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System of multi-valued variational inequalities

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Abstract. In this paper we give an existence result for a system of two variational inequalities defined by two multi-valued mappings. We show that by our result, Brouwer's fixed point theorem can be easily deduced.

1. Introduction

Variational inequalities with a mapping defined on a subset of a certain Banach space have been extensively studied in the literature (see e.g. [4], [9], and the references therein). Usually, in these results, the monotonicity or a generalized monotonicity property of the mapping involved plays a crucial role.

Let X and Y be two reflexive real Banach spaces and $A \subseteq X$, $B \subseteq Y$ be nonempty closed convex sets. Denote by X^* and Y^* the dual spaces of X and Y, respectively. Consider two multi-valued mappings, $F : A \times B \to 2^{X^*}$ and $G : A \times B \to 2^{Y^*}$. The aim of this paper is to establish an existence result for the following problem (system of two variational inequalities):

Problem 1. Find $(a, b) \in A \times B$ such that

$$(1) \qquad \sup_{w \in F(a,b)} \langle w, x - a \rangle \geq 0, \quad \forall x \in A$$

and

$$(2) \qquad \sup_{z \in G(a,b)} \langle z, y - b \rangle \geq 0, \quad \forall y \in B.$$

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The idea of considering a system of type (1)–(2) originates in the following problem known as existence of a Nash equilibrium point. Let $K_1, \ldots, K_n$ $(n \geq 2)$ be nonempty sets, and $f_i : K_1 \times \cdots \times K_n \to \mathbb{R}$ $(i = 1, \ldots, n)$ be given functions. A point $(x_1, \ldots, x_n) \in K_1 \times \cdots \times K_n$ is called a *Nash equilibrium point* if

$$f_i(x_1, \ldots, x_i, \ldots, x_n) \leq f_i(x_1, \ldots, y_i, \ldots, x_n), \quad \forall y_i \in K_i$$

holds for $i = 1, \ldots, n$. An important existence theorem is due to NASH [6]. This notion has been turned out to be very useful in game theoretical and economical applications. The result of Nash offers a sufficient condition for the existence of an equilibrium point, however, equilibrium points may exist even if the conditions imposed by Nash (continuity and quasiconvexity in the i-th variable) are not satisfied. This fact motivated the authors in [3] to introduce the notion of Nash stationary point, i.e. such a point in which a certain kind of derivative is nonnegative. In the paper mentioned, the autors established results where the Nash stationary points are obtained as solutions of a system of inequalities defined by those kind of derivatives. That system (in case $n = 2$) is a particular case of our Problem 1 above, where the mappings F and G are of *subdifferential type*.

In this paper, the multi-valued mappings F and G are in general not subdifferentials of certain functions, permitting thus to extend the applicability of Problem 1. The main result (Theorem 2.1) gives a sufficient condition for the existence of a solution of Problem 1. As an application, we establish a solvability result for a certain type of variational inequalities (Theorem 2.2), from which Brouwer's fixed point theorem can easily be deduced (Corollary 2.1).

We first recall some definitions. Let K be a nonempty subset of a Banach space B and $T : K \to 2^{B^*}$. The operator T is called *monotone*, if for every $x, y \in K$ and every $u \in T(x)$, $v \in T(y)$ we have that

$$\langle u - v, x - y \rangle \geq 0.$$

Note that the (single-valued) operator $T : K \to X^*$ is called *pseudomonotone* if for every net (x_α) converging weakly to the element x $(x_\alpha, x \in K)$ such that

$$\limsup_\alpha \langle T(x_\alpha), x_\alpha - x \rangle \leq 0,$$

we also have

$$\langle T(x), x - z\rangle \leq \liminf_{\alpha} \langle T(x_\alpha), x_\alpha - z\rangle, \quad \forall z \in K.$$

If B and B' are two Banach spaces, then the mapping $T : B \to 2^{B'}$ is said to be *upper semicontinuous* (on B) if for every $x_0 \in B$ and any open set V containing $T(x_0)$, there exists a neighbourhood U of x_0 in B such that $T(x) \subseteq V$ for all $x \in U$. The mapping T is said to be *lower semicontinuous* (on B) if for every $x_0 \in B$, any $y_0 \in T(x_0)$ and any neighbourhood V of y_0, there exists a neighbourhood U of x_0 such that $T(x) \cap V \neq \emptyset$ for every $x \in U$.

The following famous lemma of Ky Fan [2] plays a crucial role in the proof of our result.

Lemma 1.1. *Let Z be a nonempty subset of a Hausdorff topological vector space E. For each $z \in Z$, let $F(z)$ be a closed subset of E such that the convex hull of every finite subset $\{z_1, \dots, z_n\}$ of Z is contained in the corresponding union $\bigcup_{i=1}^{n} F(z_i)$. If there exists a point $z_0 \in Z$ such that $F(z_0)$ is compact, then $\bigcap_{z\in Z} F(z) \neq \emptyset$.*

We also need the following lemma of Shih and Tan [7].

Lemma 1.2. *Let K be a nonempty convex subset of a Banach space B, let $T : K \to 2^{B^*}$ be such that each $T(x)$ is a weak* compact subset of B^*, and let T be upper semicontinuous from the line segments of K in the weak* topology of B^*. Then for $y \in K$ the inequality*

$$\inf_{u\in T(x)} \langle u, x - y\rangle \geq 0, \quad \forall x \in K$$

implies

$$\sup_{w\in T(y)} \langle w, x - y\rangle \geq 0, \quad \forall x \in K.$$

In order to establish an existence result for Problem 1, let us introduce an auxiliary problem.

Problem 2. Find $(a, b) \in A \times B$ such that

$$\inf_{u\in F(x,b)} \langle u, x - a\rangle \geq 0, \quad \forall x \in A \tag{3}$$

and

$$\inf_{v\in G(a,y)} \langle v, y - b\rangle \geq 0, \quad \forall y \in B. \tag{4}$$

Then we have the following

Proposition 1.1. *Suppose that for every $x \in A$ and $y \in B$ the mappings $F(\,.\,, y) : A \to 2^{X^*}$ and $G(x,\,.\,) : B \to 2^{Y^*}$ are monotone and upper semicontinuous from the line segments of A (resp. B) in the weak topology of X^* (resp. Y^*).*

Then Problems 1 and 2 are equivalent, namely they have the same set of solutions.

PROOF. Suppose first that the pair $(a, b) \in A \times B$ is a solution of Problem 1. Fix $x \in A$. Observe, by reflexivity, that the weak and the weak* topologies of X^* (and Y^*) coincide. Since $F(a, b)$ is weakly compact, there exists $\bar{u} \in F(a, b)$ such that

$$\langle \bar{u}, x - a\rangle = \sup_{w \in F(a,b)} \langle w, x - a\rangle \geq 0.$$

By the monotonicity of $F(\,.\,, b)$ we also have

$$\langle u, x - a\rangle \geq 0, \quad \forall u \in F(x, b),$$

and so (3) holds. A similar argument shows that (4) also holds, by using the monotonicity of $G(a,\,.\,)$.

The reverse implication, i.e. that each solution of Problem 2 is also a solution of Problem 1 follows by Lemma 1.2, applying it first to the mapping $F(\,.\,, b)$ and then to the mapping $G(a,\,.\,)$. □

2. Existence results

Recall that X and Y are (as in Section 1) two reflexive Banach spaces, $A \subseteq X$, $B \subseteq Y$ are nonempty closed convex sets, and $F : A \times B \to 2^{X^*}$, $G : A \times B \to 2^{Y^*}$ two multi-valued mappings.

To establish the main result of this paper we need the following weak continuity concept.

Condition (C). Let $x \in A$ and $y \in B$ be fixed. We say that the mapping pair $F(x,\,.\,) : B \to 2^{X^*}$ and $G(\,.\,, y) : A \to 2^{Y^*}$ satisfies condition (C) if for any nets (a_α) and (b_α) converging weakly to the elements a and b, respectively, where $a_\alpha, a \in A$ and $b_\alpha, b \in B$, and for each $u_\alpha \in F(x, b_\alpha)$

and $v_\alpha \in G(a_\alpha, y)$ such that u_α converges weakly to $u \in F(x,b)$ and v_α converges weakly to $v \in G(a,y)$, we have

$$\langle u, a\rangle + \langle v, b\rangle \le \limsup_\alpha [\langle u_\alpha, a_\alpha\rangle + \langle v_\alpha, b_\alpha\rangle].$$

If X and Y are finite dimensional, then condition (C) is clearly satisfied. Below we specify two situations, each of which them guarantees condition (C).

1. Let $F : A \times B \to X^*$, $G : A \times B \to Y^*$ be given (single-valued mappings) and for each $x \in A$ and $y \in B$ consider the mapping $\mathcal{H}_{x,y} : A \times B \to X^* \times Y^*$ given by

$$\mathcal{H}_{x,y}(a,b) := (F(x,b), G(a,y)).$$

If here $\mathcal{H}_{x,y}$ is pseudomonotone, then condition (C) is satisfied. Indeed, let a_α and b_α be two nets converging weakly to the elements a and b, where $a_\alpha, a \in A$ and $b_\alpha, b \in B$ are such that the corresponding nets $F(x, b_\alpha)$ and $G(a_\alpha, y)$ converge weakly to the elements $F(x,b)$ and $G(a,y)$, respectively. We distinguish two cases. First, if

$$\limsup_\alpha [\langle F(x,b_\alpha), a_\alpha - a\rangle + \langle G(a_\alpha, y), b_\alpha - b\rangle] \le 0, \tag{5}$$

then by pseudomonotonicity (see Section 1) one obtains that

$$\langle F(x,b), a\rangle + \langle G(a,y), b\rangle \le \liminf_\alpha [\langle F(x,b_\alpha), a_\alpha\rangle + \langle G(a_\alpha,y), b_\alpha\rangle].$$

Secondly, if the relation (5) does not hold, then we have that

$$0 < \limsup_\alpha [\langle F(x,b_\alpha), a_\alpha - a\rangle + \langle G(a_\alpha,y), b_\alpha - b\rangle]$$
$$= \limsup_\alpha [\langle F(x,b_\alpha),\, a_\alpha\rangle + \langle G(a_\alpha,y), b_\alpha\rangle] - \langle F(x,b), a\rangle - \langle G(a,y), b\rangle.$$

Clearly, in both cases, the relation required in condition (C) follows.

2. Suppose that for each $x \in A$ and $y \in B$ the (single-valued) mappings $F(x, .) : B \to X^*$ and $G(. , y) : A \to Y^*$ are completely continuous, i.e. continuous from the weak topology to the strong topology. Then condition (C) is satisfied. To show this, consider the nets (a_α) and (b_α)

converging weakly to the elements a and b respectively, where $a_\alpha, a \in A$ and $b_\alpha, b \in B$. By the hypothesis, $F(x, b_\alpha)$ converges strongly to $F(x,b)$, while $G(a_\alpha, y)$ converges strongly to $G(a,y)$. Then

$$\langle F(x,b), a\rangle + \langle G(a,y), b\rangle = \lim_\alpha [\langle F(x,b_\alpha), a_\alpha\rangle + \langle G(a_\alpha, y), b_\alpha\rangle],$$

which shows that condition (C) is satisfied.

Now we can state the main result of this paper.

Theorem 2.1. *Suppose that the following conditions are satisfied:*

(i) *for every $x \in A$ and $y \in B$ the mappings $F(\,.\,,y) : A \to 2^{X^*}$ and $G(x,\,.\,) : B \to 2^{Y^*}$ are monotone and upper semicontinuous from the line segments of A (resp. B) in the weak topology of X^* (resp. Y^*);*

(ii) *for each $x \in A$ and $y \in B$ the mappings $F(x,\,.\,) : B \to 2^{X^*}$ and $G(\,.\,,y) : A \to 2^{Y^*}$ are weakly lower semicontinuous and satisfy condition (C);*

(iii) *for each $x \in A$ and $y \in B$ the sets $F(x,y)$ and $G(x,y)$ are weakly compact subsets of X^*, respectively Y^*;*

(iv) *there exist bounded sets $C \subseteq A$ and $D \subseteq B$, and $x_0 \in C$, $y_0 \in D$ such that*

$$\sup_{u \in F(x,y)} \langle u, x_0 - x\rangle + \sup_{v \in G(x,y)} \langle v, y_0 - y\rangle < 0, \quad \forall (x,y) \in (A \times B) \setminus (C \times D). \tag{6}$$

Then Problem 1 admits a solution.

If in addition $F(a,b)$ and $G(a,b)$ are convex sets, then there exist $\bar u \in F(a,b)$ and $\bar v \in G(a,b)$ such that

$$\langle \bar u, x - a\rangle \geq 0, \quad \forall x \in A \tag{7}$$

and

$$\langle \bar v, y - b\rangle \geq 0, \quad \forall y \in B. \tag{8}$$

Proof. Introduce, for each $(x,y) \in A \times B$, the sets

$$\mathcal{S}(x,y) := \{(a,b) \in A \times B : \sup_{w \in F(a,b)} \langle w, x - a\rangle + \sup_{z \in G(a,b)} \langle z, y - b\rangle \geq 0\}$$

and

$$\mathcal{T}(x,y) := \{(a,b) \in A \times B : \inf_{u \in F(x,b)} \langle u, x-a \rangle + \inf_{v \in G(a,y)} \langle v, y-b \rangle \geq 0\}.$$

By monotonicity of $F(\,.\,,b)$ and $G(a,\,.\,)$ it is easy to see that $\mathcal{S}(x,y) \subseteq \mathcal{T}(x,y)$ for each $(x,y) \in A \times B$. Furthermore, since the solution sets of Problem 1 and Problem 2 coincide with the sets $\bigcap_{(x,y)\in A\times B} \mathcal{S}(x,y)$ and $\bigcap_{(x,y)\in A\times B} \mathcal{T}(x,y)$, respectively, we have by Proposition 1.1 that $\bigcap_{(x,y)\in A\times B} \mathcal{S}(x,y) = \bigcap_{(x,y)\in A\times B} \mathcal{T}(x,y)$.

Now, for each finite set of pairs $(x_1,y_1),\ldots,(x_n,y_n) \in A \times B$, the inclusion

$$co\{(x_1,y_1),\ldots,(x_n,y_n)\} \subseteq \bigcup_{i=1}^{n} \mathcal{S}(x_i,y_i) \tag{9}$$

holds, where *co* stands for the convex hull operator. To prove this, let Δ_n be the set $\{\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n : \lambda_i \geq 0,\ i=1,\ldots,n,\ \sum_{i=1}^n \lambda_i = 1\}$ and suppose by contradiction that there exists $\lambda = (\lambda_1,\ldots,\lambda_n) \in \Delta_n$ such that the element $(x_\lambda,y_\lambda) := \sum_{k=1}^n \lambda_k(x_k,y_k) \notin \mathcal{S}(x_i,y_i)$ for any $i \in \{1,\ldots,n\}$. Then

$$\sup_{w \in F(x_\lambda,y_\lambda)} \langle w, x_i - x_\lambda \rangle + \sup_{z \in G(x_\lambda,y_\lambda)} \langle z, y_i - y_\lambda \rangle < 0, \quad \forall i \in \{1,\ldots,n\}.$$

If we take some fixed elements $w \in F(x_\lambda,y_\lambda)$ and $z \in G(x_\lambda,y_\lambda)$, the latter relations imply

$$\langle w, x_i - x_\lambda \rangle + \langle z, y_i - y_\lambda \rangle < 0, \quad \forall i \in \{1,\ldots,n\}.$$

By multiplying each of these relations with λ_i, $i \in \{1,\ldots,n\}$ and then taking their sum we obtain the contradiction $0 < 0$, which shows that (9) holds true.

We now show that the set $\mathcal{S}(x_0,y_0)$ is bounded, where x_0 and y_0 are the elements specified in (iv). Indeed, as it can be seen, $\mathcal{S}(x_0,y_0) \subseteq C \times D$. If not, i.e. there exists a pair $(a,b) \in \mathcal{S}(x_0,y_0)$ such that $(a,b) \notin C \times D$, then we have $\sup_{u\in F(a,b)}\langle u, x_0 - a\rangle + \sup_{v \in G(a,b)}\langle v, y_0 - b\rangle \geq 0$, which contradicts (6).

Now, since X and Y were supposed to be reflexive, the weak closure $\mathrm{cl}_w\, \mathcal{S}(x_0, y_0)$ of the bounded set $\mathcal{S}(x_0, y_0)$ is weakly compact. By Lemma 1.1 of Ky Fan applied to the sets $\mathrm{cl}_w\, \mathcal{S}(x, y)$ one obtains

$$\bigcap_{(x,y)\in A\times B} \mathrm{cl}_w\, \mathcal{S}(x, y) \neq \emptyset. \tag{10}$$

Next we show that the set $\mathcal{T}(x, y)$ is weakly closed for each $(x, y) \in A \times B$. Indeed, for fixed x and y, let us consider a net $(a_\alpha, b_\alpha) \in \mathcal{T}(x, y)$, $\alpha \in I$, which converges weakly to an element $(a, b) \in A \times B$ and show that this element belongs to $\mathcal{T}(x, y)$.

Let $\bar{u} \in F(x, b)$ and $\bar{v} \in G(a, y)$ be such that $\inf_{u\in F(x,b)}\langle u, x - a\rangle = \langle \bar{u}, x-a\rangle$ and $\inf_{v\in G(a,y)}\langle v,\, y-b\rangle = \langle \bar{v},\, y-b\rangle$. (Observe that this is possible since the sets $F(x, b)$ and $G(a, y)$ are supposed to be weakly compact.)

By the weak lower semicontinuity, we can specify nets $u_\alpha \in F(x, b_\alpha)$ and $v_\alpha \in G(a_\alpha, y)$ such that u_α converges weakly to $\bar{u}$ and v_α converges weakly to $\bar{v}$.

Since $(a_\alpha, b_\alpha) \in \mathcal{T}(x, y)$ for every $\alpha \in I$, we have $\inf_{u\in F(x,b_\alpha)}\langle u, x - a_\alpha\rangle + \inf_{v\in G(a_\alpha,y)}\langle v, y-b_\alpha\rangle \geq 0$ and therefore $\langle u_\alpha, x-a_\alpha\rangle + \langle v_\alpha, y-b_\alpha\rangle \geq 0$ for each $\alpha \in I$. This (passing to the limsup) yields

$$\limsup_\alpha [\langle u_\alpha, a_\alpha\rangle + \langle v_\alpha, b_\alpha\rangle] \leq \langle \bar{u}, x\rangle + \langle \bar{v}, y\rangle.$$

Using condition (C) one obtains

$$\langle \bar{u}, a\rangle + \langle \bar{v}, b\rangle \leq \langle \bar{u}, x\rangle + \langle \bar{v}, y\rangle,$$

or

$$\langle \bar{u}, x - a\rangle + \langle \bar{v}, y - b\rangle \geq 0,$$

which shows that $(a, b) \in \mathcal{T}(x, y)$, i.e. the set $\mathcal{T}(x, y)$ is weakly closed.

Consequently, $\mathrm{cl}_w\, \mathcal{S}(x, y) \subseteq \mathcal{T}(x, y)$ for each $(x, y) \in A \times B$, and by (10) it follows that $\bigcap_{(x,y)\in A\times B} \mathcal{T}(x, y) \neq \emptyset$, and since the last set equals $\bigcap_{(x,y)\in A\times B} \mathcal{S}(x, y)$ this leads to

$$\bigcap_{(x,y)\in A\times B} \mathcal{S}(x, y) \neq \emptyset,$$

which shows that Problem 1 admits a solution.

Suppose now that the sets $F(a,b)$ and $G(a,b)$ are convex. Observe that inequality (1) can also be written as $\inf_{x\in A}\sup_{w\in F(a,b)}\langle w, x-a\rangle \geq 0$. By the well-known minimax theorem of KNESER [5], it follows that

$$\sup_{w\in F(a,b)}\inf_{x\in A}\langle w, x-a\rangle = \inf_{x\in A}\sup_{w\in F(a,b)}\langle w, x-a\rangle \geq 0, \tag{11}$$

(see also YAO [8]). Note that the real-valued function

$$w \longmapsto \inf_{x\in A}\langle w, x-a\rangle$$

is concave and upper semicontinuous, therefore it is also weakly upper semicontinuous. Since the set $F(a,b)$ is weakly compact, it follows from (11) that there exists $\bar{u} \in F(a,b)$ such that

$$\langle \bar{u}, x-a\rangle \geq 0, \quad \forall x \in A,$$

i.e. relation (7) holds. The existence of an element $\bar{v} \in G(a,b)$ satisfying relation (8) can be proved similarly, therefore we omit it. This completes the proof. □

Observe that our proof of the above theorem is based on Ky Fan's lemma (Lemma 1.1). As it is well-known, the latter is equivalent to Brouwer's fixed point theorem (see for instance ZEIDLER [9]). In the following we show that Brouwer's fixed point theorem can easily be deduced from our Theorem 2.1. To this end, we first establish an existence result for a special type of variational inequality.

Theorem 2.2. *Let X and Y be reflexive (real) Banach spaces, and $A \subseteq X$, $B \subseteq Y$ be bounded closed convex sets. Suppose $T : A \to B$ is a compact operator and $F : A \times B \to 2^{X^*}$ satisfies the following conditions:*

(i') *for each $y \in B$, the mapping $F(\,.\,, y) : A \to 2^{X^*}$ is monotone and upper semicontinuous on line segments in A to the weak topology of X^*;*

(ii') *for each $x \in A$, the mapping $F(x, \,.\,) : B \to 2^{X^*}$ is weakly lower semicontinuous and satisfies the following property: if (b_α) is a net converging strongly to the element b ($b_\alpha, b \in B$) and $(u_\alpha), u_\alpha \in F(x, b_\alpha)$, is a net converging weakly to the element $u \in F(x,b)$, then there exists a subnet (u_{α_j}) converging strongly to u;*

(iii') *for each $(x,y) \in A \times B$, the set $F(x,y)$ is a weakly compact convex subset of X^*.*

Then there exist $a \in A$ and $\bar{u} \in F(a, T(a))$ such that

$$\langle \bar{u}, x - a \rangle \geq 0, \quad \forall x \in A. \tag{12}$$

PROOF. By hypothesis, the set $B_1 := \mathrm{cl}(coT(A))$ is a compact and convex subset of B, where cl denotes the closure, while *co* denotes the convex hull operation. Let F_1 be the restriction of the mapping F to the set $A \times B_1$ and let $J : Y \to 2^{Y^*}$ be the duality mapping of the space Y. By ASPLUND's theorem ([1]) we can suppose that Y^* is strictly convex. Recall that in this case, J is single-valued and *demicontinuous*, i.e. continuous from the strong topology of Y to the weak topology of Y^*. Furthermore, since J is the subdifferential of the continuous and convex function $f : Y \to \mathbb{R}$ given by

$$f(y) := \frac{1}{2}\|y\|^2,$$

it is also monotone. Define now the mapping $G_1 : A \times B_1 \to Y^*$ by $G_1(x, y) := J(y - T(x))$. The pair (F_1, G_1) satisfies conditions (i), (iii) and (iv) of Theorem 2.1. We show that (ii) is also satisfied. Indeed, let (a_α) and (b_α) be two nets converging weakly to the elements a and b, where a_α, $a \in A$ and b_α, $b \in B$, while $u_\alpha \in F_1(x, b_\alpha)$ and $v_\alpha = G_1(a_\alpha, y)$ are converging weakly to $u \in F(x, b)$ and $v = G_1(a, y)$ respectively. Since B_1 is compact, one can choose a strongly convergent subnet of (b_α) (denoted also by (b_α)). By (ii') there exists a strongly convergent subnet (denoted in the same way) of (u_α), converging to u. Therefore we have

$$\langle u, a \rangle + \langle v, b \rangle = \lim_\alpha [\langle u_\alpha, a_\alpha \rangle + \langle v_\alpha, b_\alpha \rangle],$$

hence (ii) is satisfied.

By Theorem 2.1 there exist $(a, b) \in A \times B_1$ and $\bar{u} \in F(a, b)$ such that

$$\langle \bar{u}, x - a \rangle \geq 0, \qquad \forall x \in A \tag{13}$$

and

$$\langle J(b - T(a)), y - b \rangle \geq 0, \quad \forall y \in B_1.$$

Replacing y by $T(a)$ the latter implies that $b = T(a)$. Now by (13) we obtain (12). This completes the proof. $\square$

By the previous result we easily deduce Brouwer's fixed point theorem:

Corollary 2.1. *Let K be a convex compact subset of $\mathbb{R}^n$, and let $f : K \to K$ be continuous. Then f admits a fixed point.*

Proof. Take $F : K \times K \to \mathbb{R}^n$ given by $F(x, y) := x - f(y)$. Then, by Theorem 2.2 for T the identity operator, one obtains the existence of an element $a \in K$ such that

$$\langle F(a, a), x - a\rangle = \langle a - f(a), x - a\rangle \geq 0, \quad \forall x \in K.$$

If we put $x := f(a) \in K$, the latter reduces to $a = f(a)$. □

Observe that Corollary 2.1 can also be deduced directly from Theorem 2.1.

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