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Linear recursive sequences and power series

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Introduction

Let R_n $(n = 0, 1, \dots)$, be a sequence of complex numbers defined by a k^{th} order linear recurrence

$$R_n = A_1 R_{n-1} + A_2 R_{n-2} + \dots + A_k R_{n-k}, \qquad (n \ge k),$$

where k is a nonnegative integer, $A_1, \dots, A_k$ are fixed parameters, $A_k \ne 0$ and the initial values $R_0, \dots, R_{k-1}$ are complex numbers not all zero. The polynomial

$$c(x) = x^k - A_1 x^{k-1} - A_2 x^{k-2} - \dots - A_k$$

is called the characteristic polynomial of the sequence R_n. If $R(x)$ is a function whose formal power series expansion is of the form

$$R(x) = R_0 + R_1 x + R_2 x^2 + \dots = \sum_{n=1}^{\infty} R_n x^n,$$

then $R(x)$ is called the *generating function* of the sequence R_n $(n = 0, 1, \dots)$.

For many recurrences generating functions are known. The best known example is the generating function of the Fibonacci sequence. Let F_n, $(n = 0, 1, 2, \dots)$ be the Fibonacci sequence defined by $F_n = F_{n-1} + F_{n-2}$ $(n > 1)$ and $F_0 = 0$, and $F_1 = 1$. Then it is well known that

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n.$$

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V.E. Hoggatt, Jr. [5] and R.T. Hansen [4] gave a great number of generating functions for sequences constructed from Fibonacci and Lucas sequences.

In this paper we show that the generating function of any linear recurrent sequence is a rational function, and conversely, any rational function whose denominator is not zero at $x = 0$, is the generating function of some linear recurrence. Our results give an easy way to determine the power series expansion of rational functions. Furthermore, as a consequence of our development, we give new proofs for some known results. Similar topics were investigated by M. D'Ocagne [8], A.J. van der Poorten [11] and P. Erdős, Th. Maxsein, P.R. Smith [1].

Main theorem

For a given fixed linear recurrence it is not difficult to determine the generating function. Furthermore, from [10], pp. 96–97, it follows that if a power series $a_0 + a_1x + a_2x^2 + \dots$ represents a rational function, then the sequence a_n $(n = 0, 1, \dots)$ satisfies a linear recurrence. We give a unified proof of these results and some improvements.

Theorem 1. *Let a_n $(n = 0, 1, \dots)$, be a linear recurrence with characteristic polynomial $x^k - A_1x^{k-1} - \dots - A_k$. Then the formal power series*

$$a_0 + a_1x + a_1x^2 + \dots$$

is generated by a rational function of the form

$$a(x) = \frac{b_0 + b_1x + \dots + b_{k-1}x^{k-1}}{1 - A_1x - A_2x^2 - \dots - A_kx^k},$$

where

$$b_i = a_i - \sum_{j=1}^{i} A_j a_{i-j} \qquad (i = 0, 1, 2, \dots, k-1).$$

Conversely, every rational function

$$\frac{b_0 + b_1x + \dots + b_sx^s}{d_0 + d_1x + \dots + d_kx^k},$$

where $s \geq 0$, $k \geq 1$, $b_s \neq 0$, $d_0 \neq 0$, $d_k \neq 0$, and the denominator is not an algebraic factor of the numerator, is a generating function of some sequence a_n, satisfying a linear recurrence relation with characteristic polynomial

$$x^k + \frac{d_1}{d_0}x^{k-1} + \dots + \frac{d_k}{d_0}$$

for indices $n \geq n_0 = \max(k, 1+s)$. The terms of the sequence a_n are rational integers when $s < k$, $d_0 = 1$ and $d_1, \ldots, d_k$, $b_0, \ldots, b_s$ are rational integers.

PROOF. $\Longrightarrow$ First suppose that a_n is a k^{th} order linear recurrence satisfying the relation

$$a_n = A_1 a_{n-1} + A_2 a_{n-2} + \cdots + A_k a_{n-k} \tag{1}$$

for any $n \geq k$. If $a(x)$ is a function such that

$$a(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

then

$$\begin{aligned} &a(x) - A_1 x a(x) - A_2 x^2 a(x) - \cdots - A_k x^k a(x) = \\ =&a(x)(1 - A_1 x - A_2 x^2 - \cdots - A_k x^k) \\ =&a_0 + (a_1 - A_1 a_0)x + (a_2 - A_1 a_1 - A_2 a_0)x^2 + \cdots \\ &\cdots + (a_{k-1} - A_1 a_{k-2} - \cdots - A_{k-1} a_0)x^{k-1} \end{aligned}$$

since, by (1), the coefficients of x^k, $x^{k+1}, \ldots$ on the right-hand side are zero. From this we have

$$a(x) = \frac{a_0 + (a_1 - A_1 a_0)x + \cdots + (a_{k-1} - A_1 a_{k-2} - \cdots - A_{k-1} a_0)x^{k-1}}{1 - a_1 x - \cdots - a_k x^k}$$

which proves the first part of the theorem.

Now let

$$a'(x) = \frac{b'_0 + b'_1 x + \cdots + b'_s x^s}{d_0 + d_1 x + \cdots + d_k x^k}$$

be a rational function with complex coefficients, where $s \geq 0$, $k \geq 1$, $b'_s \neq 0$, $d_0 \neq 0$, $d_k \neq 0$ and the denominator is not a divisor of the numerator. Then $a'(x)$ can be written as

$$a'(x) = a''(x) + \frac{1}{d_0} \cdot a(x), \tag{2}$$

where $a''(x)$ is a polynomial and $a(x)$ is of the form

$$a(x) = \frac{b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}}{1 - A_1 x - \cdots - A_k x^k}$$

with b_i's not all zero and $A_i = -d_i/d_0$ for $i = 1, 2, \ldots, k$.

Let us consider the system of equations

$$\begin{aligned} a_{k-1} - A_1 a_{k-2} - A_2 a_{k-3} - \cdots - A_{k-2} a_1 - A_{k-1} a_0 &= b_{k-1} \\ a_{k-2} - A_1 a_{k-3} - \cdots - A_{k-3} a_1 - A_{k-2} a_0 &= b_{k-2} \\ &\vdots \\ a_1 - A_1 a_0 &= b_1 \\ a_0 &= b_0 \end{aligned}$$

in unknowns $a_0, a_1, \ldots, a_{k-1}$. This system has a unique solution since its determinant is 1. The solutions are integers if the A_i's and b_i's are rational integers and $a_0, \ldots, a_{k-1}$ are not all zero since $b_0, \ldots, b_{k-1}$ are not all zero. But then, as we have seen above, the sequence a_n $(n = 0, 1, \ldots)$ defined by the initial terms $a_0, \ldots, a_{k-1}$ and by the recursion

$$a_n = A_1 a_{n-1} + A_2 a_{n-2} + \cdots + A_k a_{n-k} \qquad (n \geq k), \tag{3}$$

has the generating function $a(x)$. From this, by (2), it follows that the coefficients of the power series of $a'(x)$ satisfy the recurrence relation (3) for any $n \geq n_0$, where $n_0 = k$ or $n_0 = k + 1 + \deg a''(x)$ according as $a''(x)$ is identically zero or not.

From this the theorem follows.

Remark. Theorem 1 and its proof give a method for obtaining the power series expansion of a rational function. For example let

$$a(x) = \frac{4 - 3x + 2x^2}{1 - 3x + 2x^2} = 1 + \frac{3}{1 - 3x + 2x^2}.$$

By Theorem 1 the function $\frac{3}{1-3x+2x^2}$ is the generating function of a linear recurrence a_n with characteristic polynomial $x^2 - 3x + 2$. For the initial terms, we have from the proof of the Theorem, $a_1 - 3a_0 = 0$ and $a_0 = 3$. So $a_0 = 3$, $a_1 = 9$ and

$$a(x) = a_0 + 1 + \sum_{n=1}^{\infty} a_n x^n.$$

But it is known that if α_1 and α_2 are the roots of the characteristic polynomial of a second order linear recurrence a_n and $\alpha_1 \neq \alpha_2$, then the terms can be expressed by

$$a_n = \frac{(a_1 - a_0\alpha_2){\alpha_1}^n - (a_1 - a_0\alpha_1){\alpha_2}^n}{\alpha_1 - \alpha_2}. \tag{4}$$

In our case $\alpha_1 = 2$, $\alpha_2 = 1$, $a_0 = 3$, $a_1 = 9$ and so we have

$$\frac{4 - 3x + 2x^2}{1 - 3x + 2x^2} = 4 + \sum_{n=1}^{\infty} (6 \cdot 2^n - 3) x^n.$$

Consequences of the theorem

Let R_n $(n = 0, 1, \dots)$ be a linear recurrence of order k as defined in the introduction and let $\alpha_1, \alpha_2, \dots, \alpha_m$ be the distinct roots of the characteristic polynomial $c(x)$ with multiplicities $k_1, k_2, \dots, k_m$ respectively, $(k_1 + \cdots + k_m = k)$.

Explicit forms for the terms R_n are known in special cases, the best known of which is Binet's formula for Fibonacci numbers: $F_n = (\alpha_1^n - \alpha_2^n)/(\alpha_1 - \alpha_2)$, where α_1 and α_2 are the roots of the polynomial $x^2 - x - 1$. For general second order recurrences formula (4) is also well known. For an arbitrary sequence R_n, J.A. Jeske [6] proved the existence of polynomials $r_i(x)$, $i = 1, \dots, m$, such that

$$R_n = \sum_{i=1}^{m} r_i(n)\alpha_i^n$$

for any $n \geq 0$. We show that this result and its converse follow from our Theorem.

Corollary 1. *The terms of the linear recurrence R_n can be expressed by*

$$R_n = \sum_{i=1}^{m} r_i(n)\alpha_i^n \tag{5}$$

for every $n \geq 0$, where $r_i(x)$, $(i = 1, \dots, m)$ are fixed polynomials of degree $k_i - 1$, respectively and the coefficients of these polynomials are elements of the number field generated by $R_0, \dots, R_{k-1}$, $A_1, \dots, A_k$ and $\alpha_1, \dots, \alpha_m$ over the rationals. Also conversely, if the terms of a sequence satisfy equality (5) with polynomials $r_i(x)$ of degree $k_i - 1$ $(1 \leq i \leq m)$, then the sequence satisfies a linear recurrence of order $k_1 + \ldots + k_m$ with characteristic polynomial

$$c(x) = \prod_{i=1}^{m} (x - \alpha_i)^{k_i}.$$

Proof. Let R_n be a linear recurrence with characteristic polynomial

$$c(x) = \prod_{i=1}^{m} (x - \alpha_i)^{k_i} = x^k - A_1 x^{k-1} - \cdots - A_k. \tag{6}$$

Then, by Theorem 1, there is a polynomial $g(x)$ with $\deg g(x) < k$ and such that

$$R(x) = \frac{g(x)}{1 - A_1 x - \ldots - A_k x^k} = \sum_{n=0}^{\infty} R_n x^n. \tag{7}$$

By (6) we can write

$$R(x)=\frac{g(x)}{\prod_{i=1}^{m}(1-\alpha_i x)^{k_i}}=\sum_{i=1}^{m}\frac{g_i(x)}{(1-\alpha_i x)^{k_i}} \tag{8}$$

where

$$g_i(x)=b_{i,0}+b_{i,1}x+\cdots+b_{i,k_i-1}x^{k_i-1}, \qquad (i=1,\ldots,m),$$

are polynomials.

It is known that

$$\frac{1}{(1-x)^k}=\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n,$$

from which

$$\begin{aligned}\frac{g_i(x)}{(1-\alpha_i x)^{k_i}}&=\sum_{n=0}^{\infty}\sum_{j=0}^{k_i-1}b_{i,j}\binom{n+k_i-1}{k_i-1}\alpha_i^n x^{n+j}=\\&=\sum_{n=0}^{\infty}\sum_{j=0}^{k_i-1}\frac{b_{i,j}}{\alpha_i^j}\binom{n+k_i-1}{k_i-1}\alpha_i^{n+j}x^{n+j}=\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{k_i-1}\frac{b_{i,j}}{\alpha_i^j}\binom{n-j+k_i-1}{k_i-1}\right)\alpha_i^n x^n=\\&=\sum_{n=0}^{\infty}r_i(n)\alpha_i^n x^n\end{aligned} \tag{9}$$

follows, where $r_i(x)$ is a polynomial of degree at most k_i-1. From this, by (7) and (8), equality (5) follows. The restrictions on the coefficients of the polynomials also hold.

Now let R'_n be a sequence of numbers defined by

$$R'_n=\sum_{i=1}^{m}r_i(n)\alpha_i^n \tag{10}$$

for any $n\geq 0$, where $r_i(x)$, $(i=1,\ldots,m)$, are polynomials of degree k_i-1, respectively, $m,k_1,\ldots,k_m$ are positive rational integers, and $\alpha_1,\ldots,\alpha_m$ are fixed non-zero numbers.

We show that for any polynomial

$$r(x)=c_0+c_1x+\cdots+c_{k-1}x^{k-1}, \tag{11}$$

there are numbers $t_0, t_1, \dots, t_{k-1}$ such that

$$(12)\qquad \begin{aligned} r(x) = & t_0 \binom{x+k-1}{k-1} + t_1 \binom{x-1+k-1}{k-1} + \cdots \\ & + t_{k-1} \binom{x-(k-1)+k-1}{k-1} \end{aligned}$$

for any non-negative integer x. By (11) and (12) $t_0 = c_0$ follows since $r(0) = c_0$ and $\binom{x-i+k-1}{k-1} = 0$ if $x - i < 0$. Also by (12) we have

$$t_1 = r(1) - t_0 \binom{k}{k-1}$$

and continuing this process with $x = 2, 3, \dots, k-1$ we have

$$t_i = r(i) - \sum_{j=0}^{i-1} t_j \binom{i-j+k-1}{k-1}$$

for any $1 \le i \le k-1$. So the numbers $t_0, \dots, t_{k-1}$ are uniquely determined. From this and (9) by replacing $b_{i,j}$ by $t_j \alpha^j$, it follows that

$$\sum_{n=0}^{\infty} r(n) \alpha^n x^n = \frac{g'(x)}{(1-\alpha x)^k},$$

for some polynomial $g'(x)$ of degree $k-1$, for any $\alpha \ne 0$ and polynomial $r(x)$ of degree $k-1$. So by (10)

$$(13)\qquad \sum_{i=1}^{m} \frac{g_i(x)}{(1-\alpha_i x)^{k_i}} = \sum_{i=0}^{\infty} R'_n x^n$$

follows for some polynomials $g_i(x)$ of degree $k_i - 1$, $i = 1, \dots, m$.

The right hand side of (13) is a rational function with denominator of the form

$$c'(x) = \prod_{i=1}^{m} (1-\alpha_i x)^{k_i} = 1 - A_1 x - \cdots - A_k x^k$$

and so, by the Theorem, the sequence R'_n is a linear recurrence with characteristic polynomial

$$c(x) = \prod_{i=1}^{m} (x-\alpha_i)^{k_i} = x^k - A_1 x^{k-1} - \cdots - A_k.$$

Remark. Some authors state formula (5) in the form that the degree of the polynomials $r_i(x)$ are "at most" $k_i - 1$. Corollary 1 implies that if $\deg r_i(x) < k_i - 1$ for some i, then the sequence R_n satisfies a recurrence relation of order less than $k_1 + \dots + k_m = k$.

Let $q \geq 1$, $r \geq 0$ be fixed natural numbers. Then the sequence $H_n = R_{qn+r}$, $(n = 0, 1, 2, \dots)$, is a subsequence of R_n. If R_n is a second order linear recursive sequence, then it is easy to see that H_n is also a second order linear recurrence. For the Fibonacci sequence this was shown by J.H. Halton [3]. For an arbitrary linear recurrence this result can be derived from Corollary 1. Now we give another proof of it using only our Theorem.

Corollary 2. *Let R_n, $n = 0, 1, 2, \dots$, be a k^{th} order linear recurrence. Then the subsequence H_n of R_n defined by*

$$H_n = R_{qn+r} \qquad (n \geq 0),$$

where $q > 0$ and r, $0 \leq r < q$, are fixed integers, is also a linear recurrence, of order k. If R_n is a sequence of rational integers with integer parameters, then the sequence H_n also has integer parameters.

Proof. Let R_n be a k^{th} order recurrence with characteristic polynomial

$$c(x) = x^k - A_1 x^{k-1} - \dots - A_k.$$

Then for any given $r \geq 0$ the sequence $R_r, R_{r+1}, \dots$ is also a linear recurrence with the same characteristic polynomial, $c(x)$. Let $g(x)/f(x)$ be the generating function of this sequence, i.e.,

$$\frac{g(x)}{f(x)} = R_r + R_{r+1}x + R_{r+2}x^2 + \dots$$

where, by Theorem 1, $\deg g(x) < \deg f(x) = k$ and

$$f(x) = 1 - A_1 x - \dots - A_k x^k.$$

Let $q > 0$ be a natural number and ε be a primitive q^{th} root of unity. Put

$$\frac{g(\varepsilon^i x)}{f(\varepsilon^i x)} = R_r + R_{r+1}(\varepsilon^i x) + R_{r+2}(\varepsilon^i x)^2 + \dots$$

for $i = 0, 1, \dots, q-1$. Then

$$\frac{G(x)}{F(x)} = \sum_{i=0}^{q-1} \frac{g(\varepsilon^i x)}{f(\varepsilon^i x)} = q \cdot R_r + \sum_{n=1}^{\infty} \left(R_{n+r} x^n \cdot \sum_{i=0}^{q-1} \varepsilon^{i \cdot n} \right), \tag{14}$$

where

$$F(x) = \prod_{i=0}^{q-1} f(\varepsilon^i x).$$

But

$$\sum_{i=0}^{q-1} \varepsilon^{in} = \begin{cases} 0 & \text{if } q \nmid n \\ q & \text{if } q \mid n \end{cases}$$

and so by (14) we have

$$\frac{G(x)}{F(x)} = \sum_{n=0}^{\infty} q \cdot R_{qn+r} x^{qn} = q \cdot \sum_{n=0}^{\infty} H_n x^{qn}. \tag{15}$$

By the definition of $F(x)$, for any complex number x and any integer i, $(0 \le i \le q-1)$, $F(x) = F(\varepsilon^i x)$. This implies that if $\delta \neq 0$ is a root of the equation $F(x) = 0$, then $\varepsilon\delta$, $\varepsilon^2\delta, \dots, \varepsilon^{q-1}\delta$ are also roots with the same multiplicity as δ. Since $F(0) \neq 0$, it follows that $F(x)$ is of the form

$$F(x) = f_k \cdot \prod_{i=1}^{k} (x^q - \beta_i^q) = f_0 + f_1 x^q + f_2 x^{2q} + \cdots + f_k x^{kq}, \tag{16}$$

where $\beta_1, \dots, \beta_k$ are the roots of the polynomial $f(x)$, $|f_k| = |A_k|^q$ and $|f_0| = |f_k| \cdot |\prod_{i=1}^{k} \beta_i|^q = 1$.

Let

$$G'(x) = g(x) \cdot f(\varepsilon x) \cdot f(\varepsilon^2 x) \cdot f(\epsilon^{q-1} x) = a_m x^m + \cdots + a_1 x + a_0.$$

Then

$$G(x) = \sum_{i=0}^{q-1} G'(\varepsilon^i x) = b_m x^m + \cdots + b_1 x + b_0,$$

where

$$b_i = a_i \sum_{j=0}^{q-1} \varepsilon^{ij} = \begin{cases} q \cdot a_i & \text{if } q \mid i \\ 0 & \text{if } q \nmid i \end{cases}$$

for any $0 \le i \le m$ and so $G(x)$ has the form

$$G(x) = q \cdot \sum_{i=0}^{t} g_i x^{qi} \tag{17}$$

for some integer $t \ge 0$.

By (15), (16) and (17), after replacing x^q by x, we get

$$\frac{\pm(g_0 + g_1 x + \cdots + g_t x^t)}{1 \pm (f_1 x + \cdots + f_k x^k)} = \sum_{n=0}^{\infty} H_n x^n,$$

which, by Theorem 1, implies that H_n is a linear recurrence of order k with characteristic polynomial $c(x) = x^k \pm (f_1 x^{k-1} + \cdots + f_k)$.

If R_n is a sequence of integers with rational integer parameters $A_1, \dots ,$ A_k, then by using theorems about the elementary symmetric functions of variables which are roots of a polynomial, it is easy to check that $f_1, \dots , f_k$ are integers. The initial terms of the sequence H_n are obviously integers and so the proof is complete.

Using the generating functions for the Fibonacci F_n and Lucas L_n sequences H.W. Gould [2] showed that for any fixed integer $p \geq 1$ the sequences F_n^p and L_n^p are also linear recurrences. Similar results were obtained by I.I. Kolonder [7] for general second order linear recurrences. Corollary 1 implies a generalization of these results.

Corollary 3. *Let R_n be a linear recurrence and let*

$$p(x) = a_t x^t + a_{t-1} x^{t-1} + \cdots + a_0$$

be a polynomial. Then the sequence G_n, defined by

$$G_n = p(R_n) \qquad (n = 0, 1, \dots)$$

also satisfies a linear recursion equation.

Proof. By Corollary 1, for any given complex number a and any integer $s \geq 0$, the numbers $a \cdot R_n^s$ are of the form

$$a \cdot R_n^s = a \cdot \left(\sum_{i=1}^{m} r_i(n) \alpha_i^n \right)^s = \sum_{i=1}^{v} g_i(n) \beta_i^n$$

for all $n \geq 0$. Here v is a positive integer, the $g_i(x)$'s are polynomials and the β_i's are determined by

$$\beta_i = \prod_{j=1}^{m} \alpha_j^{e_j} \quad \text{with} \quad \sum_{j=1}^{m} e_j = s.$$

But then

$$G_n = p(R_n) = \sum_{s=0}^{t} a_s R_n^s = \sum_{i=1}^{u} h_i(n) \gamma_i^n$$

for some integer u, some polynomials $h_i(x)$ and some numbers γ_i. So by Corollary 1 the sequence G_n is a linear recurrence.

Remark. One can easily check that if R_n is a sequence with integer parameters and the coefficients of $p(x)$ are integers, then the parameters of the sequence G_n are also integers.

Additional consequences

Now we list some further consequences which can be proved easily from the above results.

Corollary 4. *Let $S = \{p_1, \dots, p_s\}$ be a finite set of prime numbers and suppose that a_n $(n = 0, 1, \dots)$, is a sequence of non zero integers such that all prime divisors of the terms $a_0, a_1, a_2, \dots$ are elements of the set S. Then the sequence a_n satisfies a linear recurrence relation if and only if the terms of a_n are of the form*

$$a_{pn+r} = c_r \cdot \alpha_r^{pn+r}$$

for all $n \geq 0$ and all r, $0 \leq r < p$, where p is a positive integer and α_r, c_r $(r = 0, 1, \dots, p-1)$ are constants.

Proof. This follows from Theorem 1 and a result of G. Pólya [9]. But it can also be proved from Corollaries 1 and 2, using the known result: if a_n is a non-degenerate linear recurrence of integers, then there are infinitely many prime p such that $p|a_n$ for some n.

Corollary 5. *Let R_n be a linear recurrence of order k with generating function $g(x)/f(x)$. If $(g(x), f(x)) > 1$, then the sequence satisfies a recurrence relation of order less than k.*

Proof. From Theorem 1.

Corollary 6. *If R_n $(n = 0, 1, \dots)$ and K_n are linear recurrences of order k_1 and k_2, respectively, then the sequence $R_n + K_n$, $(n = 0, 1, \dots)$, is also a linear recurrence of order $\leq k_1 + k_2$.*

Proof. This follows from Theorem 1 by adding the generating functions of the sequences. Corollary 1 also implies this assertion.

Corollary 7. *If R_n and K_n are the sequences defined in Corollary 6, then the sequence L_n, defined by*

$$L_n = \sum_{i=0}^{n} R_i K_{n-i}$$

is also a linear recurrence of order at most $k_1 + k_2$.

Proof. It follows from Theorem 1 by multiplying the generating functions of the sequences.

Corollary 8. *If L_n is a linear recurrence and $p > 0$ is a fixed natural number, then the sequence P_n, defined by*

$$P_n = \sum_{i=0}^{n} L_i^p$$

is also a linear recurrence.

PROOF. The sequence L_n^p is a linear recurrence by Corollary 3. Hence if in Corollary 7 we replace the sequence R_n by L_n^p and let K_n be defined by $K_n = 1$ for $n \geq 0$, then the assertion follows.

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