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A note on powers of Pisot numbers

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Abstract. Let b be an algebraic number greater than one such that the fractional part of its powers tends to zero or one. We show that in the first case b is an integer and in the second case b is a certain type of Pisot number.

For $x \in \mathbb{R}$ let $\{x\}$ denote the fractional part of x, and let $\|x\|$ denote the distance from x to the nearest integer. A theorem of Koksma states that the sequence $\{b^n\}_{n \geqslant 1}$ is uniformly distributed in the interval $[0;1]$ for almost all real numbers $b > 1$. However Pisot (or Pisot–Vijayaraghavan) numbers represent a nontrivial exeptional set in this theorem. A real algebraic integer greater than 1 is called a Pisot (or PV) number if all its remaining conjugates (if any) lie strictly inside the unit circle. A theorem of Pisot and Vijayaraghavan (see, e.g., [1], [4]) implies that for an algebraic number b greater than one $\|b^n\| \to 0$ $(n \to \infty)$ iff b is a PV number. In estimating the number of lattice points below a logarithmic curve [5] the question arises whether there exists any $b > 1$ with $\{b^n\} \to 0$ as $n \to \infty$. Leaving transcendental numbers out of consideration G. Kuba [5] asked the following:

Question. *Are there any PV numbers $b \notin \mathbb{Z}$ with $\{b^n\} \to 0$ as $n \to \infty$?*

A related question is the following one: find all real numbers $b > 1$ for which $\{b^n\} \to 1$ as $n \to \infty$. If b is algebraic then this condition implies that b is a PV number. We say that b is a *strong PV number* if it is a

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PV number such that one of its remaining conjugates $b_2 > 0$ is strictly greater then any of the absolute values of the remaining ones (if any) $b_2 > \max_{3 \leqslant j \leqslant d} |b_j|$. Clearly, the strong PV numbers of degree $d = 2$ are given by $b = \frac{1}{2}\left(p + \sqrt{p^2 - 4q}\right)$, where p, q and $p^2 - 4q$ are positive integers and $p^2 - 4q$ is not a perfect square. The referee noted that the set of strong PV numbers was also considered by D. Boyd [3].

The following theorem describes the algebraic numbers with the limit of fractional part of powers zero or one. In particular, it shows that the answer to the question in [5] is negative.

Theorem. *If $b > 1$ is an algebraic number such that $\{b^n\} \to 0$ as $n \to \infty$ then $b \in \mathbb{Z}$. If with the same hypotheses $\{b^n\} \to 1$ as $n \to \infty$ then b is a strong PV number.*

Proof. Suppose first that $\{b^n\} \to 0$ as $n \to \infty$. The theorem of Pisot and Vijayaraghavan implies that either $b \in \mathbb{Z}$ or b is a PV number of degree $d \geqslant 2$. We will show that the second case is impossible. Indeed, let $b_2, b_3, \ldots, b_d$ be the conjugates of b. Clearly, the sum

$$b^n + b_2^n + b_3^n + \ldots + b_d^n$$

is an integer if n is a positive integer. Since

$$S(n) = b_2^n + b_3^n + \ldots + b_d^n$$

is a real number and $S(n) \to 0$ as $n \to \infty$, it suffices to show that $S(n)$ is positive for an infinite number of n's.

Let $M = \max_{2 \leqslant j \leqslant d} |b_j|$. A result of C. Smyth [7] shows that if b is a PV number then it has no two conjugates of equal modulus, except for pairs of complex conjugates (see [2], [6] for related results). Without loss of generality we assume that $|b_2| = M$. There are three possibilities: $b_2 = M$, $b_2 = -M$ and $b_2 = Me^{i\theta}$ with $\theta \in (0; \pi)$. In the first two cases we have $S(2n) > 0$ for all sufficiently large n's and the first part of the theorem follows. In the third case let $b_3 = Me^{-i\theta}$. Then

$$b_2^n + b_3^n = 2M^n \cos(n\theta) \geqslant M^n$$

if $\|n\theta/2\pi\| \leqslant 1/6$. By Dirichlet's theorem, for each $x \in \mathbb{R}$ and $\delta > 0$ the inequality $\|nx\| < \delta$ has infinite number of solutions in positive integers n (here $x = \theta/2\pi$, $\delta = 1/6$). Hence, if $m = \max_{4\leqslant j\leqslant d} |b_j|$ for these n we have

$$S(n) \geqslant M^n - (d-3)m^n$$

which is positive for sufficiently large n. This completes the proof of the first part of the theorem.

Suppose now that $\{b^n\} \to 1$ as $n \to \infty$. Then again $S(n) \to 0$ as $n \to \infty$. In the first case ($b_2 = M$) b is a strong PV number. In the second case ($b_2 = -M$) $S(n)$ is negative for all sufficiently large odd n which contradicts the condition $\{b^n\} \to 1$ ($n \to \infty$). In the third case

$$b_2^n + b_3^n = 2M^n \cos(n\theta) \leqslant -M^n$$

if $1/3 \leqslant \|n\theta/2\pi\| \leqslant 2/3$. Clearly, for $x \in \mathbb{R}$, $x \notin \mathbb{Z}$ the inequality $1/3 \leqslant \|nx\| \leqslant 2/3$ has infinite number of solutions in positive integers n (for $x \notin \mathbb{Q}$ this follows from the uniform distribution of $\{nx\}$ and for $x \in \mathbb{Q}$ this is an easy exercise). So that for these n we have

$$S(n) \leqslant -M^n + (d-3)m^n$$

which is negative for sufficiently large n. This completes the proof of the second part of the theorem. □

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