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Local peaks of additive functions

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Abstract. It is proved that if a completely additive arithmetical function $u(n)$ satisfies

$$u(n) \le \max(u(n+1), \dots, u(n+k)) + l(n),$$

with a monotonically decreasing function $0 < l(n)$ such that $l(2)+l(2^2)+l(2^3)+\dots < \infty$, then $u(n) = c\log n + v(n)$, where $v(n)$ is of finite support.

1. Introduction

Let $\mathcal{A}^*$ be the class of completely additive real valued functions.

Let $t \ge 1$, $i_1, i_2, \dots, i_t$ be an arbitrary permutation of the integers $1, 2, \dots, t$. We think that for all f, with the exception of some very special ones,

$$\frac{1}{x}\#\{n \le x \mid f(n+i_1) \le f(n+i_2) \le \dots \le f(n+i_t)\} \tag{1.1}$$

has a positive limit as $x \to \infty$. Since the log function is monotonic, it is exceptional.

Another type of exceptional function $f \in \mathcal{A}^*$ can be constructed by choosing $f(2) > 0$ and $f(p) = 0$ for every odd prime. Then $f(n+1) \le f(n+2) \le f(n+3)$ has no solutions.

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Conjecture. *Assume that for some $f \in \mathcal{A}^*$ there exists an integer $t \geq 1$ and a permutation $i_1, i_2, \dots, i_t$ of the integers $1, 2, \dots, t$ such that (1.1) tends to zero as $x \to \infty$. Then $f(n) = c \log n + u(n)$ with some constant c, where $u \in \mathcal{A}^*$ is of finite support.*

We are far from being able to prove this conjecture.

2. Formulation of the theorems

Let $\mathcal{F}$ be the class of those monotonically decreasing functions $l : \mathbb{N} \to [0, \infty)$ for which

$$\sum_{j=1}^{\infty} l(2^j) < \infty$$

holds. Let $\mathcal{P}$ be the set of primes.

We shall characterize those $u \in \mathcal{A}^*$ for which with a suitable $l \in \mathcal{F}$

$$u(n) \leq \max(u(n+1), \dots, u(n+k)) + l(n) \quad n \in \mathbb{N} \tag{2.1}$$

holds. Here $k \geq 1$ is an arbitrary fixed integer.

Theorem 1. *If (2.1) holds, then there exists a constant c, and $v \in \mathcal{A}^*$, such that $u(n) = c \log n + v(n)$, where $v(p) = 0$ for all but finitely many primes p. If $\mathcal{R} = \{q_1, \dots, q_r\}$ (it might be empty) is the set of the exceptional primes on which v does not equal to zero, then $v(q_j) < 0$ $(j = 1, \dots, r)$ and for every $n \in \mathbb{N}$ there exists a $j \in \{1, \dots, k\}$, for which $n + j$ is coprime to each q_l $(l = 1, \dots, r)$.*

Conversely, let $\mathcal{R} = \{q_1, \dots, q_r\}$ be such a collection of primes for which for every $n \in \mathbb{N}$ there exists at least one $j \in \{1, \dots, k\}$ such that $n + j$ is coprime to all members of $\mathcal{R}$. Let $\bar{v} \in \mathcal{A}^*$ be defined on primes as follows: $\bar{v}(q_j) = \gamma_j \leq 0$ $(j = 1, \dots, r)$, γ_j are arbitrary, $\bar{v}(p) = 0$ if $p \in \mathcal{P} \setminus \mathcal{R}$. Then $\bar{v}(n) \leq \max\{\bar{v}(n+j), j = 1, \dots, k\}$, furthermore $u(n) = c \log n + \bar{v}(n)$ satisfies (2.1), for each $c \in \mathbb{R}$, with a suitable $l \in \mathcal{F}$.

Theorem 2. *Assume that for some $u \in \mathcal{A}^*$ and $l \in \mathcal{F}$ the relation*

$$u(n) \geq \min(u(n-1), u(n+1)) - l(n) \tag{2.2}$$

holds. Then

$$u(n) = c \log n + v(n), \tag{2.3}$$

where c is a suitable constant, and either $v(n) = 0$ identically, or there is an odd prime q for which $v(q) > 0$, and $v(n) = 0$ if $(n, q) = 1$.

Conversely, all such u satisfies (2.2) *with some $l \in \mathcal{F}$.*

3. Proof of Theorem 1

The second assertion is clear, we prove the first one.

A finite set of distinct primes $\{q_1, \dots, q_r\}$ is said to be of type $\mathcal{T}$ if there exist k consecutive integers $m+1, \dots, m+k$ none of which is coprime to $q_1, q_2, \dots, q_r$.

Let

$$\delta_p := \frac{u(p)}{\log p} \quad (p \in \mathcal{P}).$$

Lemma 1. *Assume that* (2.1) *is satisfied. Let $\{q_1, \dots, q_r\} \in \mathcal{T}$. Then*

$$\delta_p \le \max\{\delta_{q_1}, \dots, \delta_{q_r}\} \quad (p \in \mathcal{P}).$$

First we observe that the theorem easily follows from Lemma 1. Indeed, it is clear that a set of k distinct primes belongs to $\mathcal{T}$, since $m + j \equiv 0 \pmod{q_j}$ $(j = 1, \dots, k)$. Thus, Lemma 1 implies that the set $\{\delta_p \mid p \in \mathcal{P}\}$ does not contain more than k values. Let ξ be the largest value of δ_p. From Lemma 1, $\{q \mid \delta_q < \xi\} \notin \mathcal{T}$, and we are ready.

Proof of Lemma 1. Let

$$\{q_1, \dots, q_r\} \in \mathcal{T},\ c := \max_{j=1,\dots,r} \delta_{q_j}, \quad u^*(n) := u(n) - c \log n.$$

It is enough to prove that $u^*(n) \le 0$ for $n \in \mathbb{N}$. Let us observe that (2.1) holds for $u^*(n)$ with some other $l \in \mathcal{T}$.

Let $K = q_1, \dots, q_r$, $q_1 < \dots < q_r$, $C := \frac{k+K}{q_1-1}$.

We have

$$\max_{j=1,\dots,r} u^*(q_j) = 0.$$

Assume that $u^*(n_0) > 0$ for some n_0. Let

$$L(x) = \max_{n \le x} u^*(n).$$

Then $L(x) \to \infty$.

Iterating the inequality (2.1), we obtain that
$u^*(n) \le \max(u^*(n+j+1),\dots,u^*(n+j+k)) + l^{(j)}(n)$ holds for every $j \in \{0,1,\dots,K\}$ with some $l^{(j)} \in \mathcal{F}$, consequently

$$u^*(n) \le \min_{j=0,\dots,K} \max(u^*(n+j+1),\dots,u^*(n+j+k)) + l_K(n), \quad l_K \in \mathcal{F}. \tag{3.1}$$

Let $x > C$, $L(x) = u^*(N_0)$.

Since $\{q_1,\dots,q_r\} \in \mathcal{T}$, therefore there exists such a j in $[1,K]$ for which $(N_0+j+l,K) > 1$ $(l=1,\dots,k)$. Let l^* be such a value for which

$$u^*(N_0+j+l^*) = \max_{l=1,\dots,k} u^*(N_0+j+l).$$

Let $N_0+j+l^* = q_s N_1$.

Then $u^*(N_0+j+l^*) = u^*(q_s) + u^*(N_1) \le u^*(N_1)$, and

$$N_1 \le \frac{N_0+K+k}{q_1}.$$

We can repeat this procedure by N_1 instead of N_0, and so on:

$$u^*(N_j) \le u^*(N_{j+1}) + l_K(N_j) \quad (j=0,\dots,t-1),$$

where t is the smallest index for which $N_t \le C$. Since $N_{j+1} \le \frac{N_j+K+k}{q_1-1}$ and $N_1, N_2, \dots, N_{t-1}$ is strictly decreasing, we obtain that t is finite, and

$$u^*(N_0) \le \max_{n \le C} u^*(n) + \sum_{j=0}^{t-1} l_K(N_j).$$

The sum on the right hand side is bounded, since $l_K \in \mathcal{F}$. Consequently $L(x)$ is bounded, so $u^*(n_0) > 0$ is not true.

The proof is complete. $\square$

4. Proof of Theorem 2

The second assertion is obvious, we prove the first one.

If u is a solution of (2.2), then so is $u(n) - c\log n$ as well, thus we may assume that $u(2) = 0$.

First we show that $u(n) \geq 0$ for every $n \in \mathbb{N}$. Assume that $u(n_0) < 0$. Let

$$L(x) = \min_{n \leq x} u(n).$$

Then $L(x) \to -\infty$.

Let x be large, $u(N_0) = L(x)$, $N_0 \leq x$. We may assume that N_0 is odd. Then

$$u(N_0) \geq \min\left(u\left(\frac{N_0-1}{2}\right),\ u\left(\frac{N_0+1}{2}\right)\right) - l(N_0).$$

Thus there is an odd integer $N_1 \leq \frac{N_0+1}{2}$ for which $u(N_1) \leq u(N_0) + l(N_0)$. Repeating this procedure, we get: $u(N_{j+1}) \leq u(N_j) + l(N_j)$ $(j = 1, \dots, t-1)$, $1 = N_t < N_{t-1} < \cdots < N_0$. Since $N_{j+1} \leq \frac{N_j+1}{2}$, therefore

$$\sum_{j=0}^{t} l(N_j)$$

is bounded, $0 = u(N_t) \leq u(N_0) + l(N_0) + \cdots + l(N_{t-1})$, thus $u(N_0) \geq -c$ with some positive c.

Thus $u(2) = 0$ and $u(n) \geq 0$ for every $n \in \mathbb{N}$.

Assume that there exist two primes q_1, q_2 for which $u(q_1) > 0$, $u(q_2) > 0$. Let

$$\Delta = \min(u(q_1),\ u(q_2)).$$

Let $l \pmod{q_1 q_2}$ be determined by $l \equiv -1 \pmod{q_1}$, $l \equiv 1 \pmod{q_2}$. Then, there is a constant c_1 such that for every $n > c_1$, $n \equiv l \pmod{q_1 q_2}$, $u(n) \geq \frac{\Delta}{2}$. Let $\mathcal{I}$ be the set of primes $p \equiv l \pmod{q_1 q_2}$ larger than c_1. Thus, $u(n) \geq \frac{\Delta}{2}$ if n has at least one prime divisor from $\mathcal{I}$.

Let $\pi \in \mathcal{P}$, $\pi \equiv 5 \pmod 8$. Then the Legendre symbol $\left(\frac{2}{\pi}\right) = -1$, thus $2^{\frac{\pi-1}{2}} \equiv -1 \pmod{\pi}$, and so $2^{\alpha_t} \equiv -1 \pmod{\pi}$, where $\alpha_t = \frac{\pi-1}{2} + t(\pi-1) = \frac{\pi-1}{2}(1+2t)$.

Let $p \equiv 3 \pmod 8$, $p \in \mathcal{P}$. Let t be such an integer for which $1+2t \equiv 0 \pmod{\frac{p-1}{2}}$.

Let $1+2t = \frac{p-1}{2}s$. For such a t, $p-1 \mid 2^{\alpha_t} - 1$, and so

$$0 = u(2^{\alpha_t}) \geq \min(u(p),\ u(\pi)) - l(2^{\alpha_t}).$$

Since α_t can be arbitrary large, $l(2^{\alpha_t}) \to 0$ $(\alpha_t \to \infty)$, we get that $\min(u(p),\ u(\pi)) = 0$.

The relative density of those primes $\pi \equiv 5 \pmod 8$, for which either $(\pi+1,\ \mathcal{I})=1$, or $(\pi-1,\ \mathcal{I})=1$ is zero. Similary, the relative density of the primes $p \equiv 3 \pmod 8$, for which either $(p+1,\ \mathcal{I})=1$, or $(p-1,\ \mathcal{I})=1$ is zero. Consequently, there exists at least one couple p, π for which $u(\pi)$, $u(p) \geq \frac{\Delta}{2}$. This is a contradiction, $\Delta > 0$ cannot hold.

The proof is complete.

5. Further theorems

If $u(n)$ satisfies (2.2), then for $v(n) = -u(n)$, we have

$$v(n) \leq \max(v(n-1),\ v(n+1)) + l(n). \tag{5.1}$$

Consequently, from Theorem 2, we have

Theorem 2'. *If* (5.1) *holds with some $l \in \mathcal{F}$, then $v(n) = c\log n - h(n)$, with some constant c, and either $h(n) = 0$ identically, or there is an odd prime q for which $h(q) > 0$, and $h(n) = 0$ for every n coprime to q.*

As a direct consequence, we have

Theorem 3. *Let f be a completely multiplicative function taking on positive values, such that*

$$2f(n) \leq f(n+1) + f(n-1) \tag{5.2}$$

holds for every large n.

Then $f(n) = n^s$ and either $s \leq 0$ or $s \geq 1$.

Proof of Theorem 3. Let $v(n) := \log f(n)$. Then, (5.2) implies that $v(n) \leq \max(v(n+1),\ v(n-1))$ for every large n, consequently the conditions of Theorem 2' are satisfied. Thus $v(n) = s\log n - h(n)$, consequently $f(n) = n^s G(n)$, where either $G(n) = 1$ identically, or there exists an odd $q \in \mathcal{P}$ for which $G(q) = e^{-h(q)} < 1$, and $G(n) = 1$, if $(n,q) = 1$. Substituting into (5.2), we get

$$2G(n) \leq \left(1+\frac{1}{n}\right)^s G(n+1) + \left(1-\frac{1}{n}\right)^s G(n-1).$$

Let $n \to \infty$ over the set of the integers for which $q \| n+1$. Then $(n,q) = (n-1,q) = 1$, and so $2 \leq G(q) + 1$.

Thus $h(n) = 0$ identically, i.e. $f(n) = n^s$.

Finally we observe that $2n^s \leq (n+1)^s + (n-1)^s$ holds for every large n, if and only if $s \leq 0$ or $s \geq 1$. $\square$

From Theorem 2 we can deduce similarly

Theorem 4. *Let f be a positive real valued completely multiplicative function such that for every large n*

$$2f(n) \geq f(n+1) + f(n-1)$$

holds. Then $f(n) = n^s$, $0 \leq s \leq 1$.

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