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Quasi-metrizability of the finest quasi-proximity

By SALVADOR ROMAGUERA (Valencia)

Abstract. A characterization of the bispaces whose finest quasi-proximity is quasi-metrizable is obtained in terms of real-valued quasi-proximally continuous functions. We also prove that for a doubly Hausdorff bispace X the following are equivalent: (i) X admits a quasi-metric for which every real-valued bicontinuous function is quasi-uniformly continuous; (ii) the finest quasi-proximity of X is quasi-metrizable; (iii) the finest quasi-uniformity of X is quasi-metrizable. Examples showing that double Hausdorffness of X cannot be omitted in this result are given.

As an application of our methods we deduce that the fine quasi-proximity (resp. quasi-uniformity) of a T_1 topological space X is quasi-metrizable if and only if X admits a quasi-metric for which every lower semicontinuous function is quasi-proximally (resp. quasi-uniformly) continuous. We also deduce that if the finest quasi-proximity of a Hausdorff topological space X is quasi-metrizable, then its fine quasi-uniformity is quasi-metrizable and, thus, X is a metrizable space with only finitely many nonisolated points.

1. Introduction

Throughout this paper the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all positive integer numbers, respectively. If (X,τ) is a topological space and A is a subset of X, then $\tau\,\mathrm{cl}(A)$ and $\tau\,\mathrm{int}(A)$ will denote the closure of A and the interior of A in (X,τ), respectively.

Our basic references for quasi-proximity spaces are [8] and [28], for quasi-uniform and quasi-metric spaces they are [8] and [15] and for bitopological spaces they are [13] and [18].

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Let us recall that a quasi-pseudometric on a (nonempty) set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, x) = 0$, and

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If, in addition, d satisfies:

(iii) $d(x, y) = 0 \Leftrightarrow x = y$,

then, d is called a quasi-metric on X.

A quasi-(pseudo)metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-(pseudo)metric on X.

Each quasi-pseudometric d on X generates a topology $T(d)$ on X, which has as a base the collection $\{S_d(x, r) : x \in X,\ r > 0\}$, where $S_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

If d is a quasi-(pseudo)metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-(pseudo)metric on X, called the conjugate of d. Then, the function $d \vee d^{-1}$ defined on $X \times X$ by $(d \vee d^{-1})(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$, is a (pseudo)metric on X.

Each quasi-pseudometric d on X generates a quasi-uniformity $\mathcal{U}_d$ on X, which has as a base the countable collection $\{U_n : n \in \mathbb{N}\}$, where $U_n = \{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$ for all $n \in \mathbb{N}$ (see [8, p. 3]).

A topological space (X, τ) is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X such that $T(d) = \tau$. In this case, we say that (X, τ) admits d (and d is said to be compatible with τ).

The notion of a bispace (bitopological space in [13]) appears in a natural way when one considers the topologies $T(d)$ and $T(d^{-1})$ generated by a quasi-pseudometric d and its conjugate d^{-1}. A bispace is an ordered triple (X, τ_1, τ_2) such that X is a (nonempty) set and τ_1 and τ_2 are topologies on X. A bispace (X, τ_1, τ_2) is said to be quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X such that $T(d) = \tau_1$ and $T(d^{-1}) = \tau_2$. In this case, we say that (X, τ_1, τ_2) admits d (and d is said to be compatible with (τ_1, τ_2)).

A UC space is a metric space for which every real-valued continuous function is uniformly continuous. UC spaces have been investigated by many authors in different contexts [1], [2], [3], [4], [5], [9], [10], [12], [19], [20], [21], [22], [23], [24], [25], [30], [31], etc. In particular, it is well known that for a metric space (X, d) the following are equivalent: (i) (X, d) is a

UC space; (ii) d is an equinormal metric on X; (iii) the uniformity generated by d is exactly the fine uniformity of (X, d). Perhaps, the most visual characterization of metrizable spaces whose fine uniformity is generated by a metric, is the following result proved by NAGATA [22]: The fine uniformity of a metrizable space is metrizable if and only if the set of the nonisolated points is compact. Later on, SHARMA [30] proved that the finest proximity of a metrizable space is metrizable if and only it admits an equinormal metric, so, it follows that the fine uniformity of a Tychonoff space is metrizable if and only if its finest proximity is metrizable. In [14], KÜNZI proved that the fine quasi-uniformity of a T_1 topological space is quasi-metrizable if and only if it is a quasi-metrizable space containing only finitely many nonisolated points.

These interesting results suggest some questions in a natural way. For instance, characterize the quasi-metric spaces for which every real-valued lower semicontinuous function is quasi-uniformly continuous, investigate the relationship between the bispaces whose finest quasi-proximity is quasi-metrizable and the bispaces whose finest quasi-uniformity is quasi-metrizable, etc. We here obtain characterizations of the bispaces whose finest quasi-proximity is quasi-metrizable both in terms of a bitopological notion of equinormality and in terms of real-valued bicontinuous functions which are quasi-proximally continuous. We observe that, contrarily to the metric case, there exist bispaces whose finest quasi-proximity is quasi-metrizable but their finest quasi-uniformity is not. However, we prove that if (X, τ_1, τ_2) is a quasi-metrizable bispace such that both τ_1 and τ_2 are Hausdorff topologies, then the following are equivalent: (i) The finest quasi-proximity of (X, τ_1, τ_2) is quasi-metrizable; (ii) The finest quasi-uniformity of (X, τ_1, τ_2) is quasi-metrizable; (iii) (X, τ_1, τ_2) admits a quasi-metric for which every real-valued bicontinuous function is quasi-uniformly continuous. We also present an example of a quasi-metrizable bispace which satisfies condition (iii) above but whose finest quasi-uniformity is not quasi-metrizable. As an application of our methods we deduce that a quasi-metric space (X, d) has the property that every real-valued lower semicontinuous function is quasi-proximally (resp. quasi-uniformly) continuous if and only the quasi-proximity (resp. the quasi-uniformity) generated by d is exactly the finest quasi-proximity (resp. the fine quasi-uniformity) of the topological space $(X, T(d))$. We also deduce, Künzi's theorem mentioned above as well as the fact that the fine quasi-uniformity of a Hausdorff topological space is quasi-metrizable if and only if its finest quasi-proximity is quasi-metrizable.

2. Bispaces whose finest quasi-proximity is quasi-metrizable

If δ is a quasi-proximity for a set X we write $A\delta B$ for $(A,B)\in\delta$ and $A^{-}\delta B$ for $(A,B)\notin\delta$.

It is well known [8, p. 12] that if $\mathcal{U}$ is a quasi-uniformity on a set X, the quasi-proximity induced by $\mathcal{U}$ is the quasi-proximity $\delta_{\mathcal{U}}$ defined by

$$A\delta_{\mathcal{U}}B \text{ if and only if for each } U\in\mathcal{U}, \quad (A\times B)\cap U\neq\emptyset.$$

Hence, if d is a quasi-pseudometric on X, we have $A\delta_{\mathcal{U}_d}B$ if and only if $d(A,B)=0$. In this case we write δ_d instead of $\delta_{\mathcal{U}_d}$ and we say that δ_d is the quasi-proximity induced by the quasi-pseudometric d.

A quasi-proximity ρ for a set X is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X such that $\delta_d=\rho$.

It is well known that every topological space (X,τ) admits a finest compatible quasi-proximity $\delta_{\mathcal{FN}}$. Moreover, $A\delta_{\mathcal{FN}}B$ if and only if $A\cap\tau\,\mathrm{cl}(B)\neq\emptyset$. In particular, if (X,τ) is T_1, $T(\delta_{\mathcal{FN}}^{-1})$ is the discrete topology on X.

Now let (X,τ_1,τ_2) be a pairwise completely regular bispace. A quasi-proximity δ for X is called compatible with (τ_1,τ_2) if $T(\delta)=\tau_1$ and $T(\delta^{-1})=\tau_2$. Similarly to the proof of [8, Proposition 1.38] one can show that every pairwise completely regular bispace admits a finest compatible quasi-proximity. If (X,τ_1,τ_2) is a pairwise Hausdorff pairwise normal bispace, the finest compatible quasi-proximity can be easily described.

Proposition 1. *Let (X,τ_1,τ_2) be a pairwise Hausdorff pairwise normal bispace. Then the relation $\delta_{\mathcal{BFN}}$ defined by*

$$A\delta_{\mathcal{BFN}}B \text{ if and only if } \tau_2\,\mathrm{cl}(A)\cap\tau_1\,\mathrm{cl}(B)\neq\emptyset$$

is the finest quasi-proximity of (X,τ_1,τ_2).

Proof. It is proved in [11] that, indeed, $\delta_{\mathcal{BFN}}$ is a quasi-proximity compatible with (τ_1,τ_2). Let ρ be any quasi-proximity for X compatible with (τ_1,τ_2) and let $A\delta_{\mathcal{BFN}}B$. We want to show that then $A\rho B$. Assume the contrary. Then there is $C\subseteq X$ such that $A^{-}\rho C$ and $(X\backslash C)^{-}\rho B$. Hence $C^{-}\rho^{-1}A$, so $C\subseteq\tau_2\,\mathrm{int}(X\backslash A)$. Moreover, $(X\backslash C)\subseteq\tau_1\,\mathrm{int}(X\backslash B)$. Therefore $\tau_2\,\mathrm{cl}(A)\cap\tau_1\,\mathrm{cl}(B)=\emptyset$, a contradiction. We conclude that $A\rho B$. $\square$

Remark 1. It is well known that if (X, τ) is a T_1 topological space, then (X, τ, D) is a pairwise Hausdorff pairwise normal bispace, where D denotes the discrete topology on X. Hence, from Proposition 1 and the comments made above it follows the known fact that if (X, τ) is a T_1 topological space, then the finest quasi-proximity of (X, τ) coincides with the finest quasi-proximity of the bispace (X, τ, D).

Definition 1. A quasi-pseudometric d on a set X is called *pairwise equinormal* if $d(A, B) > 0$ whenever A is a (nonempty) $T(d^{-1})$-closed set and B is a disjoint (nonempty) $T(d)$-closed set.

Theorem 1. *The finest quasi-proximity of a pairwise Hausdorff pairwise completely regular bispace (X, τ_1, τ_2) is quasi-metrizable if and only if it admits a pairwise equinormal quasi-metric.*

Proof. If the finest quasi-proximity of (X, τ_1, τ_2) is quasi-metrizable, there exists a quasi-metric d on X compatible with (τ_1, τ_2) such that $A\delta_d B$ if and only if $\tau_2 \operatorname{cl}(A) \cap \tau_1 \operatorname{cl}(B) \neq \emptyset$, by Proposition 1 (recall that every quasi-metrizable bispace is pairwise normal). Since $A\delta_d B$ if and only if $d(A, B) = 0$, we conclude that $d(A, B) > 0$ whenever A is a (nonempty) τ_2-closed set and B is a disjoint (nonempty) τ_1-closed set. Thus d is pairwise equinormal.

Conversely, the quasi-proximity δ_d induced by the pairwise equinormal quasi-metric d satisfies $A\delta_d B$ if and only if $d(A, B) = 0$. Consequently, $\tau_2 \operatorname{cl}(A) \cap \tau_1 \operatorname{cl}(B) \neq \emptyset$ whenever $A\delta_d B$, by the paiwise equinormality of d. Then, it follows from Proposition 1 that $A\delta_{\mathcal{BFN}} B$ whenever $A\delta_d B$. We conclude that δ_d is exactly the finest quasi-proximity of (X, τ_1, τ_2). □

Remark 2. Actually, the proof of Theorem 1 shows that if d is a quasi-metric on a set X, then d is pairwise equinormal if and only if δ_d coincides with the finest quasi-proximity of the bispace $(X, T(d), T(d^{-1}))$.

In our next theorem we shall characterize the bispaces whose finest quasi-proximity is quasi-metrizable in terms of real-valued bicontinuous functions which are quasi-proximally continuous.

Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bispaces. A function f from X to Y is said to be bicontinuous if f is continuous from (X, τ_i) to (Y, τ_i'), $i = 1, 2$.

Let (X, δ) and (Y, ρ) be two quasi-proximity spaces. A function f from X to Y is called *qp*-continuous [8, 1.48], if $f(A)\rho f(B)$ whenever $A\delta B$.

Denote by ℓ the quasi-pseudometric on $\mathbb{R}$ given by $\ell(x,y) = (x-y) \vee 0$. We say that a real-valued function f defined on a quasi-pseudometric space (X,d) is *quasi-proximally continuous* if it is *qp*-continuous from (X, δ_d) to $(\mathbb{R}, \delta_\ell)$. Thus, a real-valued function f defined on the quasi-pseudometric space (X,d) is quasi-proximally continuous if and only if $\inf\{(f(a)-f(b)) \vee 0 : a \in A,\ b \in B\} = 0$ whenever $d(A,B) = 0$.

Definition 2. A quasi-metric space (X,d) is called a *QP space* if every real-valued lower semicontinuous function (with respect to $T(d)$) is quasi-proximally continuous. A quasi-metrizable topological space (X,τ) is said to be a *QP topological space* if it admits a quasi-metric d for which (X,d) is a QP space.

A quasi-metric space (X,d) is called a *BQP space* if every real-valued bicontinuous function (from $(X, T(d), T(d^{-1}))$ to $(\mathbb{R}, T(\ell), T(\ell^{-1}))$) is quasi-proximally continuous. A quasi-metrizable bispace (X, τ_1, τ_2) is said to be a *BQP bispace* if it admits a quasi-metric d for which (X,d) is a *BQP space*.

Theorem 2. *A quasi-metric space (X,d) is a BQP space if and only if the quasi-proximity δ_d, induced by d, is the finest quasi-proximity of the bispace $(X, T(d), T(d^{-1}))$.*

Proof. Suppose that the quasi-metric space (X,d) is a BQP space. By Remark 2, it suffices to show that d is a pairwise equinormal quasi-metric on X. Let A be a (nonempty) $T(d^{-1})$-closed set and let B be a disjoint (nonempty) $T(d)$-closed set. By [13, Theorem 2.7] there is a bicontinuous function $f : X \to [0,1]$ such that $f(A) = 1$ and $f(B) = 0$. Therefore,

$$\inf\{(f(a) - f(b)) \vee 0 : a \in A,\ b \in B\} = 1.$$

Since (X,d) is a BQP space we deduce that $d(A,B) > 0$. Thus d is pairwise equinormal.

Conversely, let f be a real-valued bicontinuous function from (X,τ_1,τ_2) to $(\mathbb{R}, T(\ell), T(\ell^{-1}))$, where $\tau_1 = T(d)$ and $\tau_2 = T(d^{-1})$. Let A and B be two subsets of X such that $d(A,B) = 0$. Then $d(\tau_2 \operatorname{cl}(A), \tau_1 \operatorname{cl}(B)) = 0$. Since d is pairwise equinormal there is $x \in \tau_2 \operatorname{cl}(A) \cap \tau_1 \operatorname{cl}(B)$. We may assume the following cases:

I. $x \in A \cap B$. Then, obviously, $\inf\{(f(a) - f(b)) \vee 0 : a \in A, b \in B\} = 0$.

II. $x \in (\tau_2 \operatorname{cl}(A)\backslash A) \cap B$. In this case there is a sequence $(a_n)_{n\in\mathbb{N}}$ of distinct points in A such that $d(a_n, x) \to 0$. Since f is upper semicontinuous with respect to τ_2 and $x \in B$, we obtain that $\inf\{(f(a) - f(x)) \vee 0 : a \in A\} = 0$.

III. $x \in A \cap (\tau_1 \operatorname{cl}(B)\backslash B)$. Then, an argument similarly to the given in II, permits us to obtain that $\inf\{(f(x) - f(b)) \vee 0 : b \in B\} = 0$.

IV. $x \in (\tau_2 \operatorname{cl}(A)\backslash A) \cap (\tau_1 \operatorname{cl}(B)\backslash B)$. Then there exist a sequence $(a_n)_{n\in\mathbb{N}}$ of (distinct) points in A and a sequence $(b_n)_{n\in\mathbb{N}}$ of (distinct) points in B such that $d(a_n, x) \to 0$ and $d(x, b_n) \to 0$. Since f is bicontinuous, we immediately deduce that $\inf\{(f(a) - f(b)) \vee 0 : a \in A, b \in B\} = 0$.

We conclude that (X, d) is a BQP space. □

Corollary 1. *The finest quasi-proximity of a pairwise Hausdorff pairwise completely regular bispace is quasi-metrizable if and only if it is a BQP bispace.*

In [14, Lemma 1.1] Künzi proved that a topological space has a σ-interior preserving topology if and only if its finest quasi-proximity is quasi-pseudo-metrizable. Here we obtain the following characterizations of those quasi-metrizable topological spaces whose finest quasi-proximity is quasi-metrizable.

Corollary 2. *For a quasi-metrizable topological space (X, τ) the following statements are equivalent:*

(1) *The finest quasi-proximity of (X, τ) is quasi-metrizable.*

(2) *(X, τ) admits a quasi-metric d such that $d(A, B) > 0$ whenever A is a (nonempty) set and B is a disjoint (nonempty) closed set.*

(3) *(X, τ) is a QP topological space.*

Proof. (1) ⇒ (2): If the finest quasi-proximity of (X, τ) is quasi-metrizable we deduce, from Remark 1, that the finest quasi-proximity of (X, τ, D) is quasi-metrizable, where D denotes the discrete topology on X. By Theorem 1, (X, τ, D) admits a pairwise equinormal quasi-metric d, which, obviously, satisfies the conditions of (2).

(2) ⇒ (3): Suppose that there is a point $x \in X$ which is not $T(d^{-1})$-isolated. Then there is a sequence $(a_n)_{n\in\mathbb{N}}$ of distinct points in X such that $a_n \neq x$ for all $n \in \mathbb{N}$ and $d(a_n, x) \to 0$. Thus, $d(A, B) = 0$, where $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{x\}$, a contradiction. Hence, $T(d^{-1})$ is the discrete topology on X, and, thus, d is pairwise equinormal. By Theorem 1

and Corollary 1, (X, τ, D) is a BQP bispace, so (X, τ) is a QP topological space.

(3) $\Rightarrow$ (1): Let d be a quasi-metric on X compatible with τ for which (X, d) is a QP space. Suppose that there is a point $x \in X$ which is not $T(d^{-1})$-isolated. Then there is a sequence $(a_n)_{n\in\mathbb{N}}$ of distinct points in X such that $a_n \neq x$ for all $n \in \mathbb{N}$ and $d(a_n, x) \to 0$. Consider the function f defined on X by $f(x) = 0$ and $f(y) = 1$ for all $y \in X\backslash\{x\}$. Then f is lower semicontinuous on (X, τ) but clearly it is not quasi-proximally continuous. We conclude that $T(d^{-1})$ is the discrete topology on X, so, (X, τ, D) is a BQP bispace because (X, τ) is a QP topological space. From Corollary 1 and Remark 1 it follows that the finest quasi-proximity of (X, τ) is quasi-metrizable. □

The notion of a pairwise compact bispace was introduced in [7]. It is known that a bispace (X, τ_1, τ_2) is pairwise compact if and only if every proper τ_i-closed set is τ_j-compact, $i, j = 1, 2$; $i \neq j$.

Proposition 2. *Let (X, τ_1, τ_2) be a quasi-metrizable pairwise compact bispace. Then every compatible quasi-metric is pairwise equinormal.*

Proof. Let d be a quasi-metric on X compatible with (τ_1, τ_2). Suppose that there exist a (nonempty) τ_2-closed set A and a disjoint (nonempty) τ_1-closed set B such that $d(A, B) = 0$. Then there exist a sequence $(a_n)_{n\in\mathbb{N}}$ in A and a sequence $(b_n)_{n\in\mathbb{N}}$ in B such that $d(a_n, b_n) \to 0$. Since the bispace is pairwise compact, there exists a subsequence $(a_{k(n)})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ that is τ_1-convergent to a point $a \in A$. Moreover, $(b_{k(n)})_{n\in\mathbb{N}}$ has a τ_2-cluster point $b \in B$. It follows from the triangle inequality that $a = b$, a contradiction. We conclude that d is pairwise equinormal. □

Corollary 3. *The finest quasi-proximity of any quasi-metrizable pairwise compact bispace is quasi-metrizable.*

Example 1. Let $X = \{1/n : n \in \mathbb{N}\}$ and let d be the quasi-metric defined on X by $d(1/n, 1/m) = 1/m$ for $n \neq m$ and $d(x, x) = 0$ for all $x \in X$. Then $T(d)$ is the cofinite topology on X and $T(d^{-1})$ is the discrete topology on X. It is known (and easy to verify) that $(X, T(d), T(d^{-1}))$ is a pairwise compact bispace. Hence, every compatible quasi-metric is pairwise equinormal. So, the finest quasi-proximity of $(X, T(d), T(d^{-1}))$ is quasi-metrizable.

It is interesting to note that, by [16, Proposition 4], $(X, T(d))$ (and, hence, $(X, T(d), T(d^{-1}))$) admits a unique quasi-proximity, because it is hereditarily compact. (See [17] for an example of a non hereditarily compact T_1 topological space admitting a unique quasi-proximity.)

In [6] Brümmer showed that every topological space (X, τ) admits a finest quasi-uniformity: Basic entourages are of the form $\{(x, y) \in X \times X : d(x, y) < r\}$, where d is any quasi-pseudometric on X such that $T(d) \subseteq \tau$ and r is any positive real number. This quasi-uniformity is said to be the fine quasi-uniformity of (X, τ) (see [8]).

The bitopological counterpart of Brümmer's result was obtained by Salbany [29] who proved that every quasi-uniformizable bispace (X, τ_1, τ_2) admits a finest quasi-uniformity: Basic entourages are of the form $\{(x, y) \in X \times X : d(x, y) < r\}$, where d is any quasi-pseudometric on X such that $T(d) \subseteq \tau_1$ and $T(d^{-1}) \subseteq \tau_2$ and r is any positive real number.

In connection with these facts let us recall that a bispace is quasi-uniformizable if and only if it is pairwise completely regular [18, Theorem 4.2].

Since every quasi-uniformity with a countable base generates a quasi-pseudometric (see e.g. [8, Lemma 1.5]), we will say that the fine(st) quasi-uniformity of a (bi)space is quasi-pseudometrizable if it has a countable base.

Remark 3. Let (X, τ) be a T_1 topological space. It immediately follows from Brümmer's result and Salbany's result mentioned above that the fine quasi-uniformity of (X, τ) coincides with the finest quasi-uniformity of the bispace (X, τ, D), where D denotes the discrete topology on X (compare Remark 1).

The finest quasi-uniformity of the bispace $(X, T(d), T(d^{-1}))$ of Example 1 is not quasi-metrizable: Indeed, it follows from Künzi's theorem mentioned in Section 1 that the fine quasi-uniformity of $(X, T(d))$ is not quasi-metrizable. The conclusion now follows from Remark 3.

Therefore, an interesting question appears in a natural way: Obtain conditions under which quasi-metrizability of the finest quasi-proximity of a (bi)space implies quasi-metrizability of the fine(st) quasi-uniformity.

In the next section we shall give a solution to this question via the study of quasi-metric spaces having the property that real-valued bicontinuous functions are quasi-uniformly continuous. (In our context, this property should be considered as the analogue of property UC for metric spaces.)

3. QUC topological spaces and $BQUC$ bispaces

Let us recall [28], [8], that a real-valued function f defined on a quasi-uniform space $(X,\mathcal{U})$ is said to be quasi-uniformly continuous if for each $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $\ell(f(x), f(y)) < \varepsilon$ whenever $(x,y) \in U$. In particular, a real-valued function f defined on a quasi-pseudometric space (X,d) is said to be quasi-uniformly continuous if it is quasi-uniformly continuous for $(X,\mathcal{U}_d)$.

Definition 3. A quasi-metric space (X,d) is called a *QUC* space if every real-valued lower semicontinuous function (with respect to $(X,T(d))$ is quasi-uniformly continuous. A quasi-metrizable topological space (X,τ) is said to be a *QUC* topological space if it admits a quasi-metric d for which (X,d) is a *QUC* space.

A quasi-metric space (X,d) is called a *BQUC* space if every real-valued bicontinuous function (with respect to $(X,T(d),T(d^{-1}))$ is quasi-uniformly continuous. A quasi-metrizable bispace (X,τ_1,τ_2) is said to be a *BQUC bispace* if it admits a quasi-metric d for which (X,d) is a *BQUC* space.

In [29] Salbany showed that the finest quasi-uniformity of any pairwise completely regular bispace has the property that every real-valued bicontinuous function is quasi-uniformly continuous. From this result we immediately deduce the following result.

Proposition 3. *Every pairwise Hausdorff pairwise completely regular bispace whose finest quasi-uniformity is quasi-metrizable is a BQUC bispace.*

Proposition 4. *Let (X,d) be a BQUC space. Then d is a pairwise equinormal quasi-metric.*

Proof. By [8, Proposition 1.51] every real-valued quasi-uniformly continuous function on (X,d) is quasi-proximally continuous from (X,δ_d) to $(\mathbb{R},\delta_\ell)$. Hence (X,d) is a *BQP* space. By Theorem 2 and Remark 2, d is pairwise equinormal. □

In [14, proof of Proposition 1.13], Künzi observed that if the fine quasi-uniformity of a topological space is quasi-pseudometrizable, then its finest quasi-proximity is quasi-pseudometrizable. From Propositions 3 and 4 and Theorem 1 we here obtain the following result.

Corollary 4. *If the finest quasi-uniformity of a pairwise Hausdorff pairwise completely regular bispace is quasi-metrizable, then its finest quasi-proximity is quasi-metrizable.*

Lemma 1 [28, Corollary 3.2.3]. *Let (X, τ_1, τ_2) be a pairwise normal bispace. Let A be a τ_2-closed set, B a τ_1-closed set and $C = A \cap B$. Then every real-valued bounded bicontinuous function f on $(C, \tau_1|C, \tau_2|C)$ has a bicontinuous extension to (X, τ_1, τ_2).*

Proposition 5. *Let (X, τ_1, τ_2) be a BQUC bispace. Then every sequence of non τ_i-isolated points has a τ_j-cluster point, $i, j = 1, 2$; $i \neq j$.*

Proof. Let (X, τ_1, τ_2) be a $BQUC$ bispace and let d be a compatible quasi-metric for which every real-valued bicontinuous function is quasi-uniformly continuous. Suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of (distinct) non τ_1-isolated points without τ_2-cluster point. Then $\{x_n : n \in \mathbb{N}\}$ is a τ_2-closed set. Since each x_n is a non τ_1-isolated point, there exist a subsequence $(a_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a sequence $(b_n)_{n\in\mathbb{N}}$ of distinct points in X, such that

$$\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\} = \emptyset \quad \text{and } d(a_n, b_n) \to 0.$$

Indeed: If the sequence $(x_n)_{n\in\mathbb{N}}$ has infinitely many τ_1-cluster points in $\{x_n : n \in \mathbb{N}\}$, then we may construct two disjoint subsequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that $d(a_n, b_n) < 2^{-n}$ for all $n \in \mathbb{N}$. Otherwise, there is $n_0 \in \mathbb{N}$ such that no point in $\{x_n : n \geq n_0\}$ is a τ_1-cluster point of $(x_n)_{n\in\mathbb{N}}$. Therefore, for each $n \geq n_0$ there exists an r_n, with $0 < r_n < 2^{-n}$, and a $b_n \neq x_n$, such that $d(x_n, b_n) < r_n$ and $x_m \notin S_d(x_n, r_n)$ for all $m \in \mathbb{N}\backslash\{n\}$. (Moreover, it is not a restriction to suppose that $b_n \neq b_m$ whenever $n \neq m$, since $d(x_n, b_n) \to 0$ and $(x_n)_{n\in\mathbb{N}}$ has no τ_2-cluster points.)

Now note that $\{b_n : n \in \mathbb{N}\}$ is also a τ_2-closed set because $(b_n)_{n\in\mathbb{N}}$ has no τ_2-cluster points, and put $A = \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}$.

Define a function $f : A \to \mathbb{R}$, by $f(a_n) = 2n$ and $f(b_n) = 2n-1$, for all $n \in \mathbb{N}$. Since $\tau_2 \mid A$ is the discrete topology, f is τ_2-upper semicontinuous on A. Moreover, f is τ_1-lower semicontinuous on A, since for each $n, m \in \mathbb{N}$ such that $n < m$, we have $f(a_n) < f(b_m)$, $f(a_n) < f(a_m)$, $f(b_n) < f(a_m)$ and $f(b_n) < f(b_m)$. Therefore, the function g defined on A by $g = f/(1+f)$ is also bicontinuous on A, and $1/2 \leq g(x) < 1$ for all $x \in A$. Since

A is τ_2-closed, it follows from Lemma 1 (with $B = X$), that g has a bicontinuous extension to a function $G : X \to [0,1]$. On the other hand (see [18, p. 247–248]), there is a τ_1-upper semicontinuous and τ_2-lower semicontinuous function on X, $h : X \to [0,1]$ such that $h^{-1}(0) = A$. Consider the function $H = G/(1+h)$. Then H is a bicontinuous function on (X, τ_1, τ_2) such that for each $x \in X$, $0 \le H(x) < 1$, and $H(x) = g(x)$ for all $x \in A$.

Finally, let $F = H/(1-H)$. Then, F is also bicontinuous on (X, τ_1, τ_2) and $F(x) = f(x)$ for all $x \in A$. Thus, by the hypothesis, F is quasi-uniformly continuous on (X, d). However, $d(a_n, b_n) \to 0$ and $F(a_n) - F(b_n) = 1$ for all $n \in \mathbb{N}$, a contradiction.

We conclude that every sequence of non τ_1-isolated points has a τ_2-cluster point. A similar argument shows that every sequence of non τ_2-isolated points has a τ_1-cluster point. □

Corollary 5. *Let (X, τ_1, τ_2) be a BQUC bispace. Then the fine uniformity of $(X, \tau_1 \vee \tau_2)$ is metrizable.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of non $\tau_1 \vee \tau_2$-isolated points. From Proposition 5 it follows that there is a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, that converges to a point $x \in X$ with respect to τ_2. Since $(x_{k(n)})_{n\in\mathbb{N}}$ has also a τ_1-cluster point, we deduce that x is a $\tau_1 \vee \tau_2$-cluster point of $(x_{k(n)})_{n\in\mathbb{N}}$. The conclusion follows from Nagata's theorem mentioned in Section 1. □

Corollary 6. *Let (X, τ_1, τ_2) be a quasi-metrizable bispace with only finitely many τ_1-isolated points. If (X, τ_1, τ_2) is a BQUC bispace, then:*

(i) *(X, τ_2) is a compact space and, thus, $\tau_2 \subseteq \tau_1$.*

(ii) *(X, τ_1) is a metrizable space whose fine uniformity is metrizable.*

Proof. By Proposition 5, (X, τ_2) is a compact space and, hence, $\tau_2 \subseteq \tau_1$. The assertion (ii) is now a consequence of Corollary 5. □

Remark 4. Corollary 6 shows that the Niemytzki plane, the Kofner plane and the Sorgenfrey line (see [8]) are examples of quasi-metrizable topological spaces (X, τ) that do not admit any quasi-metric d for which $(X, \tau, T(d^{-1}))$ is a *BQUC* bispace. Hence, they do not admit any quasi-metric d for which the finest quasi-uniformity of $(X, \tau, T(d^{-1}))$ is quasi-metrizable.

Example 2. Let d be the quasi-metric defined on $\mathbb{R}$ by $d(x,y) = \min\{1, y-x\}$ if $x \leq y$, and $d(x,y) = 1$ otherwise. Then $T(d)$ is the Sorgenfrey topology on $\mathbb{R}$. Since $d \vee d^{-1}$ is the discrete metric on $\mathbb{R}$, we deduce, from Remark 4, that the converse of Corollary 5 is not true in general.

Note that Example 1 also shows that such a converse does not hold (see Proposition 5). However, the space $(X, T(d))$ of Example 2 is Hausdorff.

The following is an example of a $BQUC$ bispace whose finest quasi-uniformity is not quasi-metrizable.

Example 3. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences of distinct points such that $\{x_n : n \in \mathbb{N}\} \cap \{y_n : n \in \mathbb{N}\} = \emptyset$. Take a point $a \notin (\{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\})$ and put $X = \{a\} \cup \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. Define a quasi-metric d on X by $d(a, y_n) = 1/n$ for all $n \in \mathbb{N}$, $d(x_n, y_m) = 1/n$ for all $n, m \in \mathbb{N}$, $d(x,x) = 0$ for all $x \in X$, and $d(x,y) = 1$ otherwise.

We first show that the finest quasi-uniformity $\mathcal{BFN}$ of the quasi-metrizable bispace $(X, T(d), T(d^{-1}))$ is not quasi-metrizable. Assume the contrary. Then $\mathcal{BFN}$ has a countable base $\{V_n : n \in \mathbb{N}\}$. By Lemma 2 below, for each $x \in X$ and each $n \in \mathbb{N}$ there is an $n(x) \in \mathbb{N}$ such that $S_{d^{-1}}(x, 1/n(x)) \times S_d(x, 1/n(x)) \subseteq V_n$. Let

$$W = \left[\bigcup_{n\in\mathbb{N}} (\{x_n\} \times \{x_n\})\right] \cup [\{a\} \times S_d(a,1)]$$
$$\cup \left[\bigcup_{n\in\mathbb{N}} (S_{d^{-1}}(y_n, 1/(n(y_n)+1)) \times \{y_n\})\right].$$

By Lemma 2, $W \in \mathcal{BFN}$. However, $(x_{n(y_n)+1}, y_n) \in V_n \backslash W$ for all $n \in \mathbb{N}$, because $d(x_{n(y_n)+1}, y_n) = 1/(n(y_n)+1)$. We conclude that $\mathcal{BFN}$ has no a countable base.

Finally, we prove that (X,d) is a $BQUC$ space. Assume the contrary. Then there is a real-valued bicontinuous function f on X which is not quasi-uniformly continuous. Thus, there exist an $\varepsilon > 0$ and two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in X such that $d(a_n, b_n) < 2^{-n}$ and $f(a_n) - f(b_n) \geq \varepsilon$ whenever $n \in \mathbb{N}$. If there is a subsequence $(a_{k(n)})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $a_{k(n)} = a$ for all $n \in \mathbb{N}$, then $(b_{k(n)})_{n\in\mathbb{N}}$ will be a subsequence of (distinct) points of $(y_n)_{n\in\mathbb{N}}$. Hence, $d(a, b_{k(n)}) \to 0$. Since f is lower semicontinuous with respect to $T(d)$, we obtain a contradiction. Otherwise, we may assume that $(a_n)_{n\in\mathbb{N}}$ is a subsequence of distinct points of $(x_n)_{n\in\mathbb{N}}$. If there is a subsequence $(b_{k(n)})_{k\in\mathbb{N}}$ of $(b_n)_{n\in\mathbb{N}}$ such that for some fixed $j \in \mathbb{N}$, one has $b_{k(n)} = y_j$ whenever $n \in \mathbb{N}$, we obtain a contradiction again, because f is upper semicontinuous with respect to $T(d^{-1})$ and $d(a_{k(n)}, y_j) \to 0$. Thus it only remains to consider the case that $(b_n)_{n\in\mathbb{N}}$ is a subsequence of distinct points of $(y_n)_{n\in\mathbb{N}}$. Then, for b_1, there is $\delta_1 > 0$ such that $f(x) - f(b_1) < \varepsilon/2$ whenever $d(x, b_1) < \delta_1$. Since $d(a_n, b_1) \to 0$, there is $k(1) > 1$ such that $d(a_{k(1)}, b_1) < \delta_1$, so $f(a_{k(1)}) - f(b_1) < \varepsilon/2$. Hence, $(\varepsilon/2) + f(b_{k(1)}) \leq f(a_{k(1)}) - (\varepsilon/2) < f(b_1)$. Taking $b_{k(1)}$ we obtain, similarly, a $k(2) > k(1)$ such that $f(a_{k(2)}) - f(b_{k(1)}) < \varepsilon/2$. Hence, $(\varepsilon/2) + f(b_{k(2)}) < f(b_{k(1)})$. Following this process we can construct a strictly increasing sequence $(k(n))_{n\in\mathbb{N}}$ of natural numbers such that $(\varepsilon/2) + f(b_{k(n+1)}) < f(b_{k(n)})$ for all $k \in \mathbb{N}$. Consequently, $f(b_{k(n)}) \to -\infty$. Since $d(a, b_{k(n)}) \to 0$, we deduce that $f(a) = -\infty$, a contradiction. Hence, f is quasi-uniformly continuous and, thus, (X, d) is a $BQUC$ space.

However, in the topological case we may obtain a satisfactory result, as Theorem 3 below shows. We will use the two following lemmas.

Lemma 2 [26]**.** *The finest quasi-uniformity of a quasi-pseudometrizable bispace (X, τ_1, τ_2) consists of all $\tau_2 \times \tau_1$-neighborhoods of the diagonal in $X \times X$.*

Lemma 3. *Let d be a pairwise equinormal quasi-metric on a set X. If $T(d^{-1})$ is the discrete topology on X, then there exists an $r > 0$ such that $d(x, y) \geq r$ whenever x is a $T(d)$-isolated point and $y \neq x$.*

Proof. Assume the contrary. Then there exist two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of points in X such that each a_n is $T(d)$-isolated, $a_n \neq b_n$, and $d(a_n, b_n) < 2^{-n}$ for all $n \in \mathbb{N}$. Since $T(d^{-1})$ is the discrete topology on X and each a_n is $T(d)$-isolated, we may suppose, without loss of generality, that both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of distinct points. Put

$$A = \{a_n : n \in \mathbb{N}\} \text{ and } B = T(d)\,\mathrm{cl}(\{b_n : n \in \mathbb{N}\}).$$

Since d is pairwise equinormal and $d(A, B) = 0$, we deduce that $A \cap B \neq \emptyset$. Let $x \in A \cap B$. Then x is $T(d)$-isolated, so $x \in \{b_n : n \in \mathbb{N}\}$. If $C = A \cap B$ is a finite set we have that $A_1 = A \backslash C$ is a (nonempty) $T(d^{-1})$-closed set and $B_1 = T(d)\,\mathrm{cl}(B \backslash C)$ is a disjoint (nonempty) $T(d)$-closed set such that $d(A_1, B_1) = 0$, a contradiction. Therefore, we may assume that there exists a subsequence $(a_{k(n)})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $a_{k(m)} \in \{b_n : n \in \mathbb{N}\}$ for all $m \in \mathbb{N}$. Thus we can construct two subsequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $\{x_n : n \in \mathbb{N}\} \cap T(d)\,\mathrm{cl}(\{y_n : n \in \mathbb{N}\}) = \emptyset$ and $d(x_n, y_n) \to 0$, a contradiction.

We conclude that there exists an $r > 0$ such that $d(x, y) \geq r$ whenever x is $T(d)$-isolated and $y \neq x$. □

Theorem 3. *For a quasi-metric space (X, d) the following statements are equivalent:*

(1) *(X, d) is a QUC space.*

(2) *$(X, T(d))$ has only finitely many nonisolated points and there exists an $r > 0$ such that $d(x, y) \geq r$ whenever x is a $T(d)$-isolated point and $y \neq x$.*

(3) *The quasi-uniformity $\mathcal{U}_d$, generated by d, coincides with the fine quasi-uniformity of the topological space $(X, T(d))$.*

Proof. (1) ⇒ (2): We first show that $T(d^{-1})$ is the discrete topology on X: Suppose that there exist a point $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points in X such that $d(x_n, x) \to 0$. Then, the characteristic function for $X\backslash\{x\}$ is lower semicontinuous but not quasi-uniformly continuous. Therefore $T(d^{-1})$ is the discrete topology D on X.

Hence (X, d) is a $BQUC$ space. By Proposition 4, d is pairwise equinormal and, by Proposition 5, every sequence of non $T(d)$-isolated points has a D-cluster point. So $(X, T(d))$ has only finitely many nonisolated points. Furthermore, by Lemma 3, there exists an $r > 0$ such that $d(x, y) \geq r$ whenever x is a $T(d)$-isolated point and $y \neq x$.

(2) ⇒ (3): Denote by X' the set of non $T(d)$-isolated points of X.

If $X' = \emptyset$, $T(d) = D$, and, thus, by Remark 3 and Lemma 2, $\Delta = \{(x, x) : x \in X\}$ is a base for the fine quasi-uniformity of $(X, T(d))$. Theferore, $\{(x, y) \in X \times X : d(x, y) < r\} = \Delta$, and, consequently, $\mathcal{U}_d$ is exactly the fine quasi-uniformity of $(X, T(d))$.

If $X' \neq \emptyset$, let $X' = \{x_1, \ldots, x_j\}$. We first show that $T(d^{-1})$ is the discrete topology on X: Otherwise, there exist an $x \in X$ and a sequence $(y_n)_{n\in\mathbb{N}}$ of distinct points in X such that $d(y_n, x) \to 0$. Thus, there is an $n_0 \in \mathbb{N}$ such that $y_n \neq x$ and $d(y_n, x) < r$ for all $n \geq n_0$. So, for each $n \geq n_0$, $y_n \in X'$. Since X' is a finite set, $y_n = x$ for some $n \geq n_0$, a contradiction.

Now denote by $\mathcal{FN}$ the fine quasi-uniformity of $(X, T(d))$ and let $W \in \mathcal{FN}$. Since $\mathcal{FN}$ coincides with the finest quasi-uniformity of the quasi-metrizable bispace $(X, T(d), D)$ (see Remark 3), it follows from Lemma 2 that for each $x_i \in X'$ there is an $\varepsilon_i > 0$ such that

$$\left(\bigcup_{i=1}^{j}(\{x_i\} \times S_d(x_i, \varepsilon_i))\right) \cup \left(\bigcup_{x \notin X'}(\{x\} \times \{x\})\right) \subseteq W.$$

Put $\varepsilon = \min\{\varepsilon_i : i = 1, \ldots, j\}$ and $\delta = \min\{\varepsilon, r\}$. Then $d(x, y) \geq \delta$ whenever $x \in X \backslash X'$ and $y \neq x$. Hence $\{(x, y) \in X \times X : d(x, y) < \delta\} \subseteq W$, and, consequently, $\mathcal{U}_d$ coincides with the fine quasi-uniformity of $(X, T(d))$.

(3) $\Rightarrow$ (1): This implication is clear, because it is well known that the fine quasi-uniformity of any topological space has the property that every real-valued lower semicontinuous function is quasi-uniformly continuous [8]. □

Corollary 7. *The fine quasi-uniformity of a T_1 topological space is quasi-metrizable if and only if it is a QUC topological space.*

Corollary 8 [14]**.** *The fine quasi-uniformity of a T_1 topological space (X, τ) is quasi-metrizable if and only if (X, τ) is a quasi-metrizable space with only finitely many nonisolated points.*

Proof. We first suppose that the fine quasi-uniformity of (X, τ) is quasi-metrizable. It then follows from Remark 3 that the finest quasi-uniformity of (X, τ, D) is quasi-metrizable. So (X, τ, D) is a $BQUC$ bispace. By Proposition 5, (X, τ) has only finitely many nonisolated points. Conversely, let d be a quasi-metric on X compatible with τ and let X' be the set of the nonislated points. Define for all $x, y \in X$, $e(x, y) = \min\{d(x, y), 1\}$ if $x \in X'$, $e(x, y) = 1$ if $x \in X \backslash X'$ and $x \neq y$, and $e(x, x) = 0$ for all $x \in X$. Since e is compatible with τ, the quasi-metric space (X, e) satisfies the conditions of Theorem 3(2) (with $r = 1$). Therefore, the fine quasi-uniformity of (X, τ) coincides with $\mathcal{U}_e$, so, it is quasi-metrizable. □

Note that the topologies $T(d)$ and $T(d^{-1})$ of the bispace $(X, T(d), T(d^{-1}))$ of Example 3 are not comparable. Moreover, $T(d)$ is a Hausdorff topology but $T(d^{-1})$ is not. These facts are not accidental as our two next theorems show.

Theorem 4. *Let (X, τ_1, τ_2) be a quasi-metrizable bispace such that $\tau_1 \subseteq \tau_2$. Then the following statements are equivalent:*

(1) *The finest quasi-uniformity of (X, τ_1, τ_2) is quasi-metrizable.*

(2) *(X, τ_1, τ_2) is a BQUC bispace.*

(3) *The set of the non τ_1-isolated points is τ_2-compact.*

Proof. (1) $\Rightarrow$ (2): Apply Proposition 3.

(2) $\Rightarrow$ (3): By Proposition 5, every sequence $(x_n)_{n\in\mathbb{N}}$ in X of non τ_1-isolated points has a τ_2-cluster point, which is also a τ_1-cluster point of $(x_n)_{n\in\mathbb{N}}$ because $\tau_1 \subseteq \tau_2$. Since every countably compact quasi-metrizable topological space is compact [8, Corollary 2.29], we conclude that the set of the non τ_1-isolated points is τ_2-compact.

(3) $\Rightarrow$ (1): Denote by X' the set of the non τ_1-isolated points of X. If $X' = \emptyset$, then both τ_1 and τ_2 coincide with the discrete topology on X. By Lemma 2 , $\{\Delta\}$ is a base for the finest quasi-uniformity of (X, τ_1, τ_2).

Hence, we will suppose that $X' \neq \emptyset$. In this case, choose any quasi-metric d on X compatible with (τ_1, τ_2). For each $n \in \mathbb{N}$, define

$$V_n = \{(x, y) \in X \times X : \text{there is } z \in X' \text{ such that } d(x, z) < 2^{-2n} \text{ and } d(z, y) < 2^{-2n}\}$$

and

$$U_n = V_n \cup \{(x, x) \in X \times X : x \notin X'\}.$$

Since for each $n \in \mathbb{N}$, $\Delta \subseteq U_n$ and $U_{n+1}^3 \subseteq U_n$, $\{U_n : n \in \mathbb{N}\}$ is a base for a quasi-uniformity $\mathcal{U}$ on X. Clearly, $T(\mathcal{U}) \subseteq \tau_1$ and $T(\mathcal{U}^{-1}) \subseteq \tau_2$. Moreover, for each $x \in X$, $U_{n+1}(x) \subseteq S_d(x, 2^{-2n})$ and $U_{n+1}^{-1}(x) \subseteq S_{d^{-1}}(x, 2^{-2n})$. Hence, $\mathcal{U}$ is compatible with (τ_1, τ_2). We want to show that $\mathcal{U}$ is exactly the finest quasi-uniformity of (X, τ_1, τ_2). To this end, let V be a $\tau_2 \times \tau_1$-neighborhood of the diagonal in $X \times X$. Then, for each $x \in X$ there is a τ_i-neighborhood $W_i(x)$ of x, $(i = 1, 2)$, such that

$$W = \bigcup\{W_2(x) \times W_1(x) : x \in X\} \subseteq V.$$

Hence, it suffices to show that $U_n \subseteq W$ for some $n \in \mathbb{N}$. Assume the contrary. Then, for each $n \in \mathbb{N}$ there is a pair (a_n, b_n) in $U_n \backslash W$. Thus, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X' such that $d(a_n, x_n) \to 0$ and $d(x_n, b_n) \to 0$. Since X' is τ_2-compact and $\tau_1 \subseteq \tau_2$, we deduce that there are a point $y \in X'$

and a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(d\vee d^{-1})(y,x_{k(n)})\to 0$. So $d(a_{k(n)},y)\to 0$ and $d(y,b_{k(n)})\to 0$. Therefore $(a_{k(n)})_{n\in\mathbb{N}}$ is eventually in $W_2(y)$ and $(b_{k(n)})_{n\in\mathbb{N}}$ is eventually in $W_1(y)$, which contradicts the fact that $(a_n,b_n)\notin W$ for all $n\in\mathbb{N}$. We conclude that the finest quasi-uniformity of (X,τ_1,τ_2) coincides with $\mathcal{U}$, so it is quasi-metrizable. □

A bispace (X,τ_1,τ_2) is called *doubly Hausdorff* if both τ_1 and τ_2 are Hausdorff topologies on X. A quasi-metric space (X,d) is said to be *doubly Hausdorff* if $(X,T(d),T(d^{-1}))$ is a doubly Hausdorff bispace.

Theorem 5. *For a doubly Hausdorff quasi-metric space (X,d) the following statements are equivalent:*

(1) *(X,d) is a BQUC space.*

(2) *The quasi-proximity δ_d, induced by d, is the finest quasi-proximity of the bispace $(X,T(d),T(d^{-1}))$.*

(3) *The quasi-uniformity $\mathcal{U}_d$, generated by d, is the finest quasi-uniformity of the bispace $(X,T(d),T(d^{-1}))$.*

Proof. (1) ⇒ (2): Apply Proposition 4 and Remark 2.

(2) ⇒ (3): We first show that every sequence of non $T(d)$-isolated points has a $T(d^{-1})$-cluster point. Assume the contrary. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct non $T(d)$-isolated points without $T(d^{-1})$-cluster point. Let $F=\{x_n : n\in\mathbb{N}\}$. Then F is $T(d^{-1})$-closed. For each $n\in\mathbb{N}$ put $F_n=F\backslash\{x_n\}$. Note that F_n is $T(d^{-1})$-closed whenever $n\in\mathbb{N}$. Given x_1 there is $r_1<2^{-1}$ $(r_1>0)$ such that $S_{d^{-1}}(x_1,r_1)\cap F_1=\emptyset$. Choose a $y_1\neq x_1$ with $d(x_1,y_1)<r_1$. Put $k(1)=1$. Let $k(2)$ be the first positive integer greater than 1 such that $x_{k(2)}\neq y_1$. Choose $0<r_2<\min\{r_1,2^{-2}\}$ such that $S_{d^{-1}}(x_{k(2)},r_2)\cap(F_{k(2)}\cup\{y_1\})=\emptyset$. Choose a $y_2\neq x_{k(2)}$ with $d(x_{k(2)},y_2)<r_2$. Let $k(3)$ be the first positive integer greater than $k(2)$ such that $x_{k(3)}\notin\{y_1,y_2\}$. Choose $0<r_3<\min\{r_2,2^{-3}\}$ such that $S_{d^{-1}}(x_{k(3)},r_3)\cap(F_{k(3)}\cup\{y_1,y_2\})=\emptyset$. Following this process we can construct a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, a sequence $(y_n)_{n\in\mathbb{N}}$ of points in X, a subsequence $(F_{k(n)})_{n\in\mathbb{N}}$ of $(F_n)_{n\in\mathbb{N}}$ and a strictly decreasing sequence of positive real numbers $(r_n)_{n\in\mathbb{N}}$ such that $r_n<2^{-n}$, $d(x_{k(n)},y_n)<r_n$ and

$$S_{d^{-1}}(x_{k(n)},r_n)\cap(F_{k(n)}\cup\{y_1,\dots,y_{n-1}\})=\emptyset\quad\text{for all } n>1.$$

Therefore, $x_{k(n)} \neq y_m$ for all $n, m \in \mathbb{N}$: Indeed, if $m > n$, from $d(x_{k(m)}, x_{k(n)}) \leq d(x_{k(m)}, y_m) + d(y_m, x_{k(n)})$, it follows that $r_n < r_m + d(y_m, x_{k(n)})$, so $d(y_m, x_{k(n)}) > r_n - r_m > 0$. If $m < n$, $y_m \notin S_{d^{-1}}(x_{k(n)}, r_n)$.

Now put $A = \{x_{k(n)} : n \in \mathbb{N}\}$ and $B = T(d)\,\mathrm{cl}(\{y_n : n \in \mathbb{N}\})$. Note that A is $T(d^{-1})$-closed because $(x_{k(n)})_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. If $A \cap B = \emptyset$, we obtain a contradiction because, by Remark 2, d is pairwise equinormal and $d(x_{k(n)}, y_n) \to 0$. Otherwise, there exist an $x_{k(m)} \in A$ and a subsequence $(y_{j(n)})_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $d(x_{k(m)}, y_{j(n)}) \to 0$. Since $T(d)$ is a Hausdorff topology, $H = \{x_{k(m)}\} \cup \{y_{j(n)} : n \in \mathbb{N}\}$ is a $T(d)$-closed set. Put $G = A \backslash \{x_{k(m)}\}$. Then G is a $T(d^{-1})$-closed set such that $G \cap H = \emptyset$. However, $d(G, H) = 0$ because $d(x_{k(j(n))}, y_{j(n)}) \to 0$, a contradiction. We conclude that every sequence of non $T(d)$-isolated points has a $T(d^{-1})$-cluster point. Similarly, we prove that every sequence of non $T(d^{-1})$-isolated points has a $T(d)$-cluster point.

Now put, for each $n \in \mathbb{N}$, $U_n = \{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$, and suppose that there exist a $T(d^{-1}) \times T(d)$-neighborhood W of the diagonal in $X \times X$ and a sequence $((a_n, b_n))_{n\in\mathbb{N}}$ of points in $X \times X$, such that $(a_n, b_n) \in U_n \backslash W$ for all $n \in \mathbb{N}$. Then, we may assume that both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of distinct points. We consider the two following cases:

I. The sequence $(a_n)_{n\in\mathbb{N}}$ has no $T(d^{-1})$-cluster point. Hence, we may assume, without loss of generality, that each a_n is a $T(d)$-isolated point. Put $A = \{a_n : n \in \mathbb{N}\}$ and $B = T(d)\,\mathrm{cl}(\{b_n : n \in \mathbb{N}\})$. Since $d(A, B) = 0$ and d is pairwise equinormal we deduce that $A \cap B \neq \emptyset$. If $A \cap B$ is a finite set, then, an argument similiar to the one used in the proof of Lemma 3, permits us to reach a contradiction. Otherwise, as in the proof of Lemma 3 again, we can construct two subsequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $\{x_n : n \in \mathbb{N}\} \cap T(d)\,\mathrm{cl}(\{y : n \in \mathbb{N}\}) = \emptyset$ and $d(x_n, y_n) \to 0$, a contradiction.

II. The sequence $(a_n)_{n\in\mathbb{N}}$ has a $T(d^{-1})$-cluster point $a \in X$. Then there is a subsequence $(a_{k(n)})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $d(a_{k(n)}, a) \to 0$. Thus $A = \{a\} \cup \{a_{k(n)} : n \in \mathbb{N}\}$ is $T(d^{-1})$-closed because $T(d^{-1})$ is a Hausdorff topology. Put $B = \{b_{k(n)} : n \in \mathbb{N}\}$. It is not a restriction to suppose that for each $n \in \mathbb{N}$, $b_{k(n)} \neq a$ because $(a_{k(n)}, b_{k(n)}) \notin W$ (and, hence, there is a subsequence of $(b_{k(n)})_{n\in\mathbb{N}}$ consisting of points which are different from a). If the sequence $(b_{k(n)})_{n\in\mathbb{N}}$ has no $T(d)$-cluster point, then $A \cap B \neq \emptyset$ because d is pairwise equinormal and, thus, we may suppose that there is a subsequence $(c_n)_{n\in\mathbb{N}}$ of $(b_{k(n)})_{n\in\mathbb{N}}$ such that each c_n is in A. Therefore, we can construct two subsequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ (where $(y_n)_{n\in\mathbb{N}}$ is also a subsequence of $(c_n)_{n\in\mathbb{N}}$), such that

$$\{x_n : n \in \mathbb{N}\} \cap T(d)\,\mathrm{cl}(\{y_n : n \in \mathbb{N}\}) = \emptyset \quad \text{and} \quad d(x_n, y_n) \to 0,$$

a contradiction. Otherwise, there exists a subsequence $(c_n)_{n\in\mathbb{N}}$ of $(b_{k(n)})_{n\in\mathbb{N}}$, which is $T(d)$-convergent to a point $c \in X$. Then $c \neq a$, since $(a_n, b_n) \notin W$ whenever $n \in \mathbb{N}$. Let $r > 0$ such that $S_d(c,r) \cap S_{d^{-1}}(a,r) = \emptyset$. Choose an $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $a_{k(n)} \in S_{d^{-1}}(a,r)$ and $c_n \in S_d(c,r)$, respectively. Then, $A_0 = \{a\} \cup \{a_{k(n)} : n \geq n_0\}$ is $T(d^{-1})$-closed, $C = \{c\} \cup \{c_n : n \geq n_0\}$ is $T(d)$-closed, $A_0 \cap C = \emptyset$ and $d(A_0, C) = 0$, so we have reached a contradiction. We conclude that $\mathcal{U}_d$ is exactly the finest quasi-uniformity of $(X, T(d), T(d^{-1}))$.

(3) $\Rightarrow$ (1): It follows from Salbany's theorem [29] mentioned above that the finest quasi-uniformity of any pairwise completely regular bispace has the property that every real-valued bicontinuous function is quasi-uniformly continuous. □

Corollary 9. *The finest quasi-uniformity of a doubly Hausdorff pairwise completely regular bispace is quasi-metrizable if and only if its finest quasi-proximity is quasi-metrizable.*

Corollary 10. *Let (X, τ_1, τ_2) be doubly Hausdorff pairwise completely regular bispace whose finest quasi-proximity is quasi-metrizable. Then the fine uniformity of $(X, \tau_1 \vee \tau_2)$ is metrizable.*

Proof. Apply Corollaries 9 and 5. □

Fletcher and Lindgren proved in [8, Proposition 2.34] (see also [19]) that the fine quasi-uniformity of a regular Hausdorff topological space is quasi-metrizable if and only if it is a metrizable space with only finitely many nonisolated points. This result is generalized in the following way.

Corollary 11. *For a Hausdorff topological space (X, τ) the following statements are equivalent:*

(1) *The finest quasi-proximity of (X, τ) is quasi-metrizable.*

(2) *The fine quasi-uniformity of (X, τ) is quasi-metrizable.*

(3) *(X, τ) is a metrizable space with only finitely many nonisolated points.*

Proof. (1) $\Rightarrow$ (2): It is a consequence of Theorem 5, (2) $\Rightarrow$ (3), since the finest quasi-proximity (resp. quasi-uniformity) of (X, τ) coincides with the finest quasi-proximity (resp. quasi-uniformity) of the bispace (X, τ, D) (see Remarks 1 and 3).

(2) $\Rightarrow$ (3): By [14, Proposition 1.12] (see Corollary 8), (X, τ) is a quasi-metrizable space with only finitely many nonisolated points. Since (X, τ) is a Hausdorff space, we immediately deduce that (X, τ) is regular. By [8, Proposition 2.34] mentioned above, (X, τ) is a metrizable space with only finitely many nonisolated points.

(3) $\Rightarrow$ (1): By [8, Proposition 2.34], the fine quasi-uniformity of (X, τ) is quasi-metrizable. Hence, its finest quasi-proximity is also quasi-metrizable [14, Proof of Proposition 1.13]. □

Remark 5. The first part of the proof of (2) $\Rightarrow$ (3) in Theorem 5, shows that if d is a pairwise equinormal quasi-metric on a set X such that $T(d)$ is a Hausdorff topology, then every sequence of non $T(d)$-isolated points has a $T(d^{-1})$-cluster point. Since the topological spaces (X, τ) of Remark 4 are Hausdorff and they do not have isolated points, it follows that they do not admit any compatible quasi-metric d such that the finest quasi-proximity of the bispace $(X, \tau, T(d^{-1}))$ is quasi-metrizable (otherwise $T(d^{-1})$ would be compact, so $T(d^{-1}) \subset \tau$ and thus, (X, τ) would be metrizable, a contradiction).

Subsequently, we present three examples that deal with some natural conjectures that may be considered in the light of the obtained results. Thus, in Example 4 we obtain a doubly Hausdorff quasi-metrizable non $BQUC$ bispace (X, τ_1, τ_2) such that every sequence of non τ_i-isolated points has a τ_j-cluster point, $i, j = 1, 2$; $i \neq j$. In Example 5, we shall give an example of a doubly Hausdorff bispace (X, τ_1, τ_2) whose finest quasi-uniformity is quasi-metrizable but the finest quasi-proximity of (X, τ_1) is not quasi-metrizable. Finally, Example 6 will show that the condition "doubly Hausdorff" cannot be omitted in Corollary 10.

Example 4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of distinct points. For each $n \in \mathbb{N}$ consider a sequence $(y_m^{(n)})_{m\in\mathbb{N}}$ of points such that $y_m^{(n)} \neq y_k^{(j)}$ and $y_m^{(n)} \neq x_k$ for all $n, m, k, j \in \mathbb{N}$. Put $Y = \{x_n : n \in \mathbb{N}\} \cup \{y_m^{(n)} : n, m \in \mathbb{N}\}$. Choose a point $a \notin Y$ and let $X = Y \cup \{a\}$. Now define a quasi-metric d on X as follows: $d(a, x_n) = 1/n$ for all $n \in \mathbb{N}$; $d(y_m^{(n)}, x_n) = 1/m$ for all $n, m \in \mathbb{N}$; $d(x, x) = 0$ for all $x \in X$, and $d(x, y) = 1$ otherwise.

Clearly, $(X, T(d), T(d^{-1}))$ is a doubly Hausdorff bispace. The point a is the unique non $T(d)$-isolated point and every sequence of (distinct) non

$T(d^{-1})$-isolated points is a subsequence of $(x_n)_{n\in\mathbb{N}}$, which converges to a with respect to $T(d)$.

Now we show that the finest quasi-uniformity $\mathcal{BFN}$ of $(X, T(d), T(d^{-1}))$ has no countable base. Indeed, assume the contrary and let $\{V_n : n \in \mathbb{N}\}$ be a base for $\mathcal{BFN}$. Then, for each $x \in X$ and each $n \in \mathbb{N}$ there is an $n(x) \in \mathbb{N}$ such that $S_{d^{-1}}(x, 1/n(x)) \times S_d(x, 1/n(x)) \subseteq V_n$. Let

$$W = \left[\bigcup_{n,m\in\mathbb{N}} (\{y_m^{(n)}\} \times \{y_m^{(n)}\})\right] \cup [\{a\} \times S_d(a,1)]$$
$$\cup \left[\bigcup_{n\in\mathbb{N}} (S_{d^{-1}}(x_n, 1/(n(x_n)+1)) \times \{x_n\})\right].$$

Then, $W \in \mathcal{BFN}$ (compare Example 3). However, $(y^{(n)}_{n(x_n)+1}, x_n) \in V_n \backslash W$ for all $n \in \mathbb{N}$, because $d(y^{(n)}_{n(x_n)+1}, x_n) = 1/(n(x_n)+1)$. Therefore, $\mathcal{BFN}$ is not quasi-metrizable. By Corollary 9, the finest quasi-proximity of $(X, T(d), T(d^{-1}))$ is not quasi-metrizable.

Example 5. Let X be the set of Example 4. Define a quasi-metric d on X as follows:

$$\begin{aligned}
d(a, x_n) &= d(x_n, a) = 1/n \text{ for all } n \in \mathbb{N},\\
d(a, y_m^{(n)}) &= (1/n) + (1/m) \text{ for all } n, m \in \mathbb{N},\\
d(x_n, x_k) &= |(1/n) - (1/k)| \text{ for all } n, k \in \mathbb{N},\\
d(x_n, y_m^{(n)}) &= 1/m \text{ for all } n, m \in \mathbb{N},\\
d(x_n, y_m^{(k)}) &= (1/m) + |(1/n) - (1/k)| \text{ for all } n, m, k \in \mathbb{N} \text{ with } n \neq k,\\
d(x, x) &= 0 \text{ for all } x \in X,\\
d(x, y) &= 2, \text{ otherwise.}
\end{aligned}$$

An easy computation of the different cases shows that, indeed, d is a quasi-metric on X. Note also that $(X, T(d), T(d^{-1}))$ is a doubly Hausdorff bispace such that $T(d) \subset T(d^{-1})$. Moreover, since $\{a\} \cup \{x_n : n \in \mathbb{N}\}$ is the set of non $T(d)$-isolated points and $d(x_n, a) \to 0$, we deduce that the set of the non $T(d)$-isolated points is $T(d^{-1})$-compact. So, by Theorem 4, the finest quasi-uniformity of $(X, T(d), T(d^{-1}))$ is quasi-metrizable. On the other hand, it follows from Corollary 11(3), that the finest quasi-proximity of $(X, T(d))$ is not quasi-metrizable.

Example 6. In [7] it is given an example of a quasi-metrizable pairwise compact bispace (X, τ_1, τ_2) such that τ_1 is not Hausdorff, $\tau_1 \subset \tau_2$ and the fine uniformity of (X, τ_2) is not metrizable. Thus, this example shows that the condition "doubly Hausdorff" cannot be omitted in the statement of Corollary 10.

In the light of the preceding example (see also Examples 1 and 2) it seems interesting to study the problem of characterizing those quasi-metrizable bispaces (X, τ_1, τ_2) for which the fine uniformity of $(X, \tau_1 \vee \tau_2)$ is metrizable. We conclude the paper with a solution to this question. Let us recall [8], [19], that a metric d on a set X is equinormal provided that $d(A, B) > 0$ whenever A and B are disjoint (nonempty) closed sets.

Theorem 6. *Let (X, τ_1, τ_2) be a quasi-metrizable bispace. Then, the fine uniformity of $(X, \tau_1 \vee \tau_2)$ is metrizable if and only if (X, τ_1, τ_2) admits a quasi-metric d such that $d \vee d^{-1}$ is an equinormal metric.*

Proof. *Sufficiency*: Since the equinormal metric $d \vee d^{-1}$ is compatible with $\tau_1 \vee \tau_2$, the fine uniformity of $(X, \tau_1 \vee \tau_2)$ is metrizable (see, for instance, [8, Theorem 2.33]).

Necessity: If the fine uniformity of $(X, \tau_1 \vee \tau_2)$ is metrizable, then it has a compatible equinormal metric p [8, Theorem 2.33]. Let q be a quasi-metric on X compatible with (τ_1, τ_2). For each $x \in X$ there is a sequence $(r_n(x))_{n \in \mathbb{N}}$ of positive real numbers with $5r_{n+1}(x) < r_n(x) < 2^{-n}$ and

$$S_q(x, r_n(x)) \cap S_{q^{-1}}(x, r_n(x)) \subseteq S_p(x, 2^{-n}) \quad \text{for all } n \in \mathbb{N}.$$

Put

$$V_n = \bigcup \{S_{q^{-1}}(x, r_n(x)/3) \times S_q(x, r_n(x)/3) : x \in X\}$$

for all $n \in \mathbb{N}$. Similarly to the proof of [27, Theorem 2.1], there exists a quasi-metric d on X compatible with (τ_1, τ_2) such that

$$V_{n+1} \subseteq \{(x, y) \in X \times X : d(x, y) < 2^{-n}\} \subseteq V_n$$

for all $n \in \mathbb{N}$. Finally, let A and B bet two disjoint (nonempty) closed sets in $(X, \tau_1 \vee \tau_2)$ such that $(d \vee d^{-1})(A, B) = 0$. Then, there exist a sequence $(a_n)_{n \in \mathbb{N}}$ in A and a sequence $(b_n)_{n \in \mathbb{N}}$ in B such that $(d \vee d^{-1})(a_n, b_n) < 2^{-n}$

for all $n \in \mathbb{N}$. Thus, there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X such that

$$q(a_n, x_n) < r_n(x_n)/3, \qquad q(x_n, b_n) < r_n(x_n)/3,$$
$$q(b_n, y_n) < r_n(y_n)/3 \quad \text{and} \quad q(y_n, a_n) < r_n(y_n)/3$$

for all $n \in \mathbb{N}$. Assume, without loss of generality, that $r_n(y_n) \le r_n(x_n)$ for all $n \in \mathbb{N}$. Then, $q(x_n, a_n) < r_n(x_n)$, $q(x_n, b_n) < r_n(x_n)$, $q(a_n, x_n) < r_n(x_n)$ and $q(b_n, x_n) < r_n(x_n)$. Hence, $p(a_n, b_n) < 2^{-(n-1)}$ for all $n \in \mathbb{N}$, which contradicts the fact that p is equinormal. We conclude that $d \vee d^{-1}$ is equinormal. □

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SALVADOR ROMAGUERA
ESCUELA DE CAMINOS
DEPARTAMENTO DE MATEMÁTICA APLICADA
UNIVERSIDAD POLITÉCNICA DE VALENCIA
APARTADO 22012
46071 VALENCIA
SPAIN

E-mail: sromague@mat.upv.es

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