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On the stability of monomial functional equations

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Abstract. In the present paper a certain form of the Hyers–Ulam stability of monomial functional equations is studied. This kind of stability was investigated in the case of additive functions by Th. M. Rassias and Z. Gajda.

1. Introduction

The first solution for S. Ulam's stability problem concerning the functional equation $f(x+y) = f(x)+f(y)$ was given by D. H. Hyers [6] in the following form: if X and Y are real normed spaces, Y is complete and for a function $f : X \to Y$ the expression $f(x+y)-f(x)-f(y)$ is bounded by a non-negative real number ε then there exists a unique function $a : X \to Y$ satisfying $a(x+y)-a(x)-a(y) = 0$, such that the difference $f-a$ is bounded by ε. (Cf. also [8].) Th. M. Rassias [9] investigated Ulam's problem in a more general form and proved the following: if, for a function $f : X \to Y$, there exist a real number $\varepsilon \geq 0$ and an $\alpha < 1$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon\,(\|x\|^\alpha + \|y\|^\alpha) \quad (x, y \in X),$$

then there exists a unique additive mapping $a : X \to Y$ for which

$$\|f(x)-a(x)\| \leq \delta\|x\|^\alpha \quad (x \in X),$$

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where $\delta = 2/(2-2^\alpha)$. Concerning the remaining cases, Z. Gajda [2] showed that the statement above holds for $\alpha > 1$ with $\delta = 2/(2^\alpha - 2)$, but it is not valid for $\alpha = 1$ (see [10], [11], too). The stability of the so called square-norm functional equation in a similar sense was studied by St. Czerwik [1].

In the present work we consider the above stability problem for monomial functional equations. Throughout the paper Δ denotes the difference operator, which is defined, for a function f mapping from a linear space X into Y and for a positive integer n, by

$$\Delta_y^1 f(x) = f(x+y) - f(x) \quad (x, y \in X)$$

and

$$\Delta_y^{n+1} f(x) = \Delta_y^1 \Delta_y^n f(x) \qquad (x, y \in X).$$

We call f a monomial function of degree n if

$$\Delta_y^n f(x) - n! f(y) = 0 \quad (x, y \in X).$$

Using this termimology we prove that if n is a positive integer, $\alpha \neq n$ is a real number, the normed space Y is complete and, for a function $f : X \to Y$, there exists a non-negative real number ε such that

$$\|\Delta_y^n f(x) - n! f(y)\| \leq \varepsilon \left(\|x\|^\alpha + \|y\|^\alpha\right) \quad (x, y \in X),$$

then there exists a real constant c and a unique monomial function $g : X \to Y$ of degree n for which

$$\|f(x) - g(x)\| \leq c\varepsilon \|x\|^\alpha \quad (x \in X).$$

Additionally, we show that a weak regularity condition for f implies that g is homogeneous of degree n, i.e. $g(tx) = t^n g(x)$ for all $t \in \mathbb{R}$ and $x \in X$. Moreover, by giving some counterexamples, we verify that the statement above does not hold in the case when $\alpha = n$. Obviously, our results for $n = 1$ yield Th. M Rassias' and Z. Gajda's stability theorems, in the case when $n = 2$ they imply the stability of the square-norm equation in the sense investigated in this paper (cf. [1]), furthermore, for $\alpha = 0$ they give the known Hyers–Ulam stability of monomial functional equations ([12], [5]; cf. also [7]).

2. Results and proofs

Lemma 1. *For $n, \lambda \in \mathbb{N}$, $\lambda \geq 2$ write*

$$A = \begin{pmatrix} \alpha_0^{(0)} & \dots & \alpha_0^{(\lambda n)} \\ \vdots & \ddots & \vdots \\ \alpha_{(\lambda-1)n}^{(0)} & \dots & \alpha_{(\lambda-1)n}^{(\lambda n)} \end{pmatrix},$$

where for $i = 0, \dots, (\lambda-1)n$ and $j = -i, \dots, \lambda n - i$

$$\alpha_i^{(i+j)} = \begin{cases} (-1)^{n-j}\binom{n}{j}, & \text{if } 0 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Let a_i denote the i^{th} row in A, $(i = 0, \dots, (\lambda-1)n)$. Furthermore, let $b = (\beta^{(0)} \dots \beta^{(\lambda n)})$, where

$$\beta^{(j)} = \begin{cases} (-1)^{n-\frac{j}{\lambda}}\binom{n}{\frac{j}{\lambda}}, & \text{if } \lambda \mid j, \\ 0, & \text{if } \lambda \nmid j, \end{cases}$$

for $j = 0, \dots, \lambda n$. Then there exist positive integers $K_0, \dots, K_{(\lambda-1)n}$ such that

$$K_0 a_0 + \dots + K_{(\lambda-1)n} a_{(\lambda-1)n} = b$$

and

$$K_0 + \dots + K_{(\lambda-1)n} = \lambda^n.$$

Proof. Cf. [3] and [4]. □

Lemma 2. *Let X and Y be linear normed spaces, $f : X \to Y$ be a function, n be a positive integer, and α be a real number. If there exists a non-negative real number ε such that*

$$\|\Delta_y^n f(x) - n! f(y)\| \leq \varepsilon \left(\|x\|^\alpha + \|y\|^\alpha\right) \quad (x, y \in X), \tag{1}$$

then, for any positive integer l, there exists a real number $c_l = c(l, n, \alpha)$ for which

$$\|f(lx) - l^n f(x)\| \leq c_l \varepsilon \|x\|^\alpha \quad (x \in X). \tag{2}$$

(Here $0^\alpha = 0$ for $\alpha \neq 0$ and $0^0 = 1$.)

PROOF. Let $\alpha \neq 0$ and $n, l \in \mathbb{N}$ be given and let $f : X \to Y$ satisfy (1). Our statement is trivial for $l = 1$, so we suppose that $l \geq 2$. We define, for $i = 0, \dots, (l-1)n$, the functions $F_i : X \to Y$ by

$$F_i(z) = \Delta_z^n f(iz) - n! f(z) \quad (z \in X)$$

and the function $G : X \to Y$ by

$$G(z) = \Delta_{lz}^n f(0) - n! f(lz) \quad (z \in X).$$

If we replace (x, y) by $(0, z), (z, z), \dots, ((l-1)nz, z)$ in (1) we get

$$\|F_i(z)\| \leq (i^\alpha + 1)\varepsilon \|z\|^\alpha \quad (i = 0, \dots, (l-1)n,\ z \in X). \tag{3}$$

Writing $x = 0$ and $y = lz$ in (1) yields

$$\|G(z)\| \leq l^\alpha \varepsilon \|z\|^\alpha \quad (z \in X). \tag{4}$$

With the notation of Lemma 1 for $\lambda = l$ we have

$$F_i(z) = \sum_{j=0}^{ln} \alpha_i^{(j)} f(jz) - n! f(z) \quad (i = 0, \dots, (l-1)n,\ z \in X)$$

and

$$G(z) = \sum_{j=0}^{ln} \beta^{(j)} f(jz) - n! f(lz) \quad (z \in X).$$

By Lemma 1 there exist positive integers $K_0, \dots, K_{(l-1)n}$ such that

$$K_0 + \cdots + K_{(l-1)n} = l^n$$

and

$$G(z) = K_0 F_0(z) + \cdots + K_{(l-1)n} F_{(l-1)n}(z) + l^n n! f(z) - n! f(lz) \quad (z \in X).$$

Therefore, using (3) and (4) we get

$$\|l^n f(z) - f(lz)\| \leq \frac{l^\alpha + \sum_{i=0}^{(l-1)n} K_i (i^\alpha + 1)}{n!} \varepsilon \|z\|^\alpha \quad (z \in X) \tag{5}$$

which implies our statement in the case when $\alpha \neq 0$.

If $\alpha = 0$ we get the following in a similar way

$$\|l^n f(z) - f(lz)\| \le \frac{2(l^n+1)}{n!}\varepsilon\|z\|^\alpha \quad (z \in X). \tag{6}$$ □

Theorem 1. *Let X be a linear normed space, Y be a Banach space, n be a positive integer and $\alpha \neq n$ be a real number. If, for a function $f : X \to Y$, there exists a non-negative real number ε with the property*

$$\|\Delta_y^n f(x) - n! f(y)\| \le \varepsilon\,(\|x\|^\alpha + \|y\|^\alpha) \quad (x, y \in X), \tag{7}$$

then there exists a real number $c = c(n, \alpha)$ and a monomial function $g : X \to Y$ of degree n such that

$$\|f(x) - g(x)\| \le c\varepsilon\|x\|^\alpha \quad (x \in X). \tag{8}$$

Moreover, there is only one monomial function of degree n, for which there exists a $c \in \mathbb{R}$ with this property. If, for each fixed $x \in X$, there exists a measurable, bounded set $M_x \subseteq \mathbb{R}$ with positive Lebesgue measure such that the function $h : \mathbb{R} \to Y$, $h(t) = f(tx)$ is bounded over M_x, then the mapping g is homogeneous of degree n.

Proof. I. At first we prove the existence part of the theorem in the case when $\alpha < n$. Let $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\alpha < n$ be given and let f satisfy (7). By Lemma 2, for a fixed integer $l \ge 2$, there exists a $c_l \in \mathbb{R}$ such that

$$\left\| \frac{1}{l^n} f(lx) - f(x) \right\| \le \frac{1}{l^n} c_l \varepsilon \|x\|^\alpha \quad (x \in X).$$

It can be verified by induction on m and using the triangle inequality that

$$\left\| \frac{f(l^m x)}{l^{mn}} - f(x) \right\| \le c_l \varepsilon l^{-n} \left(\sum_{j=0}^{m-1} l^{j(\alpha-n)} \right) \|x\|^\alpha \quad (x \in X,\ m \in \mathbb{N}). \tag{9}$$

Let us consider the functions $g_m : X \to Y$

$$g_m(x) = \frac{f(l^m x)}{l^{mn}} \quad (x \in X,\ m \in \mathbb{N}).$$

Because of $\alpha < n$ we have

$$\sum_{j=0}^{\infty} l^{j(\alpha-n)} = \frac{l^n}{l^n - l^\alpha},$$

thus,

$$\|g_k(x) - g_m(x)\| \le l^{m(\alpha-n)} c_l \varepsilon \frac{1}{l^n - l^\alpha} \|x\|^\alpha \quad (x \in X),$$

for $k, m \in \mathbb{N}$, $k > m$. Therefore, $(g_k(x))$ is a Cauchy sequence for each fixed $x \in X$. Since Y is complete, the definition $g : X \to Y$

$$g(x) = \lim_{m\to\infty} g_m(x) \quad (x \in X)$$

is correct. Assumption (7) gives

$$\|\Delta^n_{l^m y} f(l^m x) - n! f(l^m y)\| \le l^{\alpha m} \varepsilon \left(\|x\|^\alpha + \|y\|^\alpha\right) \quad (x, y \in X,\ m \in \mathbb{N}).$$

Dividing this inequality by l^{mn} and taking $m \to \infty$ we obtain

$$\Delta^n_y g(x) - n! g(y) = 0 \quad (x, y \in X),$$

thus, g is a monomial function of degree n. From (9) we get

$$\left\| \frac{f(l^m x)}{l^{mn}} - f(x) \right\| \le c_l \varepsilon \frac{1}{l^n - l^\alpha} \|x\|^\alpha \quad (x \in X,\ m \in \mathbb{N}),$$

therefore,

$$\|g(x) - f(x)\| \le c_l \varepsilon \frac{1}{l^n - l^\alpha} \|x\|^\alpha \quad (x \in X), \tag{10}$$

that is, (8) holds.

II. Let now $n \in \mathbb{N}$, $\alpha > n$ and we choose an arbitrary integer $l \ge 2$. By Lemma 2, for a function $f : X \to Y$ satisfying (7), we have

$$\left\| f(x) - l^n f\left(\frac{x}{l}\right) \right\| \le \frac{1}{l^\alpha} c_l \varepsilon \|x\|^\alpha \quad (x \in X).$$

It can be shown by induction on m that

$$\left\| l^{mn} f\left(\frac{x}{l^m}\right) - f(x) \right\| \le l^{-n} c_l \varepsilon \left(\sum_{j=1}^{m} l^{j(n-\alpha)} \right) \|x\|^\alpha \quad (x \in X,\ m \in \mathbb{N}). \tag{11}$$

Since $\alpha > n$ we have

$$\sum_{j=1}^{\infty} l^{j(n-\alpha)} = \frac{l^n}{l^\alpha - l^n},$$

therefore, for the functions $g_m : X \to Y$

$$g_m(x) = l^{mn} f\left(\frac{x}{l^m}\right) \quad (x \in X,\ m \in \mathbb{N}),$$

we get

$$\|g_k(x) - g_m(x)\| \le l^{m(n-\alpha)} c_l \varepsilon \frac{1}{l^\alpha - l^n} \|x\|^\alpha \quad (x \in X),$$

for all $k, m \in \mathbb{N}$, $k > m$. Thus, $(g_m(x))$ is a Cauchy-sequence for each $x \in X$, so we define the function $g : X \to Y$ by

$$g(x) = \lim_{m\to\infty} g_m(x) \quad (x \in X).$$

By (7)

$$\left\|\Delta^n_{\frac{y}{l^m}} f\left(\frac{x}{l^m}\right) - n! f\left(\frac{y}{l^m}\right)\right\| \le \frac{1}{l^{\alpha m}} \varepsilon \left(\|x\|^\alpha + \|y\|^\alpha\right) \quad (x, y \in X,\ m \in \mathbb{N}).$$

Multiplying this relation by l^{mn} and with $m \to \infty$ we get that g is a monomial function of degree n. Property (11) yields

$$\left\|l^{mn} f\left(\frac{x}{l^m}\right) - f(x)\right\| \le c_l \varepsilon \frac{1}{l^\alpha - l^n} \|x\|^\alpha \quad (x \in X,\ m \in \mathbb{N}),$$

thus,

$$\|g(x) - f(x)\| \le c_l \varepsilon \frac{1}{l^\alpha - l^n} \|x\|^\alpha \quad (x \in X), \tag{12}$$

that is, (8) holds in this case, too.

III. To prove uniqueness we suppose that $g, \bar{g} : X \to Y$ are different monomial functions of degree n such that

$$\|f(x) - g(x)\| \le c\varepsilon \|x\|^\alpha \quad (x \in X)$$

and

$$\|f(x) - \bar{g}(x)\| \le \bar{c}\varepsilon \|x\|^\alpha \quad (x \in X)$$

where $c, \bar{c} \in \mathbb{R}$ are fixed. Using the triangle inequality we get

$$\|g(x) - \bar{g}(x)\| \le (c + \bar{c})\varepsilon \|x\|^\alpha \quad (x \in X). \tag{13}$$

The functions g and $\bar{g}$ are different, so there exists a $y \in X$ for which $g(y) \neq \bar{g}(y)$. Furthermore, there exists a positive rational number r such that

$$r^{n-\alpha} > \frac{(c+\bar{c})\varepsilon\|y\|^\alpha}{\|g(y)-\bar{g}(y)\|}.$$

Since g and $\bar{g}$ are monomial functions of degree n, this inequality implies

$$\|g(ry)-\bar{g}(ry)\| > (c+\bar{c})\varepsilon\|ry\|^\alpha$$

which is a contradiction to (13).

IV. Finally we prove the last statement of the theorem. Let $x \in X$ be fixed and let $M_x \subseteq \mathbb{R}$ denote the bounded set with positive Lebesgue measure over which the mapping $h : \mathbb{R} \to Y$, $h(t) = f(tx)$ is bounded. Let us consider a $\varphi \in Y^*$ (where Y^* is the dual space of Y) and we define a function $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(t) = \varphi(g(tx))$. Since φ is linear, ψ is a monomial function of degree n, i.e.

$$\Delta_s^n \psi(t) - n!\psi(s) = 0 \quad (t, s \in \mathbb{R}).$$

Moreover,

$$|\psi(t)| \leq \|\varphi\| \left(\|g(tx)-f(tx)\| + \|f(tx)\|\right) \quad (t \in M_x). \tag{14}$$

By (8) we have

$$\|g(tx)-f(tx)\| \leq c\varepsilon K^\alpha \|x\|^\alpha \quad (t \in M_x)$$

where K denotes a bound for M_x. Furthermore, the function h is bounded over M_x, so inequality (14) gives that ψ is bounded over M_x. Thus, ψ is a real monomial function of degree n bounded over a set with positive Lebesgue measure, therefore it has the form $\psi(t) = \psi(1)t^n$ $(t \in \mathbb{R})$ (cf. [13]), which implies that g is homogeneous of degree n. □

Theorem 2. *Let n be a positive integer, ε be a positive real number and let*

$$\varepsilon^* = \frac{\varepsilon}{2^n(2^n+n!)n^n}.$$

We consider the mapping $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(x) = \begin{cases} n^n\varepsilon^*, & \text{if } x \geq n, \\ \varepsilon^* x^n, & \text{if } -n < x < n, \\ (-1)^n n^n \varepsilon^*, & \text{if } x \leq -n, \end{cases}$$

and, for a fixed integer $l \geq 2$, we define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{m=0}^{\infty} \frac{\varphi(l^m x)}{l^{mn}} \quad (x \in \mathbb{R}). \tag{15}$$

For this function we have

$$|\Delta_y^n f(x) - n! f(y)| \leq \varepsilon \left(|x|^n + |y|^n\right) \quad (x, y \in \mathbb{R}) \tag{16}$$

but there does not exist a real number $c = c(n, \alpha)$ for which there exists a monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree n such that

$$|f(x) - g(x)| \leq c\varepsilon |x|^n \quad (x \in \mathbb{R}). \tag{17}$$

Proof. Let $n, l \in \mathbb{N}$, $l \geq 2$, $\varepsilon > 0$ and we define $\varepsilon^* \in \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ as above. We have

$$|\varphi(x)| \leq n^n \varepsilon^* \quad (x \in \mathbb{R}),$$

therefore, the definition of the function $f : \mathbb{R} \to \mathbb{R}$ in (15) is correct, furthermore

$$|f(x)| \leq \sum_{m=0}^{\infty} \frac{n^n \varepsilon^*}{l^{mn}} \leq 2n^n \varepsilon^* \quad (x \in \mathbb{R}).$$

Since φ is continuous and the convergence in (15) is uniform, f is continuous, too.

We show that f satisfies inequality (16). In the case when $x = y = 0$, property (16) holds trivially. If $x, y \in \mathbb{R}$ are fixed and $0 < |x| + |y| < 1$ then there exists a positive integer m_0 such that

$$\frac{1}{l^{m_0}} \leq |x| + |y| < \frac{1}{l^{m_0 - 1}}.$$

Therefore, $|l^{m_0 - 1} y| < 1$ and $|l^{m_0 - 1}(x + ky)| < n$ for $k = 0, \ldots, n$. Since φ is a monomial function of degree n on the interval $(-n, n)$, we have

$$\Delta_{l^m y}^n \varphi(l^m x) - n! \varphi(l^m y) = 0$$

for $m = 0, \ldots, m_0 - 1$. Thus, using the well-known identity

$$\Delta_s^n \varphi(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \varphi(t + ks) \quad (t, s \in \mathbb{R})$$

and the property that $l^{m_0}(|x|+|y|) \geq 1$ implies $|l^{m_0}x|^n+|l^{m_0}y|^n \geq 2^{-(n-1)}$, we get

$$\begin{aligned}\frac{|\Delta_y^n f(x)-n!f(y)|}{|x|^n+|y|^n} &\leq \sum_{m=0}^{\infty}\frac{|\Delta_{l^m y}^n\varphi(l^m x)-n!\varphi(l^m y)|}{l^{mn}(|x|^n+|y|^n)}\\ &\leq \sum_{m=m_0}^{\infty}\frac{(2^n+n!)n^n\varepsilon^*}{l^{mn}(|x|^n+|y|^n)}=\sum_{m=0}^{\infty}\frac{(2^n+n!)n^n\varepsilon^*}{l^{mn}(|l^{m_0}x|^n+|l^{m_0}y|^n)}\\ &\leq \sum_{m=0}^{\infty}\frac{(2^n+n!)n^n\varepsilon^*}{l^{mn}2^{-(n-1)}}\leq\varepsilon.\end{aligned}$$

If $|x|+|y|\geq 1$ then $|x|^n+|y|^n\geq 2^{-(n-1)}$, therefore,

$$\frac{|\Delta_y^n f(x)-n!f(y)|}{|x|^n+|y|^n}\leq 2^{n-1}(2^n+n!)2n^n\varepsilon^*=\varepsilon$$

which proves (16).

Finally, we suppose that there exists a $c\in\mathbb{R}$ and a monomial function $g:\mathbb{R}\to\mathbb{R}$ of degree n satisfying (17). Since the function f is continuous, (17) implies that g is bounded over an interval of positive length, so it has the form

$$g(x)=\gamma x^n \quad (x\in\mathbb{R}),$$

where γ is a real constant. Furthermore, (17) gives

$$|f(x)-\gamma x^n|\leq c\varepsilon|x|^n \quad (x\in\mathbb{R}),$$

therefore,

$$\left|\frac{f(x)}{x^n}\right|-|\gamma|\leq c\varepsilon \quad (x\in\mathbb{R}). \tag{18}$$

However, there exists a positive integer m_0 such that $m_0\varepsilon^*>c\varepsilon+|\gamma|$ and for an arbitrary $x\in\left(0,\frac{n}{l^{m_0-1}}\right)$ we have $l^m x\in(0,n)$ for $m=0,\dots,m_0-1$, thus,

$$\left|\frac{f(x)}{x^n}\right|=\frac{f(x)}{x^n}\geq\sum_{m=0}^{m_0-1}\frac{\varphi(l^m x)}{l^{mn}x^n}=\sum_{m=0}^{m_0-1}\frac{\varepsilon^* l^{mn}x^n}{l^{mn}x^n}=m_0\varepsilon^*>c\varepsilon+|\gamma|$$

which contradicts (18). □

Remark 1. The positive integer $l \geq 2$ in the proofs of Theorem 1 and Theorem 2 can be chosen arbitrarily. Thus, it would be enough to verify Lemma 1 and Lemma 2 for $\lambda = 2$ and $l = 2$ and to take $l = 2$ in the proofs of Theorem 1 and 2, to get our main results. However, the proofs are not more complicated this way, so we give them in this more general form.

Remark 2. It is easy to see that with the help of formulas (5), (6), (10), and (12) the constants $c = c(n, \alpha) \in \mathbb{R}$ mentioned in Theorem 1 can be given exactly. E.g., for an arbitrary $n \in \mathbb{N}$, taking $l = 2$ and using the simple property that, for $\lambda = 2$, $K_i = \binom{n}{i}$ $(i = 0, \ldots, n)$ in Lemma 1 (cf. [4]) we get

$$c = \frac{2^\alpha + \sum_{i=0}^{n} \binom{n}{i}(i^\alpha + 1)}{n!} \frac{1}{2^n - 2^\alpha}$$

for $\alpha < n$, $\alpha \neq 0$ and

$$c = \frac{2^\alpha + \sum_{i=0}^{n} \binom{n}{i}(i^\alpha + 1)}{n!} \frac{1}{2^\alpha - 2^n}$$

for $\alpha > n$. In the cases of some special integers n these constants can be reduced. (E.g. in the case when $n = 1$, taking only $x = y = z$ in (3) we get the constants $c = 2(2 - 2^\alpha)^{-1}$ and $c = 2(2^\alpha - 2)^{-1}$.) This fact leaves the problem of the "best constant" concerning the stability studied in the present paper open.

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