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Constraint coefficient problems for a subclass of starlike functions

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Abstract. Let $\boldsymbol{S}^*_{\boldsymbol{R}}$ be the class of univalent starlike functions with real coefficients defined in the unit disk U. Using the Carathéodory–Toeplitz conditions, we are able to solve the constraint problems of the third and fourth coefficients of $\boldsymbol{S}^*_{\boldsymbol{R}}$ for any fixed second coefficient in $[-2,2]$.

1. Introduction

Let $\boldsymbol{H}(U)$ be the topological linear space of analytic functions in the unit disk $U=\{z:|z|<1\}$ and $\boldsymbol{H}_{\boldsymbol{R}}$ be the subclass of $\boldsymbol{H}(U)$ of functions with real coefficients. We consider the class $\boldsymbol{P}_{\boldsymbol{R}}$ of all functions $p\in\boldsymbol{H}_{\boldsymbol{R}}$ with:

$$p(0)=1 \quad\text{and}\quad \mathrm{Re}[p(z)]>0, \quad z\in U.$$

By $\boldsymbol{S}^*_{\boldsymbol{R}}$ we denote the subclass of $\boldsymbol{H}_{\boldsymbol{R}}$ of normalized univalent starlike functions. A function

$$g(z)=z+\sum_{n=2}^{\infty}g_nz^n$$

is in $\boldsymbol{S}^*_{\boldsymbol{R}}$, iff there exists a function

$$q(z)=1+\sum_{n=1}^{\infty}q_nz^n$$

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in $\boldsymbol{P_R}$ such that

$$\frac{zg'(z)}{g(z)} = q(z). \tag{1}$$

By $\boldsymbol{C_R}$ we denote the subclass of $\boldsymbol{H_R}$ of normalized univalent close-to-convex functions. A function

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n$$

is in $\boldsymbol{C_R}$ iff there exists a function $g(z)$ in $\boldsymbol{S^*_R}$ and a

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

in $\boldsymbol{P_R}$ such that

$$\frac{zf'(z)}{g(z)} = p(z). \tag{2}$$

If $t_1 \in [0,1]$, by $\boldsymbol{S^*_R}(t_1)$, $(\boldsymbol{C_R}(t_1))$ we denote the class of functions

$$g(z) = z + g_2 z^2 + g_3 z^3 + \cdots \in \boldsymbol{S^*_R}, (\boldsymbol{C_R})$$

for which

$$g_2 = -2 + 4t_1.$$

H. S. Al-Amiri and D. Bshouty in [1] considered the problem of calculating the values $\max\limits_{g \in \boldsymbol{S^*_R}(t_1)} g_n$, $\max\limits_{g \in \boldsymbol{C_R}(t_1)} g_n$. They solved this problem in the following cases:

$$n = 3\ \forall\, t_1 \in [0,1] \quad \text{and} \quad n = 4\ \forall\, t_1 \in \left[\frac{5}{6}, 1\right] \tag{i}$$

for the class $\boldsymbol{S^*_R}(t_1)$ and

$$n = 3\ \forall\, t_1 \in [0,1] \quad \text{and} \quad n = 4\ \forall\, t_1 \in \left[\frac{11}{12}, 1\right] \tag{ii}$$

for the class $\boldsymbol{C_R}(t_1)$.

In this paper we solve the problem for the class $\boldsymbol{S}^*_{\boldsymbol{R}}(t_1)$ for $n=4,5$ $\forall\, t_1 \in [0,1]$. We will also solve the corresponding problem concerning $\min\limits_{g\in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_n$ for $n=3,4,5$ $\forall\, t_1\in[0,1]$. All the above results are presented in Theorems 1, 2, 3, 4, 5.

We face a problem involving the estimation of quantities which depend on the Taylor coefficients of functions belonging to the class $\boldsymbol{P}_{\boldsymbol{R}}$. In [1], H. S. Al-Amiri and D. Bshouty used a Theorem of Dubins concerning the extreme points of crossections of convex sets.

Our first idea is to use the Carathéodory–Toeplitz conditions as they consist the strongest relations between the Taylor coefficients of the class $\boldsymbol{P}_{\boldsymbol{R}}$. A second idea is to express these relations in such a way that each Taylor coefficient can be converted separately to a polynomial of several variables.

Combining these two ideas, we transform the initial problem into finding the max (or min) of a polynomial of several variables, defined in a closed interval $[0,1]^k$, $k\le 4$. All the above are contained in Step 1 of the proof of Theorem 4. In Step 2 of the proof we calculate in a usual way, the maximum or the minimum of these polynomials making use of their particular properties.

A serious problem in this paper is the size of the polynomials which are involved in the elementary calculations. Using the computer algebra system Mathematica 2.2, we obtained all necessary results.

2. Main theorems

Theorem 1. *If* $\min\limits_{g\in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_3 = m_3(t_1)$, *then:*

$$m_3(t_1) = (1-4t_1)(3-4t_1), \quad \textit{for } t_1\in[0,1].$$

Theorem 2. *If* $\max\limits_{g\in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_4 = M_4(t_1)$, *then:*

$$M_4(t_1) = 4(-1+2t_1)(1-8t_1+8t_1^2), \quad \textit{for } t_1\in\left[0,\frac{5}{14}\right]$$

$$M_4(t_1) = \frac{1}{3}(13-45t_1+48t_1^2-4t_1^3), \quad \textit{for } t_1\in\left(\frac{5}{14},\frac{5}{6}\right)$$

$$M_4(t_1) = \frac{4}{3}(-1+2t_1)(3-4t_1+4t_1^2), \quad \text{for } t_1 \in \left[\frac{5}{6}, 1\right].$$

Theorem 3. *If* $\min\limits_{g \in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_4 = m_4(t_1)$, *then:*

$$\begin{aligned} m_4(t_1) &= \frac{4}{3}(-1+2t_1)(3-4t_1+4t_1^2), \quad &&\text{for } t_1 \in \left[0, \frac{1}{6}\right] \\ m_4(t_1) &= \frac{1}{3}(-12+39t_1-36t_1^2-4t_1^3), \quad &&\text{for } t_1 \in \left(\frac{1}{6}, \frac{9}{14}\right) \\ m_4(t_1) &= 4(-1+2t_1)(1-8t_1+8t_1^2), \quad &&\text{for } t_1 \in \left[\frac{9}{14}, 1\right]. \end{aligned}$$

Theorem 4. *If* $\max\limits_{g \in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_5 = M_5(t_1)$, *then:*

$$M_5(t_1) = \frac{1}{3}(15-56t_1+88t_1^2-64t_1^3+32t_1^4), \quad \text{for } t_1 \in [0,1].$$

Theorem 5. *If* $\min\limits_{g \in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_5 = m_5(t_1)$, *then:*

$$\begin{aligned} m_5(t_1) &= (5-20t_1+16t_1^2)(1-12t_1+16t_1^2), \\ &\qquad \text{for } t_1 \in \left[0, \frac{352-24\sqrt{66}}{704}\right] \cup \left[\frac{352+24\sqrt{66}}{704}, 1\right] \\ m_5(t_1) &= \frac{1}{486}\left(-1291+4064t_1-3552t_1^2-1024t_1^3+512t_1^4\right), \\ &\qquad \text{for} \quad t_1 \in \left(\frac{352-24\sqrt{66}}{704}, \frac{352+24\sqrt{66}}{704}\right). \end{aligned}$$

In order to prove the previous theorems we will need the following lemmas.

Lemma 1. *Let* $K_n(\boldsymbol{P_R})$ *be the set of* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *for which there exists a* $q(z) = 1 + q_1 z + q_2 z^2 + \cdots \in \boldsymbol{P_R}$ *having* $q_1 = x_1$, $q_2 = x_2, \ldots, q_n = x_n$. *Let also* A_n *be the set of* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

such that $D_k(x_1, x_2, \dots, x_k) > 0$, $k = 1, 2, \dots, n$ *where:*

$$D_k(x_1, x_2, \dots, x_k) = \begin{vmatrix} 2 & x_1 & x_2 & \dots & x_k \\ x_1 & 2 & x_1 & \dots & x_{k-1} \\ x_2 & x_1 & 2 & \dots & x_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k-1} & x_{k-2} & \dots & 2 \end{vmatrix}.$$

If $\overline{A}_n$ *is the closure of* A_n *then* $\overline{A}_n = K_n(\boldsymbol{P_R})$.

The above lemma is a part of the Carathéodory–Toeplitz Theorem (see [2], [3]).

Lemma 2. *If* $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ $(n \leq 4)$ *the following propositions are equivalent.*

(i) $x \in K_n(\boldsymbol{P_R})$

(ii) *there exists a* $(t_1, t_2, \dots, t_n) \in [0,1]^n$ *such that:* $x_1 = p_1(t_1)$, $x_2 = p_2(t_1, t_2), \dots, x_n = p_n(t_1, t_2, \dots, t_n)$ *where:*

$$\begin{aligned} p_1(t_1) &= -2 + 4t_1 \\ p_2(t_1, t_2) &= 2 + 16t1(-1 + t_1 + t_2 - t_1 t_2) \\ p_3(t_1, t_2, t_3) &= -2 + t_1(36 - 96t_1 + 64t_1^2) - 32t_1 t_2(1 - 5t_1 + 4t_1^2) \\ &\quad - 64t_1^2 t_2^2(1 + t_1) + 64(-1 + t_1)t_1(1 - t_2)t_2 t_3 \\ p_4(t_1, t_2, t_3, t_4) &= 2\big(1 + 32t_1(-1 + 5t_1) + 128t_1^3(-2 + t_1) + 32t_1 t_2(1 - 9t_1 \\ &\quad + 20t_1^2 - 12t_1^3) + 128t_1^2 t_2^2(1 - 4t_1 + 3t_1^2) + 128t_1^3 t_2^3(1 - t_1) \\ &\quad + 128t_1 t_2 t_3(1 - 3t_1 + 2t_1^2 - t_2) + 128t_1^2 t2^2 t_3(5 - 4t_1 - 2t_2 \\ &\quad + t_3 + 2t_1 t_2 - t_2 t_3) + 128t_1 t_2^2 t_3^2(-1 + t_2) \\ &\quad + 128t_1 t_2 t_3 t_4(-1 + t_1 + t_2 - t_1 t_2 + t_3 - t_1 t_3 - t_2 t_3 + t_1 t_2 t_3)\big). \end{aligned}$$

Proof. The quantity $D_k(x_1, x_2, \dots, x_k)$ can be written as polynomial of second degree in x_k of the form:

$$-D_{k-2}(x_1, x_2, \dots, x_{k-2})x_k^2 + \dots, \quad (D_k = 1 \text{ for } k \leq 0).$$

If $\rho_k \equiv \rho_k(x_1, x_2, \dots, x_{k-1})$ and $\rho_k^* \equiv \rho_k^*(x_1, x_2, \dots, x_{k-1})$ are the roots of the above polynomial it is easy to see that the relation $x \in A_n$ is equivalent to $\min(\rho_k, \rho_k^*) < x_k < \max(\rho_k, \rho_k^*)$ or

$$x_k = \rho_k + t_k(\rho_k^* - \rho_k), \quad t_k \in (0,1)\ k = 1, 2, \dots, n. \tag{3}$$

- For $k = 1$ we get $\rho_1 = -2$ and $\rho_1^* = 2$. Therefore

$$x_1 = p_1(t_1), \quad t_1 \in (0,1). \tag{4}$$

- For $k = 2$ by the equation $D_2(x_1, x_2) = 0$ we obtain $\rho_2 = -2 + x_1^2$ and $\rho_2^* = 2$. Thus combining (3) with (4) we get

$$x_2 = p_2(t_1, t_2), \quad (t_1, t_2) \in (0,1)^2. \tag{5}$$

- For $k = 3$ through equation $D_3(x_1, x_2, x_3) = 0$ we obtain

$$\rho_3 = -\frac{4 - 2x_1 - (x_1 - x_2)^2}{-2 + x_1} \quad \text{and} \quad \rho_3^* = -\frac{4 + 2x_1 - (x_1 + x_2)^2}{2 + x_1}.$$

Consequently, combining (3) with (4) and (5) we obtain after the calculations

$$x_3 = p_3(t_1, t_2, t_3), \quad (t_1, t_2, t_3) \in (0,1)^3. \tag{6}$$

In the same manner we can see that $x_4 = p_4(t_1, t_2, t_3, t_4)$. Summarizing, we have that the transform

$$(t_1, t_2, \dots, t_n) \longrightarrow \big(p_1(t_1), p_2(t_1, t_2), \dots, p_n(t_1, t_2, \dots, t_n)\big)$$

is one-to-one from $(0,1)^n$ onto A_n. After the above observation the rest of the proof is straightforward. □

3. Proof of Theorem 4

For the proof of the theorem we need the following two steps.

Step 1. Let $(\alpha_2, \alpha_3, \dots, \alpha_k) \in \mathbb{R}^{k-1}$ with $k \le 5$. The following properties are equivalent.

(i) There exists a function $g(z) = z + \sum\limits_{n=2}^{\infty} g_n z^n \in \boldsymbol{S}_{\boldsymbol{R}}^*$ such that:

$$g_2 = \alpha_2, \ \dots, \ g_k = \alpha_k.$$

(ii) There exists a $(t_1, t_2, \dots, t_{k-1}) \in [0,1]^{k-1}$, $k \le 5$ such that:

$$\begin{aligned}
\alpha_2 &= s_2(t_1)\\
\alpha_3 &= s_3(t_1,t_2)\\
\alpha_4 &= s_4(t_1,t_2,t_3)\\
\alpha_5 &= s_5(t_1,t_2,t_3,t_4)
\end{aligned}$$

with

$$\begin{aligned}
s_2(t_1) &= -2 + 4t_1\\
s_3(t_1,t_2) &= 3 + 16t_1(-1+t_1) + 8t_1t_2(1-t_1)\\
s_4(t_1,t_2,t_3) &= \frac{4}{3}\big(-3 + t_1(30 - 72t_1 + 48t_1^2) + t_1t_2(-20 + 76t_1 - 56t_1^2)\\
&\quad + 16t_1^2t_2^2(-1+t_1) + 16t_1t_2t_3(-1+t_1+t_2-t_1t_2)\big)\\
s_5(t_1,t_2,t_3,t_4) &= \frac{1}{3}\big(15 + t_1(-240 + 1008t_1 - 1536t_1^2 + 768t_1^3) + t_1t_2(184\\
&\quad - 1336t_1 + 2624t_1^2 - 1472t_1^3) + t_1^2t_2^2(416 - 1344t_1 + 928t_1^2)\\
&\quad + 192t_1^3t_2^3(1-t_1) + 320t_1t_2t_3(1 - 3t_1 + 2t_1^2 - t_2)\\
&\quad + t_1^2t_2^2t_3(1344 - 1024t_1 - 384t_2 + 384t_1t_2)\\
&\quad + 192t_1t_2^2t_3^2(-1+t_1+t_2-t_1t_2)\\
&\quad + 192t_1t_2t_3t_4(-1+t_1+t_2-t_1t_2+t_3-t_1t_3-t_2t_3+t_1t_2t_3)\big).
\end{aligned}$$

Proof of Step 1. Since $g(z) \in \boldsymbol{S}^*_{\boldsymbol{R}}$ it follows that

$$zg'(z) = g(z)q(z) \tag{7}$$

where

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \in \boldsymbol{P_R}.$$

From (3) we obtain

$$g_2 = q_1 \tag{8}$$

$$g_3 = \frac{1}{2}(q_1^2 + q_2) \tag{9}$$

$$g_4 = \frac{1}{6}(q_1^3 + 3q_1q_2 + 2q_3) \tag{10}$$

$$g_5 = \frac{1}{24}(q_1^4 + 6q_1^2q_2 + 3q_2^2 + 8q_1q_3 + 6q_4). \tag{11}$$

Conversely if $q(z) = 1 + q_1z + q_2z^2 + \cdots \in \boldsymbol{P_R}$ then there exists exactly one $g(z) \in \boldsymbol{S^*_R}$ such that $\frac{zg'(z)}{g(z)} = q(z)$. Replacing q_k by $p_k(t_1, t_2, \ldots, t_k)$ according to Lemma 2 after the calculations we attain the desired conclusion. □

Step 2. Calculation of $M_5(t_1)$.

We remark that:

1. According to Step 1, our problem is to find the maximum (or minimum) of the functions $s_k(t_1, t_2, \ldots, t_{k-1})$ $(k = 3, \ldots, 5)$, for fixed t_1 and $(t_2, \ldots, t_{k-1}) \in [0,1]^{k-2}$.

2. If we set in any function s_k $(k = 3, \ldots, 5)$, $t_i = 0$ or $t_i = 1$ $(i = 1, 2, 3)$, then the value of the function does not depend on the variables t_j when $j > i$.

3. The polynomial s_5 is linear in t_4 with corresponding coefficient $64(-1+t_1)t_1(-1+t_2)t_2(-1+t_3)t_3$ which is non-positive.

Using the above remarks we can continue as follows: Fixing t_1 we consider the extremum of the function s_5 on $(t_2, t_3, t_4) = (t_2, t_3, 0)$, $(t_2, t_3, t_4) = (t_2, 1, 0)$, $(t_2, t_3, t_4) = (t_2, 0, 0)$, $(t_2, t_3, t_4) = (1, 0, 0)$, $(t_2, t_3, t_4) = (0, 0, 0)$. We then find the critical points in the cube $[0,1]^3$. The maximum of all the above values is the needed result.

The Case $t_4 = 0$.

From:

$$\frac{\partial s_5(t_1, t_2, t_3, 0)}{\partial t_2} = 0 \quad \text{and} \quad \frac{\partial s_5(t_1, t_2, t_3, 0)}{\partial t_3} = 0, \tag{12}$$

after elementary calculations, we get

$$t_2 = \frac{19 + 112(-1+t_1)t_1}{72(-1+t_1)t_1} \quad \text{and} \quad t_3 = \frac{t_1(-41 + 68t_1 - 8t_1^2)}{19 + 112(-1+t_1)t_1}.$$

We then find the set of $t_1 \in (0,1)$ for which

$$\begin{cases} 1-t_3 & \geq 0 \\ t_3 & \geq 0 \\ 1-t_2 & \geq 0 \\ t_2 & \geq 0 \end{cases} \tag{13}$$

are all true. The above relations are polynomial quotients in t_1. Converting them to factor products, we find in a simple way that for $t_1 \in \left[\frac{-52+12\sqrt{23}}{16}, \frac{68-12\sqrt{23}}{16}\right]$, all inequalities in (13) are satisfied. Substituting t_2 and t_3 in $s_5(t_1,t_2,t_3,0)$ we have

$$L_1(t_1) = \frac{1}{162}(449 - 1504t_1 + 1632t_1^2 - 256t_1^3 + 128t_1^4). \tag{14}$$

The Case $t_3 = 1$.

From:

$$\frac{\partial s_5(t_1,t_2,1,0)}{\partial t_2} = 0 \tag{15}$$

we get $h_1(t_1,t_2) = 0$ where

$$h_1(t_1,t_2) = A_1(t_1)t_2^2 + B_1(t_1)t_2 + \Gamma_1(t_1) \tag{16}$$

with

$$\begin{aligned} A_1(t_1) &= \frac{32}{3}(-16t_1 + 61t_1^2 - 74t_1^3 + 29t_1^4) \\ B_1(t_1) &= \frac{8}{3}(63t_1 - 287t_1^2 + 408t_1^3 - 184t_1^4) \\ \Gamma_1(t_1) &= 5 - 80t_1 + 336t_1^2 - 512t_1^3 + 256t_1^4. \end{aligned}$$

We then find all possible cases for which $h_1(t_1,t_2)$ has at least one root with respect to t_2, in $(0,1)$. This is accomplished using the sign of the quantities:

$$A_1(t_1), \quad 1 + \frac{B_1(t_1)}{2A_1(t_1)}, \quad \frac{-B_1(t_1)}{2A_1(t_1)},$$

$$\big(B_1(t_1)\big)^2 - 4A_1(t_1)\Gamma_1(t_1), \quad h_1(t_1,0), \quad h_1(t_1,1).$$

By a usual procedure we obtain that:

(i) for $t_1 \in \left(\frac{-160+6\sqrt{870}}{52}, \frac{224+4\sqrt{238}}{368}\right]$ and
$\Psi_1 = (1-t_1)\sqrt{(-55+160t_1+26t_1^2)}$ the large root

$$t_{2_2} = \frac{32-90t_1+58t_1^2+\sqrt{2}\Psi_1}{36(-1+t_1^2)} \quad \text{is in } (0,1)$$

and

(ii) for $t_1 \in \left(\frac{-160+6\sqrt{870}}{52}, \frac{224-4\sqrt{238}}{368}\right]$ the small root

$$t_{2_1} = \frac{32-90t_1+58t_1^2-\sqrt{2}\Psi_1}{36(-1+t_1^2)} \quad \text{is in } (0,1).$$

After substituting the roots t_{2_2}, t_{2_1} in $s_5(t_1, t_2, 1, 0)$ we obtain respectively the functions

$$L_2(t_1) = \frac{\Phi_1 + \sqrt{2}t_1\Psi_1(-220+640t_1+104t_1^2)}{729(-1+t_1)} \tag{17}$$

for $t_1 \in \left(\frac{-160+6\sqrt{870}}{52}, \frac{224+4\sqrt{238}}{368}\right]$ and

$$L_3(t_1) = \frac{\Phi_1 + \sqrt{2}t_1\Psi_1(220-640t_1-104t_1^2)}{729(-1+t_1)} \tag{18}$$

for $t_1 \in \left(\frac{-160+6\sqrt{870}}{52}, \frac{224-4\sqrt{238}}{368}\right]$, where $\Phi_1 = -3645+18637t_1 - 31900t_1^2 + 16524t_1^3 - 176t_1^4 + 560t_1^5$.

The Case $t_3 = 0$.

Working as we did in previous Case we obtain that:

$$L_4(t_1) = \frac{\Phi_2 + \sqrt{2}\Psi_2(-524+1372t_1-952t_1^2+104t_1^3)}{729t_1} \tag{19}$$

for $t_1 \in \left[\frac{144-4\sqrt{238}}{368}, \frac{212-6\sqrt{870}}{52}\right)$ and

$$L_5(t_1) = \frac{\Phi_2 + \sqrt{2}\Psi_2(524-1372t_1+952t_1^2-104t_1^3)}{729t_1} \tag{20}$$

Figure 1.

for $t_1 \in \left[\frac{144+4\sqrt{238}}{368}, \frac{212-6\sqrt{870}}{52}\right)$, where $\Phi_2 = 6505t_1 - 22216t_1^2 + 21420t_1^3 - 2624t_1^4 + 560t_1^5$ and $\Psi_2 = \sqrt{t_1^2(131 - 212t_1 + 26t_1^2)}$.

The Cases $t_2 = 1$ and $t_2 = 0$.

$$L_6(t_1) = \frac{1}{3}(15 - 56t_1 + 88t_1^2 - 64t_1^3 + 32t_1^4) \tag{21}$$

for $t_1 \in [0, 1]$ and

$$L_7(t_1) = (5 - 20t_1 + 16t_1^2)(1 - 12t_1 + 16t_1^2) \tag{22}$$

for $t_1 \in [0, 1]$ are derived by setting in s_5, $t_2 = 1$ and $t_2 = 0$, respectively.

A hint about the form of $M_5(t_1)$ is obtained by the graphs of the functions L_i $(1 \leq i \leq 7)$ (see Figure 1). In order to give a strict proof of Theorem 4 we consider the functions $L_6(t_1) - L_i(t_1)$ $(i \neq 6)$ in the subdomains of their definition and we examine their signs. More specificly:

- Since $L_6(t_1) - L_1(t_1) = \frac{(19-40t_1+40t1^2)^2}{162}$ it is obvious that $L_6(t_1) \geq L_1(t_1)$ in $\left[\frac{-52+12\sqrt{23}}{16}, \frac{68-12\sqrt{23}}{16}\right]$.
- Also $L_6(t_1) - L_7(t_1) = \frac{184(1-t_1)t_1(-1+2t_1)^2}{3}$. Therefore $L_6(t_1) \geq L_7(t_1)$ in $[0, 1]$.

- Solving the equations $L_6(t_1) - L_i(t_1) = 0$ for $i = 2, 3, 4, 5$ in all the cases we find the simple roots $t_1 = 0$ and $t_1 = 1$. Checking the sign of the functions $L_6(t_1) - L_i(t_1)$ in the interior of the subdomains of their definition we obtain that $L_6(t_1) \geq L_i(t_1)$, $(i = 2, 3, 4, 5)$.

4. Proof of the other theorems

Proof of Theorem 5. As in the procedure of the Proof of Theorem 4 in order to find $\min\limits_{g \in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_5$ we consider the restriction of function s_5 for $(t_2, t_3, t_4) = (t_2, t_3, 1)$. The procedure of seeking local extreme points gives that for $t_1 \in \left[\frac{352-24\sqrt{66}}{704}, \frac{352+24\sqrt{66}}{704}\right]$ the corresponding value for s_5 is

$$R_1(t_1) = \frac{1}{486}(-1291 + 4064t_1 - 3552t_1^2 - 1024t_1^3 + 512t_1^4) \tag{23}$$

From Remark 2 of Step 2, it follows that in order to find the form of $m_5(t_1)$ it is sufficient to compare the values of the functions $L_i(t_1)$ $(i = 2, \dots, 7)$, to that of $R_1(t_1)$. For the comparison we folow the procedure of the Proof of Theorem 4. $\square$

Proof of Theorem 2. We observe that the polynomial s_4 is linear in t_3 having the non-positive $\frac{64}{3}(-1 + t_1)t_1(1 - t_2)t_2$ coefficient. We will achieve $\max\limits_{g \in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_4$ by considering the restriction of function s_4 for $(t_2, t_3) = (t_2, 0)$. Since

$$\frac{\partial s_4(t_1, t_2, 0)}{\partial t_2} = 0, \tag{24}$$

it follows that for

$$t_2 = \frac{-5 + 14t_1}{8t_1}$$

we obtain a local extreme point of s_4. The constraint $t_2 \in (0, 1)$ is satisfied for $t_1 \in \left(\frac{5}{14}, \frac{5}{6}\right)$. Replacing the above value of t_2 in $s_4(t_1, t_2, 0)$ we get

$$N_1(t_1) = \frac{1}{3}(13 - 45t_1 + 48t_1^2 - 4t_1^3) \tag{25}$$

for $t_1 \in \left(\frac{5}{14}, \frac{5}{6}\right)$. For $t_2 = 0$ and $t_2 = 1$ we obtain respectively

$$N_2(t_1) = 4(-1 + 2t_1)(1 - 8t_1 + 8t_1^2) \tag{26}$$

Figure 2.

for $t_1 \in [0,1]$ and

$$N_3(t_1) = \frac{4}{3}(-1+2t_1)(3-4t_1+4t_1^2) \tag{27}$$

for $t_1 \in [0,1]$. Comparing the values of the functions $N_i(t_1)$ as in the Proof of Theorem 4 we get that $\max\{N_i(t_1),\ i=1,2,3\}$, coincides with the form of Theorem 2 (see Figure 2). $\square$

PROOF of Theorem 3. According to the Proof of Theorem 2, $\min\limits_{g \in \boldsymbol{S}^*_{\boldsymbol{R}}(t_1)} g_4$ will be achieved by the restriction of the function s_4 for $(t_2,t_3) = (t_2,1)$. Since

$$\frac{\partial s_4(t_1,t_2,1)}{\partial t_2} = 0, \tag{28}$$

it is obvious that for

$$t_2 = \frac{-9+14t_1}{8(-1+t_1)}$$

we get a local extreme point of s_4. The constraint $t_2 \in (0,1)$ is satisfied for $t_1 \in \left(\frac{1}{6}, \frac{9}{14}\right)$. In this interval replacing the above value of t_2 in $s_4(t_1,t_2,1)$

we get

$$K_1(t_1) = \frac{1}{3}(-12 + 39t_1 - 36t_1^2 - 4t_1^3). \tag{29}$$

From Remark 2 of Step 2 the result follows again by comparing the values of functions $N_i(t_1)$ $(i = 2, 3)$, to that of $K_1(t_1)$ as in the Proof of Theorem 4. □

PROOF of Theorem 1. Follow the same procedure as in the Proof of Theorem 5, 3. □

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