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# Characterization of group homomorphisms having values in an inner product space

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**Abstract.** In this note we prove that for functions $f : G \to E$ from a group $G$ to an inner product space $E$, the inequality $\|f(xy)\| \geq \|f(x) + f(y)\|$ $(x, y \in G)$ implies $f(xy) = f(x) + f(y)$ $(x, y \in G)$, and ask the open question: Is this statement true also for strictly convex normed spaces $E$?

## 1. The main result

In this note we prove the following

**Theorem.** *For functions $f : G \to E$ from a group $G$ to a real or complex inner product space $E$, the inequality*

$$\|f(xy)\| \geq \|f(x) + f(y)\| \qquad (x, y \in G) \tag{1}$$

*implies*

$$f(xy) = f(x) + f(y) \qquad (x, y \in G). \tag{2}$$

Proof. Let $n$ denote the identity of the group $G$. With $x = y = n$ in (1) we get $f(n) = 0$, and with $y = x^{-1}$ we then have $f(x^{-1}) = -f(x)$ for all $x \in G$. Inequality (1) can be rewritten as

$$\|f(x)\|^2 + 2\,\mathrm{Re}\langle f(x), f(y)\rangle + \|f(y)\|^2 \leq \|f(xy)\|^2. \tag{3}$$

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Replacing $x$ and $y$ by $xy$ and $y^{-1}$, respectively, we get

$$\|f(xy)\|^2 + 2\,\mathrm{Re}\langle f(xy), f(y^{-1})\rangle + \|f(y^{-1})\|^2 \le \|f(x)\|^2, \quad \text{i.e.,}$$

$$\|f(xy)\|^2 - 2\,\mathrm{Re}\langle f(xy), f(y)\rangle + \|f(y)\|^2 \le \|f(x)\|^2.$$

Adding this inequality to (3) and dividing by 2 we have

$$\mathrm{Re}\langle f(x) + f(y) - f(xy), f(y)\rangle \le 0. \tag{4}$$

Replacing in (3) $x$ and $y$ by $x^{-1}$ and $xy$, respectively, we obtain

$$\|f(x^{-1})\|^2 + 2\,\mathrm{Re}\langle f(x^{-1}), f(xy)\rangle + \|f(xy)\|^2 \le \|f(y)\|^2, \quad \text{i.e.,}$$

$$\|f(x)\|^2 - 2\,\mathrm{Re}\langle f(x), f(xy)\rangle + \|f(xy)\|^2 \le \|f(y)\|^2.$$

Combining also this inequality with (3) yields

$$\mathrm{Re}\langle f(x) + f(y) - f(xy), f(x)\rangle \le 0. \tag{5}$$

Replacing here $x$ and $y$ by $xy$ and $y^{-1}$, respectively, we get

$$\mathrm{Re}\langle f(xy) - f(y) - f(x), f(xy)\rangle \le 0. \tag{6}$$

Finally, adding (4), (5), and (6), we obtain

$$\|f(x) + f(y) - f(xy)\|^2 \le 0,$$

which implies (2). □

## 2. Comments

**1.** The foregoing proof is essentially as in [5], where abelian groups $G$ had been treated. In that case, of course, (5) is the same as (4), when interchanging $x$ and $y$. In the non-abelian case it is clear that (5) can be derived in the same manner as (4) by considering in $G$ the group operation $(x, y) \to yx$.

**2.** In the talk mentioned in the title of [5] (this talk is called "paper" in the title of [4]) only the particular case $G = E = \mathbb{R}$ of the theorem had been treated (and the proof was more complicated).

**3.** Kurepa [4] proved the theorem for functions $f : G \to E$ satisfying the supplementary condition

$$f(xyz) = f(xzy) \qquad (x, y, z \in G).$$

From [4] it is also clear that the theorem does not hold for arbitrary semi-groups $G$: Consider the interval $[0, +\infty[$ with operation $+$. The function $f : [0, +\infty[ \to \mathbb{R}$ given by

$$f(x) = x^2 \qquad (x \geq 0) \tag{7}$$

satisfies $|f(x+y)| \geq |f(x)+f(y)|$, but we do not have $f(x+y) = f(x)+f(y)$ for all $x \geq 0$, $y \geq 0$.

**4.** Open problems. Let us start with an open problem from the literature. In [2] Fischer and Muszély conjectured the following.

Let $E$ be a strictly convex normed space, $G$ an arbitrary semi-group, and $f : G \to E$ such that

$$\|f(xy)\| = \|f(x) + f(y)\| \qquad (x, y \in G).$$

Then (2) holds.

The conjecture is true for inner product spaces $E$ (Fischer and Muszély [2]), and it is true for groups $G$ (Ger [3]). Both papers also contain other interesting results; already from [2] it follows that the conjecture is false for normed spaces $E$, which are not strictly convex (cf. [1] for the special case $E = C[0, 1]$). These facts, together with counterexample (7) for the additive semi-group $G = [0, +\infty[$ lead to the open question formulated in the abstract.

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