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On Finsler spaces of Douglas type IV: Projectively flat Kropina spaces

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1. Introduction

We have a remarkably interesting two (α, β)-metrics; one is the Randers metric $L = \alpha + \beta$ and the other is the Kropina metric $L = \alpha^2/\beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$ ([1], [6]). A Randers space is a Finsler space with the Randers metric. It is projectively flat, if and only if its associated Riemannian space with the metric α is projectively flat and the change of the metric $\alpha \to \alpha + \beta$ is projective ([4, Theorem 4.3]; [7, Theorem 2]). On the other hand, the condition for a Kropina space (i.e. for a Finsler space with the Kropina metric), to be projectively flat has been given until now only in an unsatisfactory form, asserting the existence of some coordinate systems ([7, Theorem 3]).

The purpose of the present paper is to give this condition for a Kropina space in a completely tensorial form. We obtain it as an application of a new formulation of the theorem on the projective flatness ([3, II]).

We enumerate here some symbols for the later use. $K^n = (M^n, L = \alpha^2/\beta)$ is a Kropina space on a smooth n-manifold M^n, $R^n = (M^n, \alpha)$ is the Riemannian space associated with K^n. Let (a^{ij}) and $(\gamma_j{}^i{}_k)$ be the inverse matrix of (a_{ij}) and the Christoffel symbols constructed from (a_{ij}).

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The semi-colon denotes the covariant differentiation with respect to $(\gamma_j{}^i{}_k)$.

$$\begin{aligned}
y_i &= a_{ir}y^r, & b^i &= a^{ir}b_r, & b^2 &= b_r b^r,\\
2r_{ij} &= b_{i;j} + b_{j;i}, & 2s_{ij} &= b_{i;j} - b_{j;i}, & s^i{}_j &= a^{ir}s_{rj},\\
s_i &= b^r s_{rj}, & s^i &= a^{ir}s_r.
\end{aligned}$$

The change $\alpha \to \alpha + \beta$ is projective if and only if $s_{ij} = 0$, that is, the covariant vector field b_i is gradient. The transvection by y^i is denoted by subscript 0.

2. Kropina spaces of Berwald type and of Douglas type

It is well-known that if a Finsler space is projectively flat, then it must be of Douglas type ([3, II]). As it has been recently shown [10], a Kropina space K^n is of Douglas type, if and only if

$$\sigma_{ij} = s_{ij} - (b_i s_j - b_j s_i)/b^2$$

vanishes identically.

It is natural to assume $b^2 \neq 0$ for K^n. In fact, K^n has the fundamental tensor ([12], [3, I]),

$$g_{ij} = 2\tau a_{ij} + 3\tau^2 b_i b_j - (4\tau/\beta)(b_i y_j + b_j y_i) + (4/\beta^2)y_i y_j,$$

where $\tau = (\alpha/\beta)^2$, and $\det(g_{ij}) = 2^{n-1}\tau^{n+2}b^2\det(a_{ij})$ [6]. Thus, on the assumption $b^2 \neq 0$ we get the inverse (g^{ij}) of (g_{ij}) as

$$\begin{aligned}
g^{ij} = {} & (\rho/2)a^{ij} - (\rho/2b^2)b^i b^j + (\rho^2/b^2\beta)(b^i y^j + b^j y^i)\\
& + (\rho/\beta)^2(1 - 2\rho^2/b^2)y^i y^j,
\end{aligned}$$

where $\rho = (\beta/\alpha)^2$.

We consider the Berwald connection $B\Gamma = \{G^i{}_j, G_j{}^i{}_k, 0\}$ of K^n, where $G^i{}_j = \dot\partial_j G^i$ and $G_j{}^i{}_k = \dot\partial_k G^i{}_j$. If we put $G^i = (\gamma_0{}^i{}_0/2) + B^i$ for K^n then we have [9]

$$(2.1)\quad \begin{aligned}
&(1) \qquad B^i = B_*{}^i + (r_{00}/2b^2)b^i - (s_0/b^2)y^i,\\
&(2) \qquad B_*{}^i = (\alpha^2/2\beta)\{(s_0/b_2)b^i - s^i{}_0\} - (r_{00}\beta/b^2\alpha^2)y^i.
\end{aligned}$$

Let now K^n be a Berwald space, that is let, $G_j{}^i{}_k$ be functions of (x^i) alone: This means that G^i are homogeneous polymonials in (y^i) of degree two. Thus we must take account of only the terms $B_*{}^i$ of G^i. (2) of (2.1) may be written as

$$2r_{00}\beta^2 y^i = \alpha^2\{\alpha^2(s_0 b^i - b^2 s^i{}_0) - 2b^2\beta B_*{}^i\}.$$

Assume that K^n is of Berwald type. As it has been shown in [2], $b^2 \neq 0$ implies $\alpha^2 \not\equiv 0 \pmod{\beta}$, and hence the above equation demands the existence of $u^i{}_0 = u^i{}_r(x)y^r$ satisfying

$$\text{(2.2)} \qquad \begin{aligned} &\text{(a)} \qquad \alpha^2(s_0 b^i - b^2 s^i{}_0) - 2b^2\beta B_*{}^i = \beta^2 u^i{}_0,\\ &\text{(b)} \qquad 2r_{00}y^i = \alpha^2 u^i{}_0. \end{aligned}$$

The latter gives $2r_{00}\beta = \alpha^2 b_r u^r{}_0$, which implies $b_r u^r{}_0 = 2u\beta$ for some $u(x)$. Thus (2.2b) yields

$$\text{(2.3)} \qquad (1) \quad r_{00} = u\alpha^2, \qquad (2) \quad u^i{}_0 = 2uy^i.$$

Then (2.2a) can be written as

$$\alpha^2(s_0 b^i - b^2 s^i{}_0) = 2\beta(u\beta y^i + b^2 B_*{}^i),$$

which shows the existence of $v^i(x)$ such that

$$s_0 b^i - b^2 s^i{}_0 = \beta v^i, \qquad 2(u\beta y^i + b^2 B_*{}^i) = \alpha^2 v^i.$$

The latter proves that $B_*{}^i$ are now homogeneous polynomials in (y^i) of degree two. The former is equivalent to $s_j b_i - b^2 s_{ij} = b_j v_i$, and the skew-symmetry of s_{ij} gives $b_i(s_j - v_j) = -b_j(s_i - v_i)$, which easily implies $v_i = s_i$. Consequently we obtain

$$\text{(2.4)} \qquad (1) \quad r_{ij} = ua_{ij}, \qquad (2) \quad s_{ij} = (b_i s_j - b_j s_i)/b^2.$$

Therefore we have

Theorem 1. *A Kropina space is of Berwald type, if and only if* (2.4) *holds, that is,*

$$\text{(2.4')} \qquad b_{i;j} = ua_{ij} + (b_i s_j - b_j s_i)/b^2.$$

Remark. The condition for a Kropina space to be of Berwald type has been given first by C. Shibata [12] and then S. Kikuchi [5] and one of the present authors [8]. Though they obtained their results in somewhat different forms, all of them can be expressed in the form

$$b_{i;j} = (f_r b^r)a_{ij} + b_i f_j - b_j f_i, \tag{2.5}$$

where f_i are some functions of (x^i), that is, $r_{ij} = (f_r b^r)a_{ij}$ and $s_{ij} = b_i f_j - b_j f_i$. From (2.5) we get $s_j = (b^r f_r)b_j - b^2 f_j$ and hence (2) of (2.4). If we put $u = b^r f_r$, then (1) of (2.4) is obtained. Conversely, if we put $f_i = (s_i + ub_i)/b^2$, then (2.4') is rewritten in the form (2.5).

Therefore a Kropina space K^n is of Douglas type, if and only if $\sigma_{ij} = 0$, and further it is of Berwald type, if and only if r_{00} is proportional to α^2 and $\sigma_{ij} = 0$.

We shall restrict our discussions to the two-dimensional case. If $s_{ij} = 0$, then $s_i = 0$ and hence $\sigma_{ij} = 0$. If $s_{ij} \neq 0$, that is, $s_{12} \neq 0$, then we put $(b_1 s_2 - b_2 s_1)/s_{12} = k$, which is nothing but $b_i s_j - b_j s_i = k s_{ij}$, $i, j = 1, 2$, and $k = 1/b^2$ is easily obtained. Thus we have

Theorem 2. *Any Kropina space of dimension two has $\sigma_{ij} = 0$ and is of Douglas type.*

Remark. The result that "any Kropina space of dimension two is a Douglas space" has been proved in ([3, I]) in a completely different way. This fact is also shown by the differential equation of geodesics [11], (3.3), in conformity with the definition of a Douglas space.

3. An outline of the theory of projective flatness

We shall sketch the theory of projective flatness which has been developed by the present authors in ([3, II]), and we apply it here to the Riemannian space $R^n = (M^n, \alpha)$ associated with a Kropina space $K^n = (M^n, L = \alpha^2/\beta)$. All the quantities of R^n will be denoted by putting the superscript $^\circ$.

We start our theory with the projective invariant, Q^0:

$$Q^h = G^h - Gy^h/(n+1), \qquad G = G^r{}_r.$$

Since G^h of R^n is equal to $\gamma_0{}^h{}_0/2$, the Q^0-invariant of R^n is written as

$$Q^{\circ h} = \frac{1}{2}\gamma_0{}^h{}_0 - \gamma_0{}^r{}_r y^h/(n+1). \tag{3.1}$$

From Q^h we construct the Q^1-invariant $Q^h{}_i = \dot{\partial}_i Q^h$ and the Q^2-invariant $Q_i{}^h{}_j = \dot{\partial}_j Q^h{}_i$. In R^n the latter is given by

$$Q^\circ{}_i{}^h{}_j = \gamma_i{}^h{}_j - \gamma_i \delta^h{}_j - \gamma_j \delta^h{}_i, \qquad \gamma_i = \gamma_i{}^r{}_r/(n+1). \tag{3.2}$$

Further we obtain the Q^3-invariant

$$Q_i{}^h{}_{jk} = \partial_k Q_i{}^h{}_j - (\dot{\partial}_r Q_i{}^h{}_j)Q^r{}_k + Q_i{}^r{}_j Q_r{}^h{}_k - (j/k),$$

where (j/k) denotes the interchange of subscripts j, k of the preceding terms. The Q^3-invariant of R^n is given by

$$\begin{aligned} Q^\circ{}_i{}^h{}_{jk} &= R^\circ{}_i{}^h{}_{jk} - \delta^h{}_j \gamma_{ik} + \delta^h{}_k \gamma_{ij}, \\ \gamma_{ij} &= \partial_j \gamma_i - \gamma_i{}^r{}_j \gamma_r + \gamma_i \gamma_j, \end{aligned} \tag{3.3}$$

where $R^\circ{}_i{}^h{}_{jk}$ is the curvature tensor of R^n. The symmetry of the Ricci tensor $R^\circ{}_{ij}(= R^\circ{}_i{}^r{}_{jr})$ implies the symmetry of γ_{ij}. We get another invariant $Q_{ij} = Q_i{}^r{}_{jr}$. In R^n we have

$$Q^\circ{}_{ij} = R^\circ{}_{ij} + (n-1)\gamma_{ij}. \tag{3.4}$$

Then the Π-tensor is constructed as

$$\Pi_i{}^h{}_{jk} = Q_i{}^h{}_{jk} + \{\delta^h{}_j Q_{ik} - (j/k)\}/(n-1),$$

which is nothing but the Weyl projective curvature tensor W. The Π-tensor of R^n is given in the well-known form

$$\Pi^\circ{}_i{}^h{}_{jk} = R^\circ{}_i{}^h{}_{jk} + \{\delta^h{}_j R^\circ{}_{ik} - (j/k)\}/(n-1). \tag{3.5}$$

Next, for the use in the two-dimensional case, we define

$$\Pi_{ijk} = \delta_k Q_{ij} + Q_i{}^r{}_j Q_{rk} - (j/k),$$

where $\delta_k = \partial_k - G^r{}_k \dot{\partial}_r$. For R^n we have

$$\Pi^\circ{}_{ijk} = R^\circ{}_{ij;k} - R^\circ{}_{ik;j} - (n-1)\gamma_r \Pi^\circ{}_i{}^r{}_{jk}. \tag{3.6}$$

Since the Weyl projective curvature tensor vanishes identically in the two-dimensional case, in the case of $n = 2$ we obtain for (3.6) the following

$$\Pi^\circ{}_{ijk} = R^\circ{}_{ij;k} - R^\circ{}_{ik;j}. \tag{3.6$_2$}$$

Now we give a new formulation to the fundamental theorem of projective flatness considered already in ([3, II]).

Theorem BM. *A Finsler space F^n of dimension n is projectively flat if and only if F^n is a Douglas space and*

(1) $n > 2:$ *the* Π*-tensor* $= 0,$ *(2)* $n = 2 : \Pi_{ijk} = 0.$

4. Projectively flat Kropina spaces

We concider now an n-dimensional Kropina space $K^n = (M^n, L = \alpha^2/\beta)$. Let $R^n = (M^n, \alpha)$ be the Riemannian space associated with K^n. All the quantities of R^n will be denoted by putting the superscript $\circ$ as in the last section. Since our purpose is to find the condition for K^n to be projectively flat, it should be assumed first that K^n be a Douglas space, and hence $s_{ij} = (b_i s_j - b_j s_i)/b^2$ according to Theorem BM. Then $s^h{}_0 = (s_0 b^h - \beta s^h)/b^2$ and B^h of (2.1) is reduced to

$$B^h = (\alpha^2 s^h + r_{00} b^h)/2b^2 - (\alpha^2 s_0 + \beta r_{00}) y^h / b^2 \alpha^2.$$

If we define the tensor

$$k_i{}^h{}_j = (a_{ij} s^h + r_{ij} b^h)/b^2, \tag{4.1}$$

then $k_0{}^h{}_0 = (\alpha^2 s^h + r_{00} b^h)/b^2$ and $k_{000} (= k_0{}^h{}_0 y_h) = (\alpha^2 s_0 + \beta r_{00})/b^2$. Hence B^h can be simply written as

$$B^h = k_0{}^h{}_0/2 - k_{000} y^h/\alpha^2,$$

from which we get $B^h{}_i = \dot\partial_i B^h$ and $B = B^r{}_r$ in the forms

$$B^h{}_i = k_0{}^h{}_i - \{(k_{0i0} + 2k_{00i}) y^h - (2k_{000}/\alpha^2) y_i y^h + k_{000} \delta^h{}_i\}/\alpha^2,$$
$$B = k_0{}^r{}_r - (n+1) k_{000}/\alpha^2.$$

Thus (3.1) leads to

$$Q^h = Q^{\circ h} + k_0{}^h{}_0/2 - k_0{}^r{}_r y^h/(n+1). \tag{4.2}$$

As it was remarked in ([3, II]), $Q^h(x,y)$ are certainly homogeneous polynomials in (y^i) of degree two for the Kropina space K^n of Douglas type.

From (4.2) and (3.2) we obtain

$$Q_i{}^h{}_j = Q^\circ{}_i{}^h{}_j + K_i{}^h{}_j, \tag{4.3}$$

where $K_i{}^h{}_j$ is the tensor defined by

$$K_i{}^h{}_j = k_i{}^h{}_j - (k_i{}^r{}_r\delta^h{}_j + k_j{}^r{}_r\delta^h{}_i)/(n+1). \tag{4.3a}$$

We remark that $K_i{}^r{}_r = 0$.

Then (4.3) and (3.3) yield the Q^3-invariants of K^n in the form

$$Q_i{}^h{}_{jk} = Q^\circ{}_i{}^h{}_{jk} + K_i{}^h{}_{jk} + (\delta^h{}_j K_i{}^r{}_k - \delta^h{}_k K_i{}^r{}_j)\gamma_r, \tag{4.4}$$

where $K_i{}^h{}_{jk}$ is the (1,3)-type tensor defined by

$$K_i{}^h{}_{jk} = K_i{}^h{}_{j;k} + K_i{}^r{}_j K_r{}^h{}_k - (j/k). \tag{4.4a}$$

(4.4) and (3.4) lead to

$$Q_{ij} = Q^\circ{}_{ij} + K_{ij} - (n-1)K_i{}^r{}_j\gamma_r, \tag{4.5}$$

where $K_{ij} = K_i{}^r{}_{jr}$ is the symmetric tensor given by

$$K_{ij} = K_i{}^r{}_{j;r} - K_i{}^r{}_s K_j{}^s{}_r. \tag{4.5a}$$

Finally the Π-tensor, that is the Weyl projective curvature tensor W of K^n is obtained from (4.4) and (4.5) together with (3.3) and (3.4) as

$$\Pi_i{}^h{}_{jk} = \Pi^\circ{}_i{}^h{}_{jk} + K_i{}^h{}_{jk} + (\delta^h{}_j K_{ik} - \delta^h{}_k K_{ij})/(n-1). \tag{4.6}$$

Therefore Theorem BM leads to our main result as follows:

Theorem 3. *A Kropina space K^n of dimension $n > 2$ is projectively flat, if and only if $s_{ij} = (b_i s_j - b_j s_i)/b^2$ and*

$$\Pi^\circ{}_i{}^h{}_{jk} + K_i{}^h{}_{jk} + (\delta^h{}_j K_{ik} - \delta^h{}_k K_{ij})/(n-1) = 0,$$

where Π° is the Weyl projective curvature tensor of the associated Riemannian space and $K_i{}^h{}_{jk}$ and K_{ik} are defined by (4.1), (4.3a), (4.4a) and (4.5a).

Next (4.5) and (3.6) give

$$\begin{aligned}\Pi_{ijk} = \Pi^\circ{}_{ijk} &+ K_{ijk} + \{K_i{}^r{}_j R^\circ{}_{rk} - (j/k)\} \\ &- [(n-1)K_i{}^r{}_{jk} + \{\delta^r{}_j K_{ik} - (j/k)\}]\gamma_r,\end{aligned} \tag{4.7}$$

where the tensor K_{ijk} is defined by

$$K_{ijk} = K_{ij;k} + K_i{}^r{}_j K_{rk} - (j/k). \tag{4.7a}$$

Lemma. *In the two-dimensional case*

$$\tau_i{}^h{}_{jk} = T_i{}^h{}_{jk} + \delta^h{}_j T_{ik} - \delta^h{}_k T_{ij}, \qquad T_{ij} = T_i{}^r{}_{jr},$$

vanishes identically for every tensor $T_i{}^h{}_{jk}$ which is skew-symmetric in j, k.

For instance

$$\begin{aligned}\tau_2{}^1{}_{12} &= T_2{}^1{}_{12} + T_{22} = T_2{}^1{}_{12} + (T_2{}^1{}_{21} + T_2{}^2{}_{22}) \\ &= T_2{}^1{}_{12} + (-T_2{}^1{}_{12} + 0) = 0.\end{aligned}$$

Hence, for $n = 2$ (4.7) reduces to

$$\Pi_{ijk} = \Pi^\circ{}_{ijk} + K_{ijk} + K_i{}^r{}_j R^\circ{}_{rk} - K_i{}^r{}_k R^\circ{}_{rj}. \tag{4.7$_2$}$$

Therefore Theorem BM together with Theorem 2 leads to

Theorem 4. *A Kropina space K^2 of dimension two is projectively flat, if and only if*

$$\Pi^\circ{}_{ijk} + K_{ijk} + K_i{}^r{}_j R^\circ{}_{rk} - K_i{}^r{}_k R^\circ{}_{rj} = 0,$$

where R° is the Ricci tensor of the associated Riemannian space, Π° is given by (3.6$_2$) and K_{ijk} and $K_j{}^i{}_k$ are defined by (4.7a) and (4.3a).

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