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On generalized recurrent Riemannian manifolds

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Abstract. In this paper we study the two 1-forms A and B appearing in the definition of the generalized recurrency of a Riemannian manifold. It is shown that for a non-zero constant scalar curvature, A is closed iff B is closed. For a nonconstant scalar curvature the 1-forms A and B cannot be closed both, unless A is collinear with B. Also, we have found out some results on a generalized conformally recurrent Riemannian manifold.

1. Introduction

In a recent paper [1] De and Guha introduced and studied a type of non-flat Riemannian space whose curvature tensor $K(X,Y,Z)$ of type $(1,3)$ satisfies the condition:

$$(1.1)\quad (D_U K)(X,Y,Z) = A(U)K(X,Y,Z) + B(U)[g(Y,Z)X - g(X,Z)Y]$$

where A and B are two 1-forms, B is non-zero and D denotes the operator of covariant differentiation with respect to the metric tensor g. Such a space has been called a *generalized recurrent space*. Here B is called its associated 1-form. These spaces are related to the pseudo symmetric Riemannian spaces of M. C. Chaki [7] and are special cases of the weakly symmetric Riemannian spaces of L. Tamássy and T. Q. Binh [8]. If the 1-form $B(U)$ becomes zero in (1.1), then the space reduces to a recurrent space according to Ruse [6] and Walker [5].

Contracting (1.1) with respect to 'X', we get

$$(1.2)\qquad (D_U \operatorname{Ric})(Y,Z) = A(U)\operatorname{Ric}(Y,Z) + (n-1)B(U)g(Y,Z).$$

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In this case, the Riemannian manifold M is called a *generalized Ricci recurrent space*, where A and B are as stated earlier. If the 1-form $B(U)$ becomes zero in (1.2), then the space reduces to a Ricci-recurrent space.

In this paper we have considered a non-flat n-dimensional Riemannian manifold in which the *conformal curvature tensor* C satisfies the condition:

$$(D_U C)(X,Y,Z) = A(U)C(X,Y,Z) + B(U)[g(Y,Z)X - g(X,Z)Y] \tag{1.3}$$

where A and B are two 1-forms, B is non-zero and the conformal curvature tensor C is defined by (see [2])

$$\begin{aligned} C(X,Y,Z) = K(X,Y,Z) &- \frac{1}{n-2}[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y \\ &+ g(Y,Z)R(X) - g(X,Z)R(Y)] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{1.4}$$

Here K is the curvature tensor of type $(1,3)$, Ric is the Ricci tensor of type (0,2), r is the scalar curvature and R is the Ricci tensor to type $(1,1)$, defined by

$$\mathrm{Ric}(X,Y) = g(R(X),Y). \tag{1.5}$$

Such an n-dimensional Riemannian manifold shall be called a *generalized conformally recurrent* Riemannian manifold. If the 1-form B is zero, then the manifold reduces to a conformally recurrent manifold [3].

The *conharmonic curvature tensor* N and the *concircular curvature tensor* W are given by [4]

$$\begin{aligned} N(X,Y,Z) = K(X,Y,Z) &- \frac{1}{n-2}[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y \\ &+ g(Y,Z)R(X) - g(X,Z)R(Y)] \end{aligned} \tag{1.6}$$

and

$$W(X,Y,Z) = K(X,Y,Z) - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y)], \tag{1.7}$$

respectively.

If the conharmonic curvature tensor N and concircular curvature tensor W satisfy the conditions:

$$(1.8)\quad (D_U N)(X,Y,Z) = A(U)N(X,Y,Z) + B(U)[g(Y,Z)X - g(X,Z)Y]$$

and

$$(1.9)\quad (D_U W)(X,Y,Z) = A(U)W(X,Y,Z) + B(U)[g(Y,Z)X - g(X,Z)Y],$$

respectively, where A and B are two 1-forms, then the Riemannian manifold is known as a *generalized conharmonically recurrent* manifold and a *generalized concircularally recurrent* manifold, respectively.

In Section 2 we have discussed the nature of 1-forms A and B and in Section 3 we have studied about a generalized conformally recurrent manifold.

2. Nature of the 1-forms A and B on a generalized recurrent space

Taking covariant derivative of (1.5) with respect to 'U', we have

$$(2.1)\qquad g((D_U R)(X), Y) = (D_U \operatorname{Ric})(X,Y).$$

Using (1.2) in (2.1), we have

$$g((D_U R)(X), Y) = A(U)\operatorname{Ric}(X,Y) + (n-1)B(U)g(X,Y)$$

which yields

$$(2.2)\qquad (D_U R)(X) = A(U)R(X) + (n-1)B(U)(X).$$

Contracting (2.2) with respect to 'X', we get

$$(2.3)\qquad Ur = A(U)r + n(n-1)B(U)$$

where r is the scalar curvature.

First we consider the case when the scalar curvature r is a constant and is different from zero. Then from (2.3), we have

$$(2.4)\qquad A(U)r + n(n-1)B(U) = 0.$$

Taking covariant derivative of (2.4) with respect to 'V', we get

$$(D_V A)(U)r + n(n-1)(D_V B)(U) = 0. \tag{2.5}$$

Interchanging U and V in (2.5) and then subtracting them, we get

$$[(D_V A)(U) - (D_U A)(V)]r + n(n-1)[(D_V B)(U) - (D_U B)(V)] = 0.$$

Thus we have

Theorem 1. *In a generalized recurrent space of non-zero constant scalar curvature r, the 1-form A is closed if and only if the 1-form B is closed.*

Next we consider the case when the scalar curvature r is not constant. From (2.3) it follows that

$$VUr = (D_V A)(U)r + A(U)(Vr) + n(n-1)(D_V B)(U). \tag{2.6}$$

Interchanging U and V in (2.6) and then subtracting, we get

$$\begin{aligned}[(D_V A)(U) - (D_U A)(V)]r + A(U)(Vr) - A(V)(Ur)\\ +n(n-1)[(D_V B)(U) - (D_U B)(V)] = 0.\end{aligned} \tag{2.7}$$

Using (2.3) in (2.7), we get

$$\begin{aligned}[(D_V A)(U) - (D_U A)(V)]r + n(n-1)[(D_V B)(U) - (D_U B)(V)]\\ +n(n-1)[A(V)B(U) - A(U)B(V)] = 0.\end{aligned}$$

Thus we can state the following theorem:

Theorem 2. *In a generalized recurrent space of non-constant scalar curvature r, the 1-forms A and B cannot be both closed, unless the 1-form A is collinear with the 1-form B.*

3. Generalized conformally recurrent Riemannian manifolds

Let M be a generalized recurrent smooth Riemannian manifold of dimension n. Taking covariant derivative of (1.4) with respect to 'U', we get

$$(3.1)\quad (D_U C)(X,Y,Z) = (D_U K)(X,Y,Z) - \frac{1}{n-2}[(D_U \operatorname{Ric})(Y,Z)X - (D_U \operatorname{Ric})(X,Z)Y + g(Y,Z)(D_U R)(X) - g(X,Z)(D_U R)(Y)] + \frac{Ur}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].$$

Using (1.1), (1.2) and (2.3) in (3.1), we get

$$(3.2)\quad (D_U C)(X,Y,Z) = A(U)\Bigg[K(X,Y,Z) - \frac{1}{n-2}\{\operatorname{Ric}(Y,Z)X - \operatorname{Ric}(X,Z)Y - g(Y,Z)R(X) - g(X,Z)R(Y)\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}\Bigg] + B(U)\Bigg[g(Y,Z)X - g(X,Z)Y - \frac{2(n-1)}{n-2}\{g(Y,Z)X - g(X,Z)Y\} + \frac{n(n-1)}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}\Bigg].$$

Using (1.4) in (3.2), we have

$$(D_U C)(X,Y,Z) = A(U)C(X,Y,Z)$$

which shows the condition of a conformally recurrent Riemannian manifold. Thus we get the following theorem:

Theorem 3. *A generalized recurrent Riemannian manifold is conformally recurrent for the same recurrence parameter.*

From (1.3) and (1.4) it follows that

$$(3.3)\quad (D_U K)(X,Y,Z) - A(U)K(X,Y,Z) - B(U)[g(Y,Z)X - g(X,Z)Y] = \frac{1}{n-2}[(D_U \operatorname{Ric})(Y,Z)X - (D_U \operatorname{Ric})(X,Z)Y$$

$$+g(Y,Z)(D_U R)(X) - g(X,Z)(D_U R)(Y)$$
$$-A(U)\{\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y + g(Y,Z)R(X) - g(X,Z)R(Y)\}]$$
$$+\frac{r}{(n-1)(n-2)}[A(U)\{g(Y,Z)X - g(X,Z)(Y)\}$$
$$-U\{g(Y,Z)R(X) - g(X,Z)R(Y)\}].$$

Permutting equation (3.3) twice with respect to U, X, Y; adding the three equations and using Bianchi's second identity, we have

$$\begin{aligned}
&A(U)K(X,Y,Z) + A(X)K(Y,U,Z) + A(Y)K(U,X,Z)\\
&+B(U)[g(Y,Z)X - g(X,Z)Y] + B(X)[g(U,Z)Y - g(Y,Z)U]\\
&+B(Y)[g(X,Z)U - g(U,Z)X]\\
&+\frac{1}{n-2}[(D_U\,\mathrm{Ric})(Y,Z)X - (D_U\,\mathrm{Ric})(X,Z)Y + g(Y,Z)(D_U R)(X)\\
&-g(X,Z)(D_U R)(Y) + (D_X\,\mathrm{Ric})(U,Z)Y\\
&-(D_X\,\mathrm{Ric})(Y,Z)U + g(U,Z)(D_X R)(Y)\\
&-g(Y,Z(D_X R)(U) + (D_Y\,\mathrm{Ric})(X,Z)U - (D_Y\,\mathrm{Ric})(U,Z)X\\
&+g(X,Z)(D_Y R)(U) - g(U,Z)(D_Y R)(X)\\
&-A(U)\{\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y + g(Y,Z)R(X) - g(X,Z)R(Y)\}\\
&-A(X)\{\mathrm{Ric}(U,Z)Y - \mathrm{Ric}(Y,Z)U + g(U,Z)R(Y) - g(Y,Z)R(U)\}\\
&-A(Y)\{\mathrm{Ric}(X,Z)U - \mathrm{Ric}(U,Z)X + g(X,Z)R(U) - g(U,Z)R(X)\}]\\
&+\frac{r}{(n-1)(n-2)}[A(U)\{g(Y,Z)X - g(X,Z)(Y)\}\\
&-U\{g(Y,Z)X - g(X,Z)Y\}]\\
&+A(X)\{g(U,Z)Y - g(Y,Z)U\} - X\{g(U,Z)Y - g(Y,Z)U\}\\
&+A(Y)\{g(X,Z)U - g(U,Z)X\} - Y\{g(X,Z)U - g(U,Z)X\}] = 0.
\end{aligned} \tag{3.4}$$

Contracting (3.4) with respect to 'X', we get

$$\begin{aligned}
&A(U)\,\mathrm{Ric}(Y,Z) - A(Y)\,\mathrm{Ric}(U,Z) + K(Y,U,Z,p)\\
&+(n-1)B(U)g(Y,Z) + B(Y)g(U,Z) - B(U)g(Y,Z)\\
&+(1-n)B(Y)g(U,Z) + \frac{1}{n-2}[(n-1)(D_U\,\mathrm{Ric})(Y,Z)
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&+g(Y,Z)(Ur) - g(D_U R)(Y),Z) + (D_Y \operatorname{Ric})(U,Z) \\
&-(D_U \operatorname{Ric})(Y,Z) + \frac{1}{2}g(U,Z)(Yr) - \frac{1}{2}g(Y,Z)(Ur) \\
&+(1-n)(D_Y \operatorname{Ric})(U,Z) + g(D_Y R)(U,Z) - g(U,Z)(Yr) \\
&-(n-1)A(U)\operatorname{Ric}(Y,Z) - A(U)g(Y,Z)r + A(U)\operatorname{Ric}(Y,Z) \\
&-A(Y)\operatorname{Ric}(U,Z) + A(U)\operatorname{Ric}(Y,Z) - A(R(Y))g(U,Z) \\
&+A(R(U))g(Y,Z) + (n-1)A(Y)\operatorname{Ric}(U,Z) - A(Y)\operatorname{Ric}(U,Z) \\
&+A(Y)g(U,Z)r] + \frac{r}{(n-1)(n-2)}[(n-1)A(U)g(Y,Z) \\
&-(n-1)g(Y,Z)U + A(Y)g(U,Z) - A(U)g(Y,Z) - ng(U,Z)Y \\
&+ng(Y,Z)U + (1-n)A(Y)g(U,Z) + (n-1)g(U,Z)Y] = 0
\end{aligned}
$$

where p is a vector field defined by

$$
g(X,p) = A(X). \tag{3.6}
$$

Using (1.5) in (3.5), we have

$$
\begin{aligned}
&A(U)R(Y) - A(Y)R(U) - K(Y,U,p) + (n-2)[B(U)Y - B(Y)U] \\
&+\frac{1}{n-2}[(n-1)(D_U R)(Y) + Y(Ur) - (D_U R)(Y) + (D_Y R)(U) \\
&-(D_U R)(Y) + \frac{1}{2}U(Yr) - \frac{1}{2}Y(Ur) + (1-n)(D_Y R)(U) + (D_Y R)(U) \\
&-U(Yr) - (n-1)A(U)R(Y) - A(U)Yr + A(U)R(Y) - A(Y)R(U) \\
&+A(U)R(Y) - A(R(Y))U + A(R(U))Y + (n-1)A(Y)R(U) \\
&-A(Y)R(U) + A(Y)Ur] + \frac{r}{(n-1)(n-2)}[(n-1)A(U)Y - (n-1)YU \\
&+A(Y)U - A(U)Y - nUY + nYU + (1-n)A(Y)U + (n-1)UY] = 0
\end{aligned}
$$

or

$$
\begin{aligned}
&K(Y,U,p) - (n-2)[B(U)Y - B(Y)U] \\
&= \frac{1}{n-2}[(n-3)(D_U R)(Y) - (n-3)(D_Y R)(U) \\
&+A(U)R(Y) - A(Y)R(U) + A(R(U))Y - A(R(Y))U] \\
&+\frac{r}{(n-1)(n-2)}[A(Y)U - A(U)Y].
\end{aligned} \tag{3.7}
$$

Contracting (3.7) with respect to 'Y', we get

$$\mathrm{Ric}(U,p) - (n-1)(n-2)B(U) = \frac{1}{n-2}\Bigg[(n-3)Ur - \frac{1}{2}(n-3)(Ur)$$

$$+A(U)r - A(R(U)) + (n-1)A(R(U))\Bigg] + \frac{r}{(n-1)(n-2)}[(n-1)A(U)]$$

or

$$2A(U)r + 2(n-1)(n-2)^2B(U) = (n-3)(Ur).$$

Thus we have

Theorem 4. *The necessary and sufficient condition that the scalar curvature r of a generalized conformally recurrent Riemannian manifold be constant is that*

$$A(U)r + (n-1)(n-2)^2B(U) = 0.$$

From (1.4), (1.6) and (1.7), we have

$$C(X,Y,Z) = N(X,Y,Z) + \frac{n}{n-2}[K(X,Y,Z) - W(X,Y,Z)]. \tag{3.8}$$

Taking covariant derivative of (3.8) with respect to 'U', we get

$$(D_UC)(X,Y,Z) = (D_UN)(X,Y,Z) + \frac{n}{n-2}[(D_UC)(X,Y,Z) - (D_UW)(X,Y,Z)]. \tag{3.9}$$

Using (1.1) in (3.9), we get

$$(D_UC)(X,Y,Z) - (D_UN)(X,Y,Z) + \frac{n}{n-2}(D_UW)(X,Y,Z) = \frac{n}{n-2}[A(U)K(X,Y,Z) + B(U)\{g(Y,Z)X - g(X,Z)Y\}]. \tag{3.10}$$

From (3.10) it is evident that if any two of the equations (1.3), (1.8) and (1.9) hold then the third also holds.

This leads the following:

Theorem 5. *Let M be a generalized recurrent Riemannian manifold of dimension n. If any two of the following hold then the third also holds:*

(i) *It is generalized conformally recurrent manifold.*

(ii) *It is generalized conharmonically recurrent manifold.*

(iii) *It is generalized concircularally recurrent manifold.*

For the same recurrence parameters.

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