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On the number of solutions of the equation $1^k + 2^k + \dots + (x-1)^k = y^z$

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To Professor K. Győry on his 60th birthday

1. Introduction

In 1956 Schäffer proved

Theorem A (Schäffer [13]). *For fixed $k > 0$ and $m > 1$ the equation*

$$1^k + 2^k + \dots + (x-1)^k = y^m \tag{1}$$

has an infinite number of solutions in positive integers x and y only in the cases

(i) $k = 1,\ m = 2$; (ii) $k = 3,\ m \in \{2, 4\}$; (iii) $k = 5,\ m = 2$.

For further generalisations and improvements we refer to [3], [5], [7], [9], [14], [15], [16] and [17]. One of the most surprising results is

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Theorem B (Győry, Tijdeman and Voorhoeve [17]). *Let $R(x)$ be a fixed polynomial with rational integer coefficients. Let $b \neq 0$ and $k \geq 2$ be fixed rational integers such that $k \notin \{3, 5\}$. Then the equation*

$$1^k + 2^k + \cdots + x^k + R(x) = by^z$$

in integers $x, y \geq 1$ and $z > 1$ has only finitely many solutions.

It is known that the sum $S_k(x) = 1^k + 2^k + \cdots + (x-1)^k$ can be expressed in terms of Bernoulli polynomials, hence the equation (1) can be considered as a superelliptic equation and one can derive an upper bound for the size of the solutions by using Baker's method. Dealing with the number of solutions, Brindza obtained

Theorem C (Brindza [4]). *For any given $m \notin \{1, 2, 4\}$, the equation* (1) *has at most e^{7k} solutions.*

The purpose of this paper is to handle the more general case when the exponent m is also unknown; i.e. we consider the equation

(2) $S_k(x) = y^z$ in positive integers $x, y > 1,\ z > 2$ and $(k, z) \notin (3, 4)$.

Theorem 1. *The equation* (2) *has at most*

$$\max\{c_1, e^{3k}\}$$

solutions, where c_1 is an effectively computable absolute constant.

A reasonable upper bound in the hyperelliptic case (i.e. when $z = 2$), which is not covered by Theorem C is provided by

Theorem 2. *If k is even then the equation*

$$S_k(x) = y^2 \tag{3}$$

possesses at most $\max\{c_2, 9^k\}$ solutions in positive integers x and y, where c_2 is an effectively computable absolute constant.

2. Preliminaries

The proofs are based upon several classical properties of Bernoulli polynomials and some up-to-date results. For the following known properties we refer to [11] (pages 4–22).

Let $B_n(X)$ denote the nth Bernoulli polynomial and $B_n = B_n(0)$, $n = 0, 1, 2, \ldots$; moreover, let D_n be the denominator of B_n. Then we have

$$(A) \qquad B_n(X) = \sum_{i=0}^{n} \binom{n}{i} B_i X^{n-i}, \quad (B_0 := 1),$$

$$(B) \qquad 1^k + 2^k + \cdots + (x-1)^k = \frac{1}{k+1}(B_{k+1}(x) - B_{k+1}),$$

$$(C) \qquad B_n(X) = (-1)^n B_n(1-X),$$

$$(D) \qquad B_{2n+1} = 0, \quad n = 1, 2, \ldots,$$

$$(E) \qquad \text{(Von Staudt and Clausen)},\ D_{2n} = \prod_{p-1|2n} p, \ \ p \text{ prime};$$

$$(F) \qquad \frac{2 \cdot (2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{(2n)!}{12(2\pi)^{2n-2}} \qquad (n > 1).$$

$$(G) \qquad \text{If } k \text{ is odd then } 0,\ 1,\ \frac{1}{2} \text{ are simple zeros of } B_k(X).$$

Moreover, we shall use the following lemmas.

Lemma 1. *For any $n \geq 3$ and positive integers a, b the diophantine equation*

$$|ax^n - by^n| = 1$$

has at most one solution in positive integers x and y.

Proof. See [2]. □

In the sequel, $c_3, c_4, \ldots,$ will denote effectively computable absolute constants.

Lemma 2. *The equation* (2) *implies*

$$z < c_3 k^2 \log 2k.$$

Proof. See [10, Theorem] and the subsequent remark. □

Lemma 3. *For any given k, m with $k > c_4$, $m > 2$ the equation* (1) *possesses at most $e^{2.85k}$ solutions.*

Proof. Using Brindza's approach (see [4]), one can see that (1) leads to at most $T(k)$ equations of type

$$au^m - bv^m = 1 \text{ in positive integers } u, v,$$

where

$$T(k) \le \left(\frac{k}{3}\right)^{\epsilon k\left(2+\frac{\log 3}{\log 2}\right)} \left(\prod_{\substack{p<2^{1/\epsilon}\\ p \text{ prime}}} \frac{2}{\epsilon \log 2}\right)^{2+\frac{\log 3}{\log 2}} \qquad (k > c_5).$$

We mention that the upper bound for $T(k)$ is valid for all $\epsilon > 0$. Applying the well-known inequality

$$\pi(n) < \frac{2n}{\log n}, \quad n > 1 \text{ (cf. [12])}, \tag{4}$$

and on taking

$$\epsilon = \frac{\log 2}{\log k/2}$$

we have

$$\begin{aligned} T(k) &\le \exp\left\{\frac{\log k/3}{\log k/2} \log 2 \left(2 + \frac{\log 3}{\log 2}\right) k \right. \\ &\qquad \left. + \frac{\log 2 + \log\log k/2 - 2\log\log 2}{\log k/2} \left(2 + \frac{\log 3}{\log 2}\right) k\right\} \\ &\le \exp\left\{\left(2 + \frac{\log 3}{\log 2}\right)(\log 2 + 0.1)k\right\} \le e^{2.85k}, \end{aligned}$$

for $k > c_6$. Finally, Lemma 1 completes the proof.

Lemma 4. *If a, b and c are positive integers then the simultaneous equations*

$$ax^2 - by^2 = 1, \quad by^2 - cz^2 = 1$$

has at most one solution (x, y, z) in positive integers.

Proof. This is the main result of [1]. □

Lemma 5. *Let $d(n)$ denote, as usual, the number of positive divisors of n. Then for every positive ϵ and positive integer n*

$$d(n) \le n^\epsilon \prod_{\substack{p^\epsilon < 2 \\ p \text{ prime}}} \frac{2}{\epsilon \log 2}.$$

Proof. See e.g. [8], page 111. □

3. Proofs of Theorems

Proof of Theorem 1. Since all the parameters of the kth Bernoulli polynomials depend only on k thus the unknowns x, y and z are bounded by a function of k.

If $k \le c_7$ then our result is an easy consequence of Lemma 2 and Lemma 5. Now supposing $k > c_7$ Theorem 1 is a straightforward corollary of Lemmata 1, 2 and 3. □

Proof of Theorem 2. We may assume again that $k > c_8$, otherwise Theorem 2 is proved by Lemma 5. If k is even then

$$\begin{aligned}(k+1)y^2 &= (k+1)S_k(x) = B_{k+1}(x) \\ &= \binom{k+1}{1} B_k x + \binom{k+1}{3} B_{k-2} x^3 + \cdots + x^{k+1}.\end{aligned}$$

On setting $P_k = \prod_{\substack{p \le k \\ p \text{ prime}}} p$ we obtain

$$(k+1)P_k y^2 = x\big((k+1)B_k P_k + x^2 f(x)\big),$$

where $f(x) \in \mathbb{Z}[x]$ and thus

$$x = au^2, \quad a \mid (k+1)B_k P_k$$

and a is a square-free and positive. We get similarly

$$x - 1 = bv^2 \quad \text{and} \quad 2x - 1 = cz^2,$$

where b and c also square-free positive rational integers and divide $(k+1)B_kP_k$. Using Lemma 4, it is enough to give an upper bound for the number of triplets (a,b,c) denoted by $N(a,b,c)$. It is plain that

$$N(a,b,c) \leq d\big(|(k+1)B_kP_k|\big)^3$$

and a simple calculation gives

$$\begin{aligned}|(k+1)B_kP_k| &\leq (k+1)\frac{k!}{12(2\pi)^{k-2}}\,2.8^k\\ &\leq (k+1)\left(\frac{k}{e}\right)^k\sqrt{2\pi k}\,e^{1/12}\frac{4\pi^2}{12}\left(\frac{2.8}{2\pi}\right)^k < \left(\frac{k}{4}\right)^k.\end{aligned}$$

Now we have, by Lemma 5 and inequality (4),

$$d\big(|(k+1)B_kP_k|\big) \leq \left(\frac{k}{4}\right)^{k\epsilon}\prod_{p<2^{1/\epsilon}}\frac{2}{\epsilon\log 2} \leq \left(\frac{k}{4}\right)^{k\epsilon}\left(\frac{2}{\epsilon\log 2}\right)^{\frac{2\cdot 2^{1/\epsilon}}{1/\epsilon\cdot\log 2}}.$$

On choosing

$$\epsilon = \frac{\log 2}{\log k/2}$$

we obtain

$$\begin{aligned}&d\big(|(k+1)B_kP_k|\big)\\ &\quad\leq \exp\left\{\frac{\log k/4}{\log k/2}k\log 2 + \frac{k}{\log k/2}(\log 2 + \log\log k - 2\log\log 2)\right\}\\ &\quad\leq \exp\{k(\log 2 + 0.03)\} \leq 9^{k/3}.\end{aligned}$$

□

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