ is either of the form

$$\text{(a)} \qquad \phi([x_{ij}]) = \sigma A[f(x_{ij})]A^{-1} + p([x_{ij}])I, \qquad [x_{ij}] \in M_n,$$

or

$$\text{(b)} \qquad \phi([x_{ij}]) = \sigma A[f(x_{ij})]^{\mathrm{tr}} A^{-1} + p([x_{ij}])I, \quad [x_{ij}] \in M_n.$$

In the case $n = 2$ there exists a ring automorphism $f : \mathbb{C} \to \mathbb{C}$, an additive function $p : M_2 \to \mathbb{C}$ and a linear mapping $L : M_2 \to M_2$ which leaves the subspace $\{\lambda I,\ \lambda \in \mathbb{C}\}$ invariant, such that ϕ is of the form

$$\phi([x_{ij}]) = L\left([f\left(x_{ij}\right)]\right) + p\left([x_{ij}]\right) I, \quad [x_{ij}] \in M_2.$$

Remarks. 1. This note contains also the proof for $n = 2$, the case that is exceptional, and was not considered in the previously mentioned papers.

2. If we add the assumption of injectivity, the result for $n \geq 3$ follows from [4].

3. Not only do we not need injectivity, in this particular case, studying the mappings on M_n, the proof is much shorter, and involves only simple linear algebra tools.

Proof. We will show that ϕ is not "very far" from being linear. As ϕ preserves commutativity we have that

$$\phi\left(\alpha X\right) \leftrightarrow \phi\left(X\right) \tag{4}$$

for all complex numbers α, and $X \in M_n$. Let $\mu \in \mathbb{C}$, $\mu \neq 0$, be fixed. Since ϕ is surjective, we can get for every pair of indices i, j a matrix Z_{ij} with $\phi\left(Z_{ij}\right) = \mu E_{ij}$. All block matrices in the proof will be partitioned according to $\mathbb{C}^n = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \operatorname{Span}\left(e_3, \dots, e_n\right)$ where $\{e_i,\ 1 \leq i \leq n\}$ is the standard basis of $\mathbb{C}^n$. The blocks that are not of dimension 1×1 will be denoted using capital letters. We have divided the proof into three steps.

Step 1. If $\phi(X) = \mu E_{ij} + \delta I$, $\delta \in \mathbb{C}$, $1 \leq i,\ j \leq n$, then $\phi(\alpha X)$, $\alpha \in \mathbb{C}$, is a sum of a scalar multiple of E_{ij} and a diagonal matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ satisfying $d_i = d_j$.

If $n = 2$ this is a straightforward consequence of (3) and (4). Let $n \geq 3$ and $i = j$. As E_{ii} is similar to E_{11} (by a permutation matrix) we may assume $i = 1$ with no loss of generality. Because of (4) $\phi(\alpha X) \leftrightarrow E_{11}$, and is therefore of the form

$$\phi(\alpha X) = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}.$$

By the same argument, we have that

$$\phi\left(\alpha Z_{22}\right) = \begin{bmatrix} b_{11} & 0 & B_{13} \\ 0 & b_{22} & 0 \\ B_{31} & 0 & B_{33} \end{bmatrix},$$

and by the additivity of ϕ

$$\phi\left(\alpha\left(X + Z_{22}\right)\right) = \begin{bmatrix} a_{11} + b_{11} & 0 & B_{13} \\ 0 & a_{22} + b_{22} & A_{23} \\ B_{31} & A_{32} & A_{33} + B_{33} \end{bmatrix}.$$

For the same reason, $\phi\left(\alpha\left(X + Z_{22}\right)\right)$ commutes with $\phi\left(X\right) + \phi\left(Z_{22}\right)$, and therefore also with $E_{11} + E_{22}$. This forces A_{23} and A_{32} to be zero. Replacing Z_{22} by Z_{kk}, $k \geq 3$, in the previous consideration, we get that A_{33} is a diagonal matrix. In particular, $\phi\left(\alpha Z_{ii}\right)$, $1 \leq i \leq n$, is a diagonal matrix for every $\alpha \in \mathbb{C}$.

Assume now $i \neq j$. Without loss of generality, we can fix $(i, j) = (1, 2)$. Since $\phi(\alpha X)$ commutes with E_{12}, we may write its block matrix as

$$\phi(\alpha X) = \begin{bmatrix} c_{11} & c_{12} & C_{13} \\ 0 & c_{11} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix}.$$

The matrix $\phi\left(\alpha\left(X + Z_{11} + Z_{22}\right)\right)$, which is of the form

$$\begin{aligned}
\phi\big(\alpha\left(X + Z_{11} + Z_{22}\right)\big) &= \phi(\alpha X) + \phi\left(\alpha Z_{11}\right) + \phi\left(\alpha Z_{22}\right) \\
&= \begin{bmatrix} c_{11} & c_{12} & C_{13} \\ 0 & c_{11} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix} \\
&= \begin{bmatrix} c_{11} + a_{11} + b_{11} & c_{12} & C_{13} \\ 0 & c_{11} + a_{22} + b_{22} & 0 \\ 0 & C_{32} & C_{33} + A_{33} + B_{33} \end{bmatrix}
\end{aligned}$$

commutes with $\phi(X+Z_{11}+Z_{22}) = \phi(X)+\phi(Z_{11})+\phi(Z_{22})$, and consequently with $E_{12}+E_{11}+E_{22}$. This implies

$$C_{13}=0 \quad \text{and} \quad C_{32}=0.$$

In order to get that C_{33} is diagonal, we choose k, $3 \leq k \leq n$, and compute

$$\begin{aligned}
\phi(\alpha(X+Z_{kk})) &= \phi(\alpha X)+\phi(\alpha Z_{kk}) \\
&= \begin{bmatrix} c_{11} & c_{12} & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} + \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \\
&= \begin{bmatrix} c_{11}+d_{11} & c_{12} & 0 \\ 0 & c_{11}+d_{22} & 0 \\ 0 & 0 & C_{33}+D_{33} \end{bmatrix}.
\end{aligned}$$

We know that the last matrix commutes with $\phi(X+Z_{kk})=\mu(E_{12}+E_{kk})+\delta I$, and therefore also with E_{kk}. Moreover, D_{33} is diagonal, which yields the desired conclusion.

Step 2. If $\phi(X)=\mu E_{ij}+\delta I$, $\delta \in \mathbb{C}$, $1 \leq i, j \leq n$, then there exists a ring automorphism $f:\mathbb{C}\to\mathbb{C}$ (independent of i, j and δ) and a complex valued function v_{ij}, such that for every $\alpha \in \mathbb{C}$ holds

$$\phi(\alpha X) = f(\alpha)\,\mu E_{ij} + v_{ij}(\alpha,\delta)\,I, \tag{5}$$

and v_{ij} is additive in the first argument.

It suffices to get (5) for $i=1$ if $i=j$, and for $(i,j)=(1,2)$ in the case $i \neq j$. Suppose first $i=j=1$. If $n \geq 3$ choose $r \geq 3$. Using the additivity of ϕ and applying Step 1 leads to

$$\begin{aligned}
\phi(\alpha(X+Z_{2r})) &= \phi(\alpha X)+\phi(\alpha Z_{2r}) \\
&= \operatorname{diag}(a_1,\dots,a_n) + (\operatorname{diag}(d_1,\dots,d_n)+u_{2r}E_{2r}),\ d_2=d_r,
\end{aligned}$$

and because of (4), the matrix $\phi(\alpha(X+Z_{2r}))$ commutes with $E_{11}+E_{2r}$. Therefore, $a_r=a_2$ for all $r \geq 3$. Now, there exist functions u_{11} and v_{11}, both additive in the first argument, such that

$$\phi(\alpha X) = u_{11}(\alpha,\delta)\,E_{11} + v_{11}(\alpha,\delta)\,I. \tag{6}$$

Next, we shall derive a similar equation to the above one for $(i,j)=(1,2)$. Let $\phi(X)=\mu E_{12}+\delta I$. For any k, $3\le k\le n$, $\phi\left(\alpha\left(X+Z_{1k}\right)\right)$ commutes with $E_{12}+E_{1k}$. This gives

$$0=\left[\phi\left(\alpha\left(X+Z_{1k}\right)\right),E_{12}+E_{1k}\right]_{1k}=\phi(\alpha X)_{11}-\phi(\alpha X)_{kk},$$

and, applying the assertion of Step 1, we get the existence of the functions u_{12} and v_{12}, such that

$$\phi\left(\alpha X\right)=u_{12}\left(\alpha,\delta\right)E_{12}+v_{12}\left(\alpha,\delta\right)I.$$

Hence, if $\phi(X)=\mu E_{ij}+\delta I$, $1\le i,\ j\le n$, there exist functions u_{ij} and v_{ij} such that

$$\phi(\alpha X)=u_{ij}(\alpha,\delta)E_{ij}+v_{ij}(\alpha,\delta)I. \tag{7}$$

Functions u_{ij} and v_{ij} are additive in the first argument and unique. In the case $n=2$, and $i\ne j$, the relation (7) is a straightforward consequence of Step 1. If $n=2$ and $i=j$, $\phi(\alpha X)$ is a diagonal matrix by Step 1, and as $\mu\ne 0$ was fixed, we get (7) with u_{ii} and v_{ii}, $i=1,2$, unique.

Since ϕ is surjective, there exist matrices X_{ij} with $\phi\left(X_{ij}\right)=E_{ij}$. Fix the set $\{X_{ij};\ \phi\left(X_{ij}\right)=E_{ij}\}$. In the previous consideration $\mu\ne 0$ was fixed but arbitrary. In particular, the application of (7) at $\mu=1$ and $\delta=0$ guaranties the existence of uniquely defined additive functions f_{ij} and g_{ij} with

$$\phi\left(\alpha X_{ij}\right)=f_{ij}(\alpha)E_{ij}+g_{ij}(\alpha)I. \tag{8}$$

We will now show that the functions f_{ij} are independent of i and j. Let $i\ne j$. By (4), and the additivity of ϕ, we have

$$\phi\left(\alpha\left(X_{ii}+X_{ij}\right)\right)\leftrightarrow E_{ii}+E_{ij},$$

and

$$\phi\left(\alpha\left(X_{ii}+X_{ij}\right)\right)=f_{ii}\left(\alpha\right)E_{ii}+f_{ij}\left(\alpha\right)E_{ij}+\left(g_{ii}(\alpha)+g_{ij}(\alpha)\right)I.$$

Therefore,

$$0=\left[f_{ii}\left(\alpha\right)E_{ii}+f_{ij}\left(\alpha\right)E_{ij},E_{ii}+E_{ij}\right]=\left(f_{ii}\left(\alpha\right)-f_{ij}\left(\alpha\right)\right)\left[E_{ii},E_{ij}\right],$$

which implies $f_{ii} = f_{ij}$. Replacing E_{ij} by E_{ji} in the above computation, we also obtain $f_{ii} = f_{ji}$ for all $1 \leq i,\, j \leq n$, $n \geq 2$. From now on f will be written instead of f_{ij}.

Our next goal is to show that

$$u_{ij}(\alpha, \delta) = f(\alpha)\mu \tag{9}$$

for all $1 \leq i,\, j \leq n$. Let $\phi(X) = \mu E_{ij} + \delta I$, and take k, $k \neq j$. As $\phi\left(\alpha\left(X + X_{ik}\right)\right) \leftrightarrow \mu E_{ij} + E_{ik}$, we have that

$$\begin{aligned} 0 &= \left[u_{ij}\left(\alpha, \delta\right) E_{ij} + f(\alpha) E_{ik}, \mu E_{ij} + E_{ik}\right] \\ &= \left(u_{ij}(\alpha, \delta) - \mu f(\alpha)\right)\left[E_{ij}, E_{ik}\right]. \end{aligned}$$

Certainly, we can always choose k, $k \neq j$, such that $[E_{ij}, E_{ik}] \neq 0$, and the desired conclusion now follows.

What is left is to show that f is multiplicative and surjective. For all complex α and β we have

$$\phi(\alpha\beta X_{12}) = f(\alpha\beta) E_{12} + g_{12}(\alpha\beta) I \tag{10}$$

and

$$\phi(\beta X_{12}) = f(\beta) E_{12} + g_{12}(\beta) I.$$

If $f(\beta) = 0$ the relation $\phi(\alpha(\beta X_{12} + X_{lk})) \leftrightarrow \phi(\beta X_{12}) + E_{lk}$, $1 \leq l, k \leq n$, implies that $\phi(\alpha\beta X_{12})$ is a scalar matrix, and thus $f(\alpha\beta) = 0$. Take now $\mu = f(\beta) \neq 0$, $\delta = g_{12}(\beta)$ and $(i, j) = (1, 2)$. Combining equations (7) with X being replaced by βX_{12}, and (9) we obtain

$$\phi(\alpha(\beta X_{12})) = f(\alpha) f(\beta) E_{12} + v_{12}\left(\alpha, g_{12}(\beta)\right) I. \tag{11}$$

Comparing the last equation to (10) gives the multiplicativity of f.

It is routine to show that the set $\{X_{ij},\ \phi\left(X_{ij}\right) = E_{ij}\}$, that has already been fixed before, forms a basis of M_n, $n \geq 2$. For details we refer the reader to [11, p. 208]. From the linear independence of the set $\{X_{ij}\}$, the relation (8) and the surjectivity of ϕ, the surjectivity of f is now easily obtained.

Note that for every $X \in M_n$ there exist unique numbers α_{ij}, $1 \leq i$, $j \leq n$, such that $X = \sum_{i,j} \alpha_{ij} X_{ij}$.

Step 3. There exists a surjective linear mapping $L : M_n \to M_n$, $n \geq 2$, that preserves commutativity, and an additive function q on M_n such that

$$\phi\left([x_{ij}]\right) = L\left(\left[f\left(x_{ij}\right)\right]\right) + q\left([x_{ij}]\right) I.$$

Let us first define an additive mapping $\phi_1 : M_n \to M_n$,

$$\phi_1(X) = \phi_1\left(\sum_{i,j} \alpha_{ij} X_{ij}\right) = \left[f\left(\alpha_{ij}\right)\right],$$

which is surjective (since f is surjective), and preserves commutativity because of

$$\begin{aligned} \phi(X) &= \phi\left(\sum_{i,j} \alpha_{ij} X_{ij}\right) \\ &= \phi_1(X) + \left(\sum_{i,j} g_{ij}\left(\alpha_{ij}\right)\right) I \\ &= \phi_1(X) + p(X) I. \end{aligned} \tag{12}$$

Clearly, p just involved is additive. Furthermore, we observe that ϕ_1 is f-quasilinear as

$$\begin{aligned} \phi_1(\alpha X) &= \phi_1\left(\sum_{i,j} \alpha \alpha_{ij} X_{ij}\right) \\ &= f(\alpha)\left[f\left(\alpha_{ij}\right)\right] \\ &= f(\alpha)\phi_1(X) \end{aligned}$$

for every $\alpha \in \mathbb{C}$. Let ψ denote the mapping on M_n defined by

$$\psi\left([x_{ij}]\right) = \left[f\left(x_{ij}\right)\right]$$

which is additive, bijective and preserves commutativity in both directions. Finally, we define $L = \phi_1 \circ \psi^{-1}$, and observe that it is homogeneous. Indeed,

$$L(\alpha X) = \phi_1\left(\psi^{-1}(\alpha X)\right) = \phi_1\left(f^{-1}(\alpha)\psi^{-1}(X)\right) = \alpha L(X).$$

Moreover, L is additive, surjective and preserves commutativity which establishes the assertion of Step 3.

Since L is linear, surjective and preserves commutativity, we then clearly have $L(I) = cI$, for some $c \neq 0$. If $n = 2$, the relation $\phi_1 = L \circ \psi$ substituted in (12) gives the desired conclusion. If $n \geq 3$ we end the proof of the theorem by substituting the well known form of a surjective linear commutativity-preserving mapping [3], [13] in (12). □

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