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Transcendence and algebraic independence connected with Mahler type numbers

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Abstract. Let $g, h \geq 2$ be fixed integers and let $H(b)$ denote the digit expansion of the positive integer b in base h. For a given infinite sequence $A = (a_n)_{n=0}^{\infty}$ of non-negative integers, we consider the real number $M_h(g; A)$ defined by the digit expansion

$$M_h(g; A) := 0.H\left(g^{a_1}\right) \dots H\left(g^{a_n}\right) \dots \text{ in base } h.$$

We prove transcendence and algebraic independence results on numbers including $M_h(g; A)$.

1. Introduction and results

Let $g, h \geq 2$ be fixed integers and let $H(b)$ denote the digit expansion of the positive integer b in the base h. For a given infinite sequence $A = (a_n)_{n=0}^{\infty}$ of non-negative integers, we consider the real number $M_h(g; A)$ defined by the digit expansion

$$M_h(g; A) := 0.H\left(g^{a_0}\right) H\left(g^{a_1}\right) \dots H\left(g^{a_n}\right) \dots$$

in the base h. This means that the digit expansion of $M_h(g; A)$ is obtained by concatenation of the digit expansions $H\left(g^{a_0}\right)$, $H\left(g^{a_1}\right)$, … .

In 1981 Mahler [5] proved the irrationality of $M_{10}(g; N_0)$, where N_0 is the sequence $(n)_{n=0}^{\infty}$. Bundschuh [2] solved the case of an arbitrary

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base $h \geq 2$. Shan [10] provided a much simpler proof for Bundschuh's result, and shortly thereafter, Niederreiter [6], Shan and Wang [11] and Yu [15] gave generalizations to different kinds of unbounded sequences A.

Of particular interest is the method of proof of [11], in that $M_h(g; A)$ is irrational if A is strictly increasing. The irrationality assertion is deduced from a result on the finiteness of the number of integer solutions (x, y) of the Thue equation $ax^r - by^r = c$ with non-zero integers a, b, c and r satisfying $ab > 0$ and $r \geq 3$. Such a link between the irrationality of Mahler type numbers $M_h(g; A)$ and the finiteness question for a certain diophantine equation appeared already in the proof of [2]. This conclusion was later much more exploited by the work of Becker [1], Sander [9] and Shorey and Tijdeman [13] on the same subject.

After this short survey of irrationality results on Mahler type numbers we will now give a more explicit expression for these $M_h(g : A)$. This expression allows us to define a function, holomorphic in the unit disk, which we wish to investigate here from the arithmetical point of view. More precisely, we shall prove transcendence and algebraic independence results on numbers including $M_h(g; A)$ under stronger hypotheses on g, h and A than in the pure irrationality case.

Let $A = (a_n)_{n=0}^{\infty}$ be a sequence as before and write g^{a_n} in the base h as

$$g^{a_n} = \sum_{j=0}^{k_n} \delta_j^{(n)} h^j \tag{1}$$

with digits $\delta_j^{(n)} \in \{0, \ldots, h-1\}$ and $\delta_{k_n}^{(n)} \neq 0$. Clearly we have

$$k_n = \left[a_n \frac{\log g}{\log h} \right]. \tag{2}$$

For $n = 0, 1, \ldots,$ we now define polynomials

$$\Delta_n(z) := \sum_{j=0}^{k_n} \delta_{k_n - j}^{(n)} z^j$$

and a power series

$$f(z; A) := \sum_{n=0}^{\infty} \Delta_n(z) z^{k_0 + \cdots + k_{n-1} + n + 1} \tag{3}$$

with radius of convergence 1.

From $\Delta_n\left(\frac{1}{h}\right) = g^{a_n} h^{-k_n}$ we see

$$f\left(\frac{1}{h}; A\right) = \sum_{n=0}^{\infty} g^{a_n} h^{-(k_0+\cdots+k_n+n+1)} = M_h(g; A).$$

Our main result on transcendence of the values of the function $f(z; A)$ is contained in the following

Theorem 1. *Let $g, h \geq 2$ be fixed, multiplicatively dependent integers, and let the sequence $A = (a_n)_{n=0}^{\infty}$ of non-negative integers satisfy the conditions*

$$\limsup_{n\to\infty} \frac{a_n}{a_0 + \cdots + a_{n-1}} = \infty \tag{4}$$

and

$$\liminf_{n\to\infty} \frac{a_0 + \cdots + a_{n-1}}{n} > 0. \tag{5}$$

If the function $f(z; A)$ is defined in $|z| < 1$ by (3), then the number $f(\alpha; A)$ is transcendental for each algebraic α satisfying

(i) *α is real and $0 < \alpha < 1$*

or

(ii) $0 < |\alpha| \leq \frac{1}{h}$

or

(iii) $0 < |\alpha| < 1$ *under the additional hypothesis*

$$\lim_{n\to\infty} a_n = \infty. \tag{6}$$

Remarks. 1. It is easily seen that, if A satisfies

$$\lim_{n\to\infty} \frac{a_n}{a_0 + \cdots + a_{n-1}} = \infty, \tag{7}$$

then all three conditions (4), (5) and (6) hold. Therefore, under hypothesis (7) $f(\alpha; A)$ is transcendental for each non-zero algebraic α in the unit disk.

2. It should be noted that (4), (5) and (6) together do not imply (7), as can be easily seen form the following example: Take $a_n := 2^n$ if n is a power of 2, and $a_n := n$, otherwise.

3. None of the conditions (4), (5) implies the other. This is evident from the two sequences $a_n := 1$ for all n, and $a_n := n$ if n is of the shape 2^{2^m} with some non-negative integer m and $a_n := 0$ otherwise.

Concerning algebraic independence we assert

Theorem 2. *Let g, h and A be as in Theorem 1, but with the hypothesis (7). If $\alpha_1, \ldots, \alpha_t$ are non-zero algebraic numbers of distinct absolute values in the unit disk, and if ℓ is any non-negative integer, then the numbers*

$$f^{(\lambda)}(\alpha_\tau; A) \quad (\tau = 1, \ldots, t; \lambda = 0, \ldots, \ell)$$

are algebraically independent (over Q, the set of rational numbers). In particular, $f(\alpha_1; A), \ldots, f(\alpha_t; A)$ are algebraically independent.

Remark 4. It is almost sure that condition (7) here can be weakened to the simultaneous conditions (4), (5) and (6) from Theorem 1, but we do not intend to explore this question further.

2. Some auxiliary results

Since both theorems are concerned with the case of multiplicatively dependent $g, h \geq 2$, we suppose from now on

$$g^r = h^s \tag{8}$$

with some positive integers r, s. The importance of this hypothesis in our context is revealed in

Lemma 1. *If (8) holds, then for any integer $a \geq 0$ exactly one digit in the canonical h-adic expansion of g^a is different from zero, and moreover one has*

$$g^a = \beta h^{[as/r]}$$

with some $\beta \in \{1, \ldots, h-1\}$.

Proof. Suppose $g = p_1^{\mu_1} \ldots p_m^{\mu_m}$, $h = p_1^{\nu_1} \ldots p_m^{\nu_m}$ with different primes $p_1, \ldots, p_m$ and positive exponents μ, ν. From (8) we conclude $r\mu_i = s\nu_i$ for $i = 1, \ldots, m$ and thus $a\mu_i = a\frac{s}{r}\nu_i$ for the same i. Therefore we have $a\mu_i = \left[a\frac{s}{r}\right]\nu_i + \lambda_i$ with some $\lambda_i \in \{0, \ldots, \nu_i - 1\}$ and from

$$g^a = p_1^{a\mu_1} \ldots p_m^{a\mu_m} = p_1^{\lambda_1} \ldots p_m^{\lambda_m} h^{[as/r]}$$

we get the assertion of Lemma 1. □

Lemma 2. *Suppose that* (8) *holds, then the function* $f(z;A)$ *from* (3) *has the shape*

$$f(z;A)=\sum_{n=0}^{\infty}\beta_n z^{e_n} \tag{9}$$

with $\beta_n\in\{1,\dots,h-1\}$ *and* $e_n:=k_0+\cdots+k_{n-1}+n+1$ *where* $k_n=\left[a_n\frac{s}{r}\right]$ *for* $n=0,1,\dots$. *For the same* n *and for any complex number* z *with* $|z|<1$ *the equality*

$$\left|f(z;A)-\sum_{i=0}^{n}\beta_i z^{e_i}\right|=\beta_{n+1}|z|^{e_{n+1}}\left(1+\gamma\frac{h-1}{1-|z|}|z|^{1+k_{n+1}}\right) \tag{10}$$

holds; here $\gamma=\gamma(n,z)$ *is a real number satisfying* $|\gamma|\le 1$. *If* $0<|z|<1$ *and* $k_i>0$ *for at least one* $i\ge n+2$ *hold, then one can even guarantee* $|\gamma|<1$.

Proof. From (2) and (8) we get $k_n=\left[a_n\frac{s}{r}\right]$, and therefore $g^{a_n}=\beta_n h^{k_n}$ with $\beta_n\in\{1,\dots,h-1\}$, from Lemma 1. Of course, this means that we have $\delta_j^{(n)}=0$ for $j=0,\dots,k_n-1$ and $\delta_{k_n}^{(n)}=\beta_n$ in (1). Thus (3) implies (9).

From the estimate

$$\left|\sum_{i=n+2}^{\infty}\beta_i z^{e_i}\right|\le (h-1)|z|^{e_{n+2}}/(1-|z|), \tag{11}$$

(with strong inequality for $0<|z|<1$ and if $e_{i+1}-e_i>1$ for at least one $i\ge n+2$) we easily deduce

$$\left|\left|f(z,A)-\sum_{i=0}^{n}\beta_i z^{e_i}\right|-\beta_{n+1}|z|^{e_{n+1}}\right|\le (h-1)\frac{|z|^{e_{n+2}}}{1-|z|}, \tag{12}$$

and this implies (10). Note here that strong inequality in (11) leads to strong inequality in (12), giving $|\gamma|<1$. □

Remark 5. If g and h are multiplicatively dependent integers, and if the sequence $A=(a_n)_{n=0}^{\infty}$ is unbounded, then $f\left(\frac{1}{h},A\right)=M_h(g;A)$ is irrational, and this is exactly Niederreiter's Theorem 1 [6]. Namely, since $e_{n+1}-e_n=k_n+1>a_n\frac{s}{r}$ and since A is not bounded, the canonical h-adic expansion of the numbers under consideration has arbitrarily long gaps.

The next lemma contains just a Liouville type estimate, a proof of which can be found, e.g. in Shidlovskii's book [12], p. 32.

Lemma 3. *For distinct algebraic numbers ξ and η the following inequality holds:*

$$|\xi - \eta| > c^{\partial(\xi)} H(\xi)^{-\partial(\eta)}.$$

Here c is a positive real constant depending only on η, and $\partial(\,\cdot\,)$, $H(\,\cdot\,)$ denote the degree and the (usual) height of an algebraic number, respectively.

The following lemma can be found in Cijsouw's dissertation [4], p. 3.

Lemma 4. *The inequality*

$$H(\xi) \leq \left(2\nu(\xi) \max\left(1, \overline{|\xi|}\right)\right)^{\partial(\xi)}$$

holds for any algebraic number ξ. Here $\nu(\,\cdot\,)$ and $\overline{|\cdot|}$ denote the denominator and the house (i.e. the maximum of the absolute values of ξ and of all its conjugates) of an algebraic number.

Finally, for the proof of Theorem 2, we quote Corollary 2 from [3] as

Lemma 5. *Let $(e_n)_{n=0}^{\infty}$ denote a strictly increasing sequence of non-negative integers, and let $(\beta_n)_{n=0}^{\infty}$ denote a sequence of non-zero algebraic numbers. Suppose that the power series $\sum_{n=0}^{\infty} \beta_n z^{e_n}$ has radius of convergence $R > 0$ and defines the function $f(z)$ in $|z| < R$. Put $S_n := [Q(\beta_0, \dots, \beta_n) : Q]$, $B_n := \max\left(1, \overline{|\beta_0|}, \cdots, \overline{|\beta_n|}\right)$, $N_n := \operatorname{lcm}(\nu(\beta_0), \dots \dots, \nu(\beta_n))$ and suppose that*

$$\lim_{n\to\infty} S_n \left(e_n + \log B_n N_n\right) / e_{n+1} = 0 \tag{13}$$

holds. If $\alpha_1, \dots, \alpha_t$ are non-zero algebraic numbers of distinct absolute values less than R, and if ℓ is any non-negative integer, then the numbers $f^{(\lambda)}(\alpha_\tau)$ $(\tau = 1, \dots, t; \lambda = 0, \dots, \ell)$ are algebraically independent.

3. Proof of the theorems

In this section $c_1, c_2, \dots$ always denote positive real constants which are independent of n.

Proof of Theorem 1. From Lemma 2, more precisely from (5), we get

$$\left| f(z; A) - \sum_{i=0}^{n} \beta_i z^{e_i} \right| \leq c_1 |z|^{e_{n+1}} \tag{14}$$

for any $n \geq 0$ and for any complex z with $|z| < 1$. We now assume that there is an algebraic number α with one of the conditions (i), (ii), or (iii), such that $f(\alpha; A)$ is also algebraic. This α will be fixed for the remaining part of the proof.

Next we have to make sure that the left-hand side of inequality (14), evaluated at $z = \alpha$, is non-zero for every sufficiently large n. In case (i) this is trivial, even for all n. In the case (ii) we have for the expression on the right-hand side of (10), again evaluated at $z = \alpha$,

$$|\gamma| \frac{h-1}{1-|\alpha|} |\alpha|^{1+k_{n+1}} \leq |\gamma| h^{-k_{n+1}} < 1 \tag{15}$$

for every $n \geq 0$. Here we are allowed to use $|\gamma| < 1$ in this case, since we have $k_i > 0$ infinitely often, by conditions (4) and (5). Therefore, from (10) again, we see the non-vanishing of the left-hand side of (14) at α for each $n \geq 0$. Finally, in the case (iii) the inequality $|\gamma| \frac{h-1}{1-|\alpha|} |\alpha|^{1+k_{n+1}} < 1$ in (15) is satisfied for any α with $|\alpha| < 1$ if n is large enough, by $1 + k_{n+1} > a_{n+1} \frac{s}{r}$ and hypothesis (6).

Now we are in a position to deduce a contradiction by estimating the left-hand side of (14) at $z = \alpha$ from below, applying Lemma 3 to $\eta := f(\alpha; A)$, $\xi := \sum_{i=0}^{n} \beta_i \alpha^{e_i}$. For this it is clear that we have to bound $H(\xi)$ from above, and to do so we use Lemma 4: Independently of n, all numbers ξ belong to the algebraic number field $Q(\alpha)$, and then we have $\partial(\xi) \leq \partial(\alpha)$. If $\nu := \nu(\alpha)$ is the denominator of α, then $\nu^{e_n}\xi$ is an algebraic integer, and thus we have $\nu(\xi) \leq \nu^{e_n}$. Finally, from the definition of ξ and the house of an algebraic number, we have

$$\overline{|\xi|} \leq \sum_{i=0}^{n} \beta_i \overline{|\alpha|}^{e_i} \leq h \sum_{i=0}^{n} \left(\max\left(2, \overline{|\alpha|}\right)\right)^{e_i} \leq c_2 \exp(c_3 e_n).$$

Combining the last estimates we deduce from Lemma 4

$$H(\xi) \leq \left(2\nu^{e_n} \max\left(1, c_2 e^{c_3 e_n}\right)\right)^{\partial(\alpha)} \leq c_4 \exp\left(c_5 e_n\right).$$

Now the generalized Liouville inequality from Lemma 3 implies

$$\left| f(\alpha; A) - \sum_{i=0}^{n} \beta_i \alpha^{e_i} \right| > c_6 \exp\left(-c_7 e_n\right) \tag{16}$$

for all sufficiently large n. Combining (14) (for $z = \alpha$) and (16), and taking logarithm, we find

$$e_{n+1} \log \frac{1}{|\alpha|} < c_7 e_n + \log \frac{c_1}{c_6}.$$

This implies that the sequence $\left(\frac{e_{n+1}}{e_n}\right)$ is bounded above. By $e_{n+1} - e_n = k_n + 1$, this means that the quotients

$$\frac{k_n + 1}{e_n} > \frac{a_n \frac{s}{r}}{(a_0 + \cdots + a_{n-1}) \frac{s}{r} + n + 1} \geq \frac{a_n}{(a_0 + \cdots + a_{n-1}) + \frac{2r}{s} n} \tag{17}$$

are bounded above. By condition (5), the inequality $a_0 + \cdots + a_{n-1} \geq tn$ holds for all large enough n, where $t > 0$ is an appropriate real constant. We deduce from (17) that $\frac{a_n}{a_0 + \cdots + a_{n-1}}$ is bounded above, and this contradicts condition (4) of our Theorem 1, which is therefore proved. □

Proof of Theorem 2. We apply Lemma 5 with $R = 1$, $S_n = 1$, $B_n = \max(\beta_0, \ldots, \beta_n)$ $(< h)$, $N_n = 1$ for all n such that condition (13) in Lemma 5 is equivalent to $\lim_{n\to\infty} \frac{e_n}{e_{n+1}} = 0$ or to $\lim_{n\to\infty} \frac{k_n+1}{e_n} = \infty$. The truth of this last relation is seen from (17), since condition (7) implies (5), and then inequality (17) says that $\frac{k_n+1}{e_n}$ is bounded below by a constant positive factor times $\frac{a_n}{a_0+\cdots+a_{n-1}}$ for all large n. But this last quotient tends to infinity as $n \to \infty$, by (7), so that Theorem 2 is proved. □

4. Two function-theoretical remarks on $f(z; A)$

Here we suppose condition (8) so that $f(z; A)$ from (3) is of the form (9).

1. From $e_{n+1} - e_n > a_n \frac{s}{r}$ we see that the power series of $f(z; A)$ has arbitrarily long gaps, if condition (6) holds. But then we can conclude from a result of Ostrowski [8] that $f(z; A)$ has at least one singularity on $|z| = 1$ which is not a pole, and thus, this function cannot be rational. Using a slightly earlier theorem of Szegő [14] on power series with only finitely many distinct coefficients, we can even deduce

from (6) that $f(z;A)$ cannot be continued analytically across the unit circle.

2. If both conditions (4) and (5) hold, then it is evident from (17) that the sequence $\left(\frac{k_n+1}{e_n}\right)$ and therefore $\left(\frac{e_{n+1}}{e_n}\right)$ is unbounded. But then we deduce from another result of Ostrowski [7] that the function $f(z;A)$ must be hypertranscendental, i.e., it does not satisfy any algebraic differential equation.

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