Publ. Math. Debrecen
56 / 1-2 (2000), 77–85

The topological structure of the set of P-sums of a sequence, II

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Abstract. Let $P = \{p_0, p_1, \dots, p_N\}$ where $p_{i-1} < p_i$ for $i = 1, 2, \dots, N$. Let $\lambda = \langle \lambda_n \rangle$ be a sequence of real numbers for which $\sum |\lambda_n|$ converges. Let

$$S(P,\lambda) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}.$$

$S(P,\lambda)$ is called the set of P-sums for the sequence λ. In the case where $P = \{0,1\}$ it is known that $S(P,\lambda)$ is one of the following: (i) a finite union of intervals; (ii) homeomorphic to the Cantor set; (iii) homeomorphic to $S(\{0,1\},\beta)$ where $\beta_{2n-1} = 3/4^n$ and $\beta_{2n} = 2/4^n$ $(n = 1, 2, \dots)$. In this paper this result is generalized to a larger class of P-sums.

1. Introduction

Let $P = \{p_0, p_1, \dots, p_N\}$ where $p_{i-1} < p_i$ for $i = 1, 2, \dots, N$. Let $\lambda = \langle \lambda_n \rangle$ be a sequence of real numbers for which $\sum |\lambda_n|$ converges and $|\lambda_n| \geq |\lambda_{n+1}| > 0$ for all n. Let

$$S(P,\lambda) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}.$$

We will call $S(P,\lambda)$ the set of P-sums for the sequence λ. In the case $P = \{0,1\}$ we will write $S(P,\lambda) = S(\lambda)$ and call this set the set of subsums of λ. In an earlier paper [2] the authors studied the topological structure

Mathematics Subject Classification: Primary: 11B05; Secondary: 11K31, 40A05.
Key words and phrases: sequence, subsum, Cantor set.

of the set $S(P,\lambda)$. In the present paper we give further results on this topic.

In [1], the following theorem was proved for positive term sequences and then in [2] the restriction that λ has only positive terms was removed.

Theorem 1.1. *$S(\lambda)$ is one of the following:*

(i) *a finite union of intervals;*

(ii) *homeomorphic to the Cantor set C;*

(iii) *homeomorphic to $S(\beta)$ where $\beta = \langle \beta_n \rangle$ with $\beta_{2n-1} = 3/4^n$ and $\beta_{2n} = 2/4^n$ $(n = 1, 2, \dots)$.*

Remark 1.1. In order to visualize the set $S(\beta)$ above we give the following example of a set which is homeomorphic to $S(\beta)$ (see [1]). Let S_n denote the union of the 2^{n-1} open middle thirds which are removed from $[0,1]$ at the n-th step in the geometric construction of C. Then

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} S_n,$$

and $S(\beta)$ is homeomorphic to $C \cup \bigcup_{n=1}^{\infty} S_{2n-1}$.

In [2] the authors gave an example of a sequence γ and a set Q for which $S(Q,\gamma)$ is not one of the three types listed above. This example will be discussed again at the end of this paper.

In this paper the authors will give some sufficient conditions on the sequence λ and/or the set P which will assure that $S(P,\lambda)$ is one of the three types given in Theorem 1.1.

2. Notation and basic results on $S(P,\lambda)$

At this point we introduce some notation and terminology which will be used later in the paper. As usual, if E is any set, $aE + b = \{ax + b : x \in E\}$. We will say E is symmetric (about a) if the condition

$$a + x \in E \Leftrightarrow a - x \in E$$

is satisfied. (It is easy to see that this is equivalent to $E = 2a - E$.)

With λ and P as given above, $|\lambda|$ will denote the sequence $\langle|\lambda_n|\rangle$, s will denote $\sum \lambda_n$ and $\bar{s}$ will denote $\sum |\lambda_n|$. Also set

$$r_n = \sum_{k=n+1}^{\infty} \lambda_k$$

and

$$\bar{r}_n = \sum_{k=n+1}^{\infty} |\lambda_k|.$$

Also ν will denote the smallest subscript for which

$$p_\nu - p_{\nu-1} = \max\{p_i - p_{i-1} : i = 1, \dots, N\}.$$

For k a positive integer, $S_k(P, \lambda)$ will denote the set

$$\left\{ \sum_{n=k+1}^{\infty} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}$$

and will be called the set of P-sums of the k-tail of λ and $F_k(P, \lambda)$ will denote

$$\left\{ \sum_{n=1}^{k} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}$$

and will be called the set of k-finite P-sums of λ.

Remark 2.1. Using the above notation, the following decomposition for $S(P, \lambda)$ is easy to see.

$$S(P, \lambda) = \bigcup_{f \in F_k(P,\lambda)} (f + S_k(P, \lambda)).$$

The following results were given in [2].

Proposition 2.1. *$S(aP + b, \lambda) = aS(P, \lambda) + bs$ for any real numbers a and b.*

Proposition 2.2. *If P is symmetric about a, then*

$$S(P, \lambda) = S(P, |\lambda|) + (s - \bar{s})a.$$

Theorem 2.3. *$S(P,\lambda)$ is a perfect set.*

The following theorem is a slight generalization of one given in [2].

Theorem 2.4. *Assume λ is a positive term sequence. $S(P,\lambda)$ is a finite union of intervals if*

$$\lambda_n \le \frac{p_N - p_0}{p_\nu - p_{\nu-1}} r_n \tag{1}$$

for n sufficiently large. (Also $S(P,\lambda)$ is an interval if (1) holds for all n.) Conversely, if

$$\frac{p_1 - p_0}{p_\nu - p_0} \ge \frac{\lambda_n}{\lambda_{n-1}} \tag{2}$$

for n sufficiently large and $S(P,\lambda)$ is a finite union of intervals, then (1) holds for n sufficiently large. (Also if (2) holds for all n and $S(P,\lambda)$ is an interval, then (1) holds for all n.)

Furthermore, if P is symmetric the requirement that λ have only positive terms can be deleted if λ_n and r_n are replaced by $|\lambda_n|$ and $\bar{r}_n$ in the above inequalities.

Proof. The first part of the theorem was proved in [2], so we only prove the second part here (the furthermore statement follows from Proposition 2.2). By replacing P by $\frac{1}{p_N - p_0}(P - p_0)$, we may assume $p_0 = 0$ and $p_N = 1$. Then, of course, (1) and (2) become

$$(p_\nu - p_{\nu-1})\lambda_n \le r_n, \tag{1'}$$

$$p_1 \lambda_{n-1} \ge p_\nu \lambda_n. \tag{2'}$$

Now suppose (2') holds for $n \ge N_0$ and that $S(P,\lambda)$ is a finite union of intervals. Assume, with the goal of obtaining a contradiction, that

$$(p_\nu - p_{\nu-1})\lambda_n > r_n$$

for infinitely many n. Then there is a sequence $\langle n_k \rangle$ such that $n_k \ge N_0$ and $(p_\nu - p_{\nu-1})\lambda_{n_k} > r_{n_k}$ for all k. We will show that

$$(p_{\nu-1}\lambda_{n_k} + r_{n_k}, p_\nu \lambda_{n_k}) \cap S(P,\lambda) = \emptyset$$

for all k, and hence the complement of $S(P,\lambda)$ has infinitely many components which completes the contradiction. Now the inequality (2'), i.e.

$p_1\lambda_{n_k-1} \geq p_\nu\lambda_{n_k}$, assures us that $0 < p_1\lambda_{n_k} < \ldots < p_{\nu-1}\lambda_{n_k} < p_\nu\lambda_{n_k}$ are the $\nu+1$ smallest elements of $F_{n_k}(P,\lambda)$ for all k. Also $p_\nu\lambda_{n_k}$ is clearly the smallest element of $p_\nu\lambda_{n_k} + S_{n_k}(P,\lambda)$. Hence, by Remark 2.1, $S(P,\lambda)$ can have no elements between $p_{\nu-1}\lambda_{n_k} + r_{n_k}$ and $p_\nu\lambda_{n_k}$.

3. Main result

The main result of this paper is the following generalization of Theorem 1.1.

Theorem 3.1. *If λ is a sequence, P is symmetric, and*

$$\frac{p_1 - p_0}{p_\nu - p_0} \geq \left|\frac{\lambda_n}{\lambda_{n-1}}\right| \tag{2}$$

for n sufficiently large, then $S(P,\lambda)$ is a set of type (i), (ii) or (iii) of Theorem 1.1.

In the proof of this theorem we will need the following lemmas.

Lemma 3.2. *Assume λ is a positive term sequence. If $p_0 s + [0,\delta] \subset S(P,\lambda)$ for some $\delta > 0$ and there is some K such that*

$$\frac{p_1 - p_0}{p_\nu - p_0} \geq \frac{\lambda_n}{\lambda_{n-1}} \tag{2}$$

for all $n \geq K$, then $S(P,\lambda)$ is a finite union of intervals.

Proof. As in the proof of Theorem 2.4 we can assume $p_N = 1$ and $p_0 = 0$. Then (2) becomes

$$p_1\lambda_{n-1} \geq p_\nu\lambda_n, \tag{2'}$$

and $p_0 s$ becomes 0. Assume, with the goal of obtaining a contradiction, that $S(P,\lambda)$ is not a finite union of intervals. By Theorem 2.4,

$$(p_\nu - p_{\nu-1})\lambda_n > r_n \tag{3}$$

for infinitely many n. Choose $n_0 \geq K$ such that (3) holds for $n = n_0$ and $p_{\nu-1}\lambda_{n_0} + r_{n_0} < \delta$. Now

$$\sum \varepsilon_i\lambda_i \leq p_{\nu-1}\lambda_{n_0} + r_{n_0} \text{ if } \varepsilon_i = 0 \text{ for } i < n_0 \text{ and } \varepsilon_{n_0} \leq p_{\nu-1}.$$

Also, by (2'),

$$\sum \varepsilon_i \lambda_i \geq p_\nu \lambda_{n_0} \text{ if } \varepsilon_i \neq 0 \text{ for some } i < n_0 \text{ or } \varepsilon_{n_0} \geq p_\nu.$$

[For $i < n$, $p_j \lambda_i \geq p_j \lambda_{n-1} \geq p_1 \lambda_{n-1} \geq p_\nu \lambda_n$ for $j > 0$.] Therefore, $(p_{\nu-1}\lambda_{n_0} + r_{n_0}, p_\nu \lambda_{n_0}) \cap S(P,\lambda) = \emptyset$ and $\delta > p_{\nu-1}\lambda_{n_0} + r_{n_0}$, a contradiction with $[0,\delta] \subset S(P,\lambda)$.

In what follows we will refer to a component of $S(P,\lambda)$ as an *interval* of $S(P,\lambda)$ and a component of $[p_0 s, p_N s] \setminus S(P,\lambda)$ as a *gap* of $S(P,\lambda)$.

Lemma 3.3. *Assume λ is a positive term sequence. If (a,b) is a gap of $S(P,\lambda)$, then for some $\varepsilon > 0$ and $\varepsilon' > 0$,*

$$(b - p_0 s) + ((p_0 s + [0,\varepsilon]) \cap S(P,\lambda)) = [b, b+\varepsilon] \cap S(P,\lambda)$$

and

$$(a - p_N s) + (p_N s + [-\varepsilon', 0]) \cap S(P,\lambda) = ([a - \varepsilon', a] \cap S(P,\lambda)).$$

Proof. As before, we will assume $p_N = 1$ and $p_0 = 0$. It is not difficult to see that b must be a finite P-sum. Suppose

$$b = \sum_{i=1}^{k} \varepsilon_i \lambda_i \in F_k(P,\lambda) \quad (\varepsilon_i \in P \text{ for } 1 \leq i \leq k)$$

with $\varepsilon_k \neq 0$. Now assume that the elements of $F_k(P,\lambda)$, in order, are

$$0 = f_1 < f_2 < \ldots < f_t = \sum_{i=1}^{k} \lambda_i$$

and suppose $b = f_j$. Recall that

$$S(P,\lambda) = \bigcup_{i=1}^{t} (f_i + S_k(P,\lambda)).$$

From this we see that $a = f_{j-1} + r_k$. Let $\varepsilon = \frac{1}{2}\min(f_2 - f_1, f_{j+1} - f_j)$. Then

$$\begin{aligned} b + ([0,\varepsilon] \cap S(P,\lambda)) &= b + ([0,\varepsilon] \cap S_k(P,\lambda)) \\ &= [b, b+\varepsilon] \cap (b + S_k(P,\lambda)) \\ &= [b, b+\varepsilon] \cap S(P,\lambda). \end{aligned}$$

Now let $\varepsilon' = \frac{1}{2}\min(f_t - f_{t-1}, f_{j-1} - f_{j-2})$. As before, and using $f_t = s - r_k$ and $f_{j-1} = a - r_k$ we have,

$$\begin{aligned}(a-s) &+ ([s-\varepsilon', s] \cap S(P,\lambda)) \\ &= (a-s) + ([s-\varepsilon', s] \cap ((s-r_k) + S_k(P,\lambda)) \\ &= [a-\varepsilon', a] \cap ((a-r_k) + S_k(P,\lambda)) \\ &= [a-\varepsilon', a] \cap S(P,\lambda).\end{aligned}$$

We now proceed to prove Theorem 3.1. The proof follows very closely the proof of the early version of Theorem 1.1 in [1].

Proof of Theorem 3.1. As before we will assume that $p_N = 1$ and $p_0 = 0$. Also since P is symmetric we can assume λ has only positive terms by Proposition 2.2. Suppose that $S(P,\lambda)$ is neither a finite union of intervals nor homeomorphic to the Cantor set. Then it is clear that the complement of $S(P,\lambda)$ must contain infinitely many intervals. $S(P,\lambda)$ must contain infinitely many intervals as well, for if there were only finitely many, then either $[0,\delta] \subset S(P,\lambda)$ for some $\delta > 0$ or $[0,\delta] \cap S(P,\lambda)$ contains no interval for some $\delta > 0$. The former cannot be true by Lemma 3.2. If the latter holds, then $S(P,\lambda) \cap [0,\delta]$ is homeomorphic to the Cantor set and then for some n, $S_n(P,\lambda)$ is homeomorphic to the Cantor set. Thus, by Remark 2.1 and the fact that a finite union of Cantor sets is a Cantor set, $S(P,\lambda)$ would be homeomorphic to the Cantor set which contradicts our initial assumption. Thus $S(P,\lambda)$ contains infinitely many intervals.

In fact, $S(P,\lambda) \cap [a,b]$ cannot be homeomorphic to the Cantor set for any $a, b \in S(P,\lambda)$ since for every n, $S_n(P,\lambda)$ must contain (infinitely many) intervals. Suppose then, that for some $x \in S(P,\lambda)$,

$$S(P,\lambda) \cap (x, x+\delta) = \emptyset \text{ for some } \delta > 0.$$

Then since $S(P,\lambda)$ is perfect,

$$S(P,\lambda) \cap (x-\delta, x) \neq \emptyset \text{ for every } \delta > 0,$$

and therefore there are intervals in $S(P,\lambda)$ arbitrarily close to x, i.e. the union of the intervals of $S(P,\lambda)$ is dense in $S(P,\lambda)$.

We now define a strictly increasing mapping f from the union of all intervals of $S(\beta)$ onto all intervals of $S(P,\lambda)$ and also all the gaps of $S(\beta)$

onto all the gaps of $S(P,\lambda)$. We define the mapping inductively. Begin by mapping the longest interval of $S(\beta)$ in a strictly increasing way onto the longest interval of $S(P,\lambda)$. There can be at most finitely many intervals of the same length in either set, so in case no one interval is the longest, we may choose the left-most interval. Denote this interval of $S(\beta)$ by I. Also, as part of the first induction step, we map the longest gap of $S(\beta)$ to the left (respectively right) of I in a strictly increasing way onto the longest gap of $S(P,\lambda)$ to the left (respectively right) of $f[I]$. If more than one gap has the same length, we again use the "left-most rule" discussed above. This completes step one of the inductive definition of f.

After the n-th step, $(4^n-1)/3$ intervals of $S(\beta)$ will have been mapped in a strictly increasing way onto $(4^n-1)/3$ intervals of $S(P,\lambda)$ and $2(4^n-1)/3$ gaps of $S(\beta)$ will have been mapped onto $2(4^n-1)/3$ gaps of $S(P,\lambda)$. We now apply the process of the first induction step to each of the spaces between any two adjacent intervals and/or gaps (of which there are 4^n-1) and to the space between 0 and the left-most of the gaps and to the space between the right-most of the gaps and 5/3. (Note that $5/3=\sum\beta_n$.) To be sure we can carry out step $n+1$ (for $n=1,2,\dots$) we need to know that in each of the 4^n spaces of $S(\beta)$ (and $S(P,\lambda)$) to which we apply the first inductive step, there are infinitely many intervals and gaps of $S(\beta)$ (and $S(P,\lambda)$). Lemma 3.3 guarantees this since there are infinitely many intervals and gaps of $S(\beta)$ (and $S(P,\lambda)$) in $[0,\varepsilon]$. Because $S(\beta)$ is symmetric and we are assuming that P is symmetric (so $S(P,\lambda)$ is symmetric), there are infinitely many intervals and gaps of $S(\beta)$ (respectively $S(P,\lambda)$) in $[5/3-\varepsilon,5/3]$ (respectively $[s-\varepsilon,s]$).

When f is defined in this way, it is a strictly increasing mapping of the union of all intervals of $S(\beta)$ onto the union of all intervals of $S(P,\lambda)$ and the union of all the gaps of $S(\beta)$ onto the union of all the gaps of $S(P,\lambda)$. Earlier in the proof it was shown that the union of all the intervals of $S(\beta)$ (respectively $S(P,\lambda)$) is dense in $S(\beta)$ (respectively $S(P,\lambda)$). Hence the union of all the intervals and gaps of $S(\beta)$ (respectively $S(P,\lambda)$) is dense in $[0,5/3]$ (respectively $[0,s]$). Hence f can be extended in a unique way to a strictly increasing mapping of $[0,5/3]$ onto $[0,s]$ which maps $S(\beta)$ onto $S(P,\lambda)$. f restricted to $S(\beta)$ is the desired homeomorphism.

4. Conjecture

As mentioned earlier, the authors gave an example in [2] of a sequence γ and a set Q for which $S(Q,\gamma)$ is not one of the three types in Theorem 1.1. In that example $\gamma_n = 1/3^n$ and $Q = \{0,1,2,9\}$. In [2] it was shown that $[0,9/8]$ is an interval of $S(Q,\gamma)$. It is not difficult to check that for any $\varepsilon' > 0$ there are infinitely many intervals and gaps of $S(Q,\gamma)$ in $[9/2-\varepsilon', 9/2]$. (Note that $9/2 = \sum p_N\gamma_n = \sum 9/3^n$.) Lemma gaps shows that when λ is a positive term sequence the structure of $S(P,\lambda)$ is highly dependent on the structure of $S(P,\lambda)$ in $[p_0 s, p_0 s+\varepsilon]$ and in $[p_N s-\varepsilon', p_N s]$. Lemma gaps implies that if λ is a positive term sequence, whenever (a,b) and (c,d) are two gaps of $S(P,\lambda)$, then, for some $\varepsilon > 0$ and $\varepsilon' > 0$,

$$(c-a) + ([a-\varepsilon, a] \cap S(P,\lambda)) = [c-\varepsilon, c] \cap S(P,\lambda)$$

and

$$(d-b) + ([b, b+\varepsilon'] \cap S(P,\lambda)) = [d, d+\varepsilon'] \cap S(P,\lambda).$$

If we can verify the above statement where *gaps* is replaced by *intervals*, we believe that a construction similar to the one in the proof of Theorem 3.1 will show that if λ is a positive term sequence, then $S(P,\lambda)$ is homeomorphic to one of three types described in Theorem 3.1 *or* homeomorphic to $S(Q,\gamma)$.

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(Received June 22, 1998)