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Characterization of field homomorphisms by functional equations

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Abstract. Let K and $\overline{K}$ be fields with a common subfield F, and let $n \in \mathbb{Z}$, $|n| \geq 2$, $A \in K^\times$, $B \in \overline{K}^\times$. We study, under mild restrictions on the cardinality and the characteristic of F, F-linear solutions $f : K \to \overline{K}$ of the functional equation

$$f(Ax^n) = Bf(x)^n \quad \text{for all } x \in K \setminus \mathrm{Ker}(f),$$

and show that either $f = 0$ or $e^{-1}f$ is a field monomorphism where $e = f(1)$.

Let K be a field of characteristic different from 2 and $f : K \to \overline{K}$ be an additive function. It is well known that f is a field homomorphism if

$$f(x^2) = f(x)^2 \quad \text{for all } x \in K.$$

The proof follows easily by calculating $f((x+y)^2)$ in two different ways. (For this and similar algebraic manipulations of functional equations related to additive functions see Kuczma [1, Ch. XIV].)

Only recently, we proved in [2] that $\pm f$ is already a field homomorphism, if

$$f\left(\frac{1}{x}\right) = \frac{1}{f(x)} \quad \text{for all } x \in K \setminus \mathrm{Ker}(f).$$

In this note, we investigate additive functions satisfying the more general functional equation (1) below. The emphasis lies here in admitting the constants A and B, and again, as in [2], in making the weaker assumption "for all $x \in K \setminus \mathrm{Ker}(f)$" instead of "for all $x \in K^\times$".

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Theorem. *Suppose that $n \in \mathbb{Z}$, $|n| \geq 2$, and*

$$n^* = \begin{cases} n, & \text{if } n > 0, \\ n^2, & \text{if } n < 0. \end{cases}$$

Let F be a field such that $\#F > n^*$ and $\mathrm{char}(F) \nmid n(n^*-1)$. Let $F \subset K$, $F \subset \overline{K}$ be extension fields, $f : K \to \overline{K}$ an F-linear map, $A \in K^\times$, $B \in \overline{K}^\times$ and

$$f(Ax^n) = Bf(x)^n \quad \text{for all } x \in K \setminus \mathrm{Ker}(f). \tag{1}$$

Then either $f = 0$, or $e = f(1) \neq 0$, and $e^{-1}f : K \to \overline{K}$ is a field monomorphism.

Proof. We set $V = \mathrm{Ker}(f)$ and $e = f(1)$. If $e \neq 0$, then (1) implies $f(A) = Be^n$, and thus $A \notin V$.

Case 1: $n > 0$. If $a, b \in K$ and $\lambda \in F$ are such that $a + \lambda b \notin V$ or $a + \lambda b = 0$, then (1) implies

$$f\left(A(a+\lambda b)^n\right) = Bf(a+\lambda b)^n.$$

Expanding both sides according to powers of λ and observing that f is F-linear, we obtain

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \lambda^\nu \left[f(Aa^{n-\nu}b^\nu) - Bf(a)^{n-\nu}f(b)^\nu\right] = 0. \tag{2}$$

Suppose first that $V = \{0\}$. If $a, b \in K$, then (2) holds for all $\lambda \in F$ and hence identically in λ, since $\#F > n$. In particular (for $\nu = 1$), we have

$$n[f(Aa^{n-1}b) - Bf(a)^{n-1}f(b)] = 0,$$

and since $\mathrm{char}(K) \nmid n$,

$$f(Aa^{n-1}b) = Bf(a)^{n-1}f(b) \quad \text{for all } a, b \in K. \tag{3}$$

In (3) we replace a by $a + \lambda$ (where $\lambda \in F$), expand according to powers of λ and obtain

$$\sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \lambda^\nu f(Aa^{n-1-\nu}b) = B\sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \lambda^\nu e^\nu f(a)^{n-1-\nu}f(b). \tag{4}$$

Again, (4) holds identically in λ, and a comparison of the coefficients of λ^{n-2} implies (since $\mathrm{char}(F) \nmid n-1$)

$$f(Aab) = Be^{n-2} f(a) f(b) \quad \text{for all } a, b \in K. \tag{5}$$

If $a = A^{-1}x$, where $x \in K$, and $b = 1$, (5) implies $f(x) = Be^{n-1} f(A^{-1}x)$. Therefore we obtain, for all $x, y \in K$,

$$\begin{aligned}(e^{-1}f)(xy) &= e^{-1} f\big(A(A^{-1}x)y\big) = Be^{n-3} f(A^{-1}x) f(y) \\ &= e^{-2} f(x) f(y) = (e^{-1}f)(x)(e^{-1}f)(y),\end{aligned}$$

which proves our assertion.

It remains to show that either $V = \{0\}$ or $V = K$. We distinguish two cases.

Case 1A: $e = 0$, $1 \in V$. If $x \in K \setminus V$, then we have $x + \lambda \in K \setminus V$ for all $\lambda \in F$, and (2) holds for $a = x$, $b = 1$ and all $\lambda \in F$. Hence (2) holds identically in λ, and we obtain (for $\nu = n-1$)

$$nf(Ax) = 0 \quad \text{and thus } x \in A^{-1}V.$$

We have proved that $K \setminus V \subset A^{-1}V$, and consequently $K \subset V \cup A^{-1}V$. Since V and $A^{-1}V$ are vector spaces over F, it follows that $V = K$.

Case 1B: $e \neq 0$, $1 \notin V$. If $x \in V$, then we have $x + \lambda \in K \setminus V$ for all $\lambda \in F^{\times}$, and (2) holds for $a = x$, $b = 1$ and all $\lambda \in F^{\times}$. Since $f(x) = 0$ and $f(A) = Be^n$, we obtain

$$\sum_{\nu=0}^{n-1} \binom{n}{\nu} \lambda^{\nu} f\left(Ax^{n-\nu}\right) = 0 \quad \text{for all } \lambda \in F^{\times}. \tag{6}$$

Since $\# F^{\times} > n-1$, (6) holds identically in λ. We focus on the coefficients of λ^1, λ^{n-2} and λ^{n-1} and observe that $\mathrm{char}(F) \nmid n(n-1)$ in order to obtain

$$\{Ax, Ax^2, Ax^{n-1}\} \subset V \tag{7}$$

and thus (by induction) also

$$A^m x \in V \quad \text{for all } m \geq 1. \tag{8}$$

We assume now that there exists an element $x \in V \setminus \{0\}$ and distinguish two cases.

Case 1Ba: $x^{-1} \in V$. Since x, x^{-1}, $(x+x^{-1}) \in V$, (7) implies Ax^2, Ax^{-2}, $A(x+x^{-1})^2 \in V$ and hence

$$A = \frac{1}{2}\left[A(x+x^{-1})^2 - Ax^2 - Ax^{-2}\right] \in V,$$

a contradiction.

Case 1Bb: $x^{-1} \notin V$. By (7), we have $Ax^{n-1} \in V$ and hence $x^{-1} + \lambda Ax^{n-1} \notin V$ for all $\lambda \in F$. We apply (2) for $a = x^{-1}$ and $b = Ax^{n-1}$ and obtain

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \lambda^\nu f\left(A^{1+\nu} x^{n(\nu-1)}\right) = 0 \quad \text{for all } \lambda \in F. \tag{9}$$

Again, (9) holds identically in λ, and we obtain (for $\nu = 1$) $nf(A^2) = 0$ and hence $A^2 \in V$. By (8), we deduce that $A^m \in V$ for all $m \geq 2$, and in particular $A^{n+1} \in V$. Since $A \notin V$, (1) implies however that $f(A^{n+1}) = f(AA^n) = Bf(A) \neq 0$, a contradiciton.

Case 2: $n < 0$. If $x \notin V$, then $f(Ax^n) = Bf(x)^n \neq 0$, hence $Ax^n \notin V$ and

$$f\left(A^{n+1}x^{n^2}\right) = f\left(A(Ax^n)^n\right) = Bf\left(Ax^n\right)^n = B^2 f(x)^{n^2}.$$

Hence it follows by Case 1 that either $f = 0$ or $e^{-1}f$ is a field monomorphism. □

Remark. The functional equation

$$f\left(\frac{1}{x^2}\right) = \frac{1}{f(x^2)} \quad \text{for all } x \in K \setminus \{0\}$$

(for an injective additive function $f : K \to \overline{K}$) can also be treated by applying f to the identity

$$\frac{1}{(2x)^2(1-x^2)^2} - \frac{1}{(2x)^2(1+x^2)^2} = \frac{1}{(1-x^4)^2}.$$

In a forthcoming paper of the first author more general such identities will be investigated to study further functional equations for additive functions.

References

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