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Sufficient condition for π–closure of a finite group

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Let π and π' be complementary sets of primes. A finite group G is called π–closed if it contains a normal π–Hall subgroup. By $\pi(n)$ we denote the set of all prime divisors of a positive integer n. Now $\pi(G) = \pi(|G|)$ where $|G|$ is the order of G.

Let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G (only finite groups are considered). By $\mathrm{Irr}_1(G)$ denote the set of all non–linear characters in $\mathrm{Irr}(G)$. Then $\mathrm{Lin}(G) = \mathrm{Irr}(G) - \mathrm{Irr}_1(G)$ is the set of all linear characters of G.

If $\chi \in \mathrm{Irr}(G)$ then

$$Z(\chi) = \{x \in G \mid |\chi(x)| = \chi(1)\},$$

the quasi–kernel of χ, and

$$\ker\chi = \{x \in G \mid \chi(x) = \chi(1)\}.$$

It is known that $Z(\chi)/\ker\chi = Z(G/\ker\chi)$ is the center of $G/\ker\chi$.

A character $\chi \in \mathrm{Irr}(G)$ may or not satisfy the following conditions:

$(*)\pi(G/Z(\chi)) \subset \pi(\chi(1)) \cup \pi$,

$(**)\pi(G/\ker\chi) \subset (\pi\chi(1)) \cup \pi$

for some fixed set π of primes.

Obviously, for given $\chi \in \mathrm{Irr}(G), (**) \Rightarrow (*)$. If $(*)$ holds then

$$\pi(G/Z(\chi)) \cap \pi' = \pi(\chi(1)) \cap \pi',$$

and analogously for $(**)$.

Theorem 1. *Suppose that π is a fixed set of primes and for all $\chi \in \mathrm{Irr}(G)$ the condition $(*)$ holds. Then*

(a) *G contains a normal π–Hall subgroup H, i.e. G is π–closed.*

(b) *G/H is nilpotent.*

Corollary 1. 1. *If* $(**)$ *holds for all* $\chi \in \mathrm{Irr}_1(G)$ *then*

(a) *G contains a normal π–Hall subgroup H.*

(b) *G/H is an abelian or a prime-power group.*

(c) *If $H < G$ and $|G : H|$ is the squarefree number, then G is solvable.*

Corollary 2. *A group G is a π–group* $\Leftrightarrow$ $(**)$ *holds for all* $\chi \in \mathrm{Irr}(G)$.

Proof of Theorem 1. If G is nilpotent then $\pi(G/Z(\chi)) = \pi(\chi(1))$. Hence we may assume that G is non-nilpotent. If G is a π–group then $(*)$ holds. Hence we may assume that G is not a π–group.

(i) G is not a π'–group.

Proof. Suppose that G is a π'–group and prove that G is nilpotent. Assume that G is a counterexample of minimal order. Then G contains only one minimal normal subgroup R. By induction, G/R is nilpotent. Since G is non-nilpotent, $R \not\leq Z(G)$. If $\chi \in \mathrm{Irr}(G)$ and $R \not\leq \ker\chi$ then $\ker\chi = 1$. Since $Z(G) = 1$ we have $Z(\chi) = 1$ and $\pi(\chi(1)) = \pi(G)$ by $(*)$.

Let R be a p–group, $p \in \pi'$. Since G is non-nilpotent, $R \not\leq \phi(G)$, the Frattini subgroup of G (Wielandt). Therefore there exists a maximal subgroup M of G with $R \not\leq M$. Then $G = MR$ and since $M \cap R$ is normal in G then $M \cap R = 1$. Take $P \in \mathrm{Syl}_p(M)$. Since $N_{PR}(P) > P$ and $M \leq N_G(P)$, then $N_G(P) = G$. In virtue of the uniqueness of R, we have $P = 1$, i.e., M is a p'–subgroup. Then by Ito's Theorem [**2**, Th. 6.15] p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$. Take $\chi \in \mathrm{Irr}(G)$ with $R \not\leq \ker\chi$; then $\ker\chi = 1$ and $\pi(\chi(1)) = \pi(G) \supseteq \{p\}$ by $(*)$, a contradiction.

Hence R is not a prime-power subgroup. Then R is non-solvable so that R is not p'–closed for all $p \in \pi(R)$. Take $P \in \mathrm{Syl}_p(G)$ for some $p \in \pi(R)$. By Tate's Theorem [**2**, Th. 6.31]

$$P \cap R \not\leq \phi(P)$$

Since $P' \leq \phi(P)$ then $P \cap R \not\leq P'$. So there exists $\lambda \in \mathrm{Lin}(P)$ such that $P \cap R \not\leq \ker\lambda$. Let $\mathrm{Irr}(\lambda^G)$ be the set of all irreducible components of the induced character λ^G. By reciprocity $\ker\chi P \leq \ker\lambda$ for $\chi \in \mathrm{Irr}(\lambda^G)$. Hence $P \cap R \not\leq \ker\chi \Rightarrow R \not\leq \ker\chi \Rightarrow \ker\chi = 1$. Since $Z(G) = 1$ then $Z(\chi) = 1$ and $\pi(G) = \pi(\chi(1))$ by $(*)$. Hence p divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(\lambda^G) \Rightarrow p$ divides $\lambda^G(1) = |G : P| \not\equiv 0 \pmod{p}$, a contradiction. Therefore if G is a π'–group, then G is nilpotent. But we assumed earlier that G is non-nilpotent. Hence G is not a π'–group. □

(ii) G is π–closed.

Proof. Assume that G is a counterexample of minimal order. Since all epimorphic images of G satisfy the condition of the Theorem, G contains only one minimal normal subgroup R. By induction G/R is π–closed. Hence $\pi(R) \not\subseteq \pi$, and there exists $p \in \pi(R) \cap \pi'$. Take $P \in \mathrm{Syl}_p(G)$.

Suppose that $P \cap R \leq \phi(P)$. Then by Tate's Theorem [**2**, Th. 6.31] R has a normal p–Hall subgroup L. Since R is a minimal normal subgroup of G then $L = 1$ and $R = P \cap R \leq \phi(P)$. Using the modulary law one obtains $R \leq \phi(G)$. By induction G/R contains a normal π–Hall subgroup T/R. Then T contains a π–Hall subgroup H, and all π–Hall subgroups are conjugate in T. Hence by Frattini's Lemma

$$G = TN_G(H) = RHN_G(H) = RN_G(H) = N_G(H),$$

and G is π–closed, a contradiction.

Therefore $P \cap R \not\leq \phi(P)$. Then, as in (i), there exists $\lambda \in \mathrm{Lin}(P)$ such that $P \cap R \not\leq \ker \lambda$. If $\chi \in \mathrm{Irr}(\lambda^G)$ then, as in (i), $\ker \chi = 1$. If $Z(G) > 1$ then $R \leq Z(G)$, and since G/R is π–closed G is π–closed, a contradiction. Therefore $Z(G) = 1$ so that $Z(\chi) = 1$. Then p divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(\lambda^G) \Rightarrow p$ divides $\lambda^G(1) = |G : P| \not\equiv 0 \pmod{p}$, a contradiction.

Hence G contains a normal π–Hall subgroup H. Since the π'–group G/H satisfies the condition of the Theorem, G/H is nilpotent by (i). The Theorem is proved. □

Proof of Corollary 1. Since $(**) \Rightarrow (*)$, G contains a normal π–Hall subgroup H and G/H is nilpotent by the Theorem. It is easy to see that a nilpotent π'–group satisfies the condition of the Theorem $\Leftrightarrow$ it is abelian or of prime-power order. Hence it remains to prove (c).

Suppose that $|G/H|$ is a squarefree number and $H < G$, but G is non-solvable. Then there exist the normal π–subgroups $K > L$ such that K/L is a non-solvable minimal normal subgroup of G/L. Let D a maximal normal subgroup of G such that $K \cap D = L$. Without loss of generality we may assume that $L = 1$. Then K is a minimal normal subgroup of G.

Suppose that $KD = G$. Then $G = K \times D$, and K is non-abelian simple, $D > 1$ is cyclic. Take $1_K \neq \varphi \in \mathrm{Irr}(R)$ and faithful $\lambda \in \mathrm{Irr}(D)$. Then $\chi = \varphi \times \lambda \in \mathrm{Irr}(G)$ is faithful but $\pi(\chi(1)) \cup \pi = \pi \not\supseteq \pi(G)$, a contradiction to $(**)$.

Suppose that $KD < G$. Then KD/D is a proper minimal normal subgroup of G/D. Without loss of generality we may assume that $D = 1$. Then by the choice of D, the subgroup K is the only minimal normal subgroup of G. Suppose that $\chi \in \mathrm{Irr}(G)$ and $K \not\leq \ker \chi$. Then $\ker \chi = 1$ and $\chi \in \mathrm{Irr}_1(G)$. Then by $(**)$ we have

$$\pi(G) \subseteq \pi \cup \pi(\chi(1)).$$

This means that $|G : H|$ divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ with $H \not\leq \ker \chi$. Then G is a Frobenius group with the core H by Corollary 2.4 in [1]. Then H is nilpotent $\Rightarrow G$ is solvable, a contradiction. Corollary 1 is proved. □

Obviously, Corollary 2 is a trivial consequence of Corollary 1.

Now we prove one criterion of solvability of a finite group G.
If $\chi \in \mathrm{Irr}(G)$ then we set

$$\sigma(\chi) = \pi(|G : Z(\chi)|/\chi(1)).$$

A group G is called a K–group if $\sigma(\chi(1)) \leq 1$ for all $\chi \in \mathrm{Irr}(G)$.
We note that $\chi \in \mathrm{Irr}(G)$ is linear iff $Z(\chi) = G$.

Theorem 2. *Suppose that G is a K–group, $\chi, \tau \in \mathrm{Irr}(G)$.*

(a) *If $Z(\chi) = Z(\tau)$ then $\sigma(\chi) = \sigma(\tau)$.*

(b) *Set $\overline{G} = G/\ker\chi$. Then $\chi = \lambda^{\overline{G}}$ where $\lambda \in Lin(\overline{P}_1 \times \overline{Z}_1)$ where $\overline{P}_1$ is a p–subgroup of $\overline{G}, \overline{Z}_1 \leq Z(\overline{G})$.*

(c) *G is solvable.*

(d) *$F(G/Z(\chi))$, the Fitting subgroup of $G/Z(\chi)$, is a p–group if $\sigma(\chi) = \{p\}$.*

Proof. (a) We may assume that χ, τ are non-linear. Suppose that $\sigma(\chi) = \{p\}$, $\sigma(\tau) = \{q\}$ and $p \neq q$ (if $\sigma(\chi)$ is empty, then χ is linear [**2**, Th. 2.30] which is impossible). Set $\overline{G} = G/Z(\chi)$, $\overline{P} \in \mathrm{Syl}_p(\overline{G})$, $\overline{Q} \in \mathrm{Syl}_q(\overline{G})$. By the condition we have

$$|\overline{P}| \geq |\overline{G}|/\chi(1), \qquad |\overline{Q}| \geq |\overline{G}|\tau,(1).$$

Since $\chi(1)^2 \leq |G/Z(\chi)| = |\overline{G}|$ for all $\chi \in \mathrm{Irr}(G)$ [**2**, Th. 2.30] we have

$$|\overline{P}|^2 \geq \left(|\overline{G}|/\chi(1)\right)^2 \geq |\overline{G}|,$$

$$|\overline{Q}|^2 \geq \left(|\overline{G}|/\tau(1)\right)^2 \geq |\overline{G}|.$$

Let $|\overline{P}| > |\overline{Q}|$. Then

$$|\overline{G}| \geq |\overline{P}||\overline{Q}| > |\overline{Q}|^2 \geq |\overline{G}|,$$

a contradiction.

(b) Set $\overline{G} = G/\ker\chi$, $\overline{P} \in \mathrm{Syl}_p(\overline{G})$, $\overline{H} = \overline{P}Z(\overline{G}) = \overline{P} \times \overline{Z}$ where the subgroup $\overline{Z} \leq Z(\overline{G})$ is cyclic. Let

$$\mathrm{Irr}(\chi_{_H}) = \{\vartheta_1, \dots, \vartheta_s\}, \qquad \vartheta_1(1) \leq \cdots \leq \vartheta_s(1).$$

The numbers $\vartheta_1(1), \dots, \vartheta_s(1)$ are powers of p[**2**, Th. 3.12].

Thus $\vartheta_1(1)$ divides $\chi(1)$. By the condition the number $|\overline{G} : \overline{H}|$ divides $\chi(1)$. Since $\vartheta_1(1)$ and $|\overline{G} : \overline{H}|$ are coprime, $|\overline{G} : \overline{H}|\vartheta_1(1) = \vartheta_1^{\overline{G}}(1)$ divides $\chi(1)$. Now $\chi \in \mathrm{Irr}(\vartheta_1^{\overline{G}})$ implies $\chi = \vartheta_1^{\overline{G}}$. Since $\overline{H}$ as a nilpotent group is an M–group (b) follows by transitivity of induction.

(c) Suppose that G is a counterexample of minimal order. Since epimorphic images of a K–group are K–groups G contains only one minimal normal subgroup R, G/R is solvable, R is non-solvable and $Z(G) = 1$. Now if $\chi \in \mathrm{Irr}(G)$ with $R \not\leq \ker\chi$, then $Z(\chi) = 1$. Let X be the set of all faithful characters in $\mathrm{Irr}(G)$. If $\chi \in X$ then $|G|/\chi(1)$ is a power of a fixed prime p by (a). Since R is non-solvable there exists $q \in \pi(R) - \{p\}$. Take $Q \in \mathrm{Syl}_p(G)$. Then $Q \cap R \not\leq \phi(Q)$ [**2**, Th. 6.31]. Since $Q' \leq \phi(Q)$ then $Q \cap R \not\leq Q'$. Therefore there exists $\lambda \in \mathrm{Irr}(Q)$ such that $Q \cap R \not\leq \ker\lambda$. If $\chi \in \mathrm{Irr}(\lambda^G)$ then by reciprocity $R \not\leq \ker\chi$ so that $\mathrm{Irr}(\lambda^G) \subseteq X$ and q divides $\lambda^G(1) = |G : Q| \not\equiv 0 \pmod{q}$, a contradiction.

(d) Suppose the contrary. Then $\overline{G} = G/Z(\chi)$ contains a non-identity normal abelian q–subgroup $\overline{Q}_1$ with a prime $q \neq p$. Take $\overline{Q} \in \mathrm{Syl}(\overline{G})$. Obviously $\overline{Q}_1 \leq \overline{Q}$. Since $\overline{G}$ is a K–group, $|\overline{Q}|$ divides $\chi(1)$ by the condition. Take in $\overline{Q}_1$ a subgroup $\overline{Q}_0$ of order q. Then $\overline{Q}_0$ is subnormal in $\overline{G}$. Let Q_0 be the inverse image of $\overline{Q}_0$ in G. Then $|Q_0/\ker\chi : Z(\chi)/\ker\chi| = |\overline{Q}_0| = q$. Since $Z(\chi)/\ker\chi = Z(G/\ker\chi)$ then $Q_0/\ker\chi$ is an abelian subnormal subgroup of $G/\ker\chi$. By Reynolds' result (see Remark after Corollary 11.29 in [**2**]) $\chi(1)$ divides $|G/\ker\chi : Q_0/\ker\chi| = |\overline{G} : \overline{Q}_0|$. Since the q-part of the number $|\overline{G} : \overline{Q}_0|$ is less than $|\overline{Q}|$ then $|\overline{Q}|$ does not divide $\chi(1)$, a contradiction. □

Remark. From assertion (b) of Theorem 2 follows that a K–group G is an M–group so that a K–group G is solvable by Taketa's Theorem [**2**, Corollary 5.13]. Our proof does not depend on Taketa's Theorem.

Note that the symmetric group S_4 is a K–group.

A group G is called a K_1–group if for each $\chi \in \mathrm{Irr}(G)$ and for each maximal abelian normal subgroup $B/\ker\chi$ of $G/\ker\chi$ we have $|\pi(|G : B|/\chi(1))| \leq 1$.

Theorem 3. *K_1–groups are solvable.*

This result is proved in the same manner as Theorem 2(c).

For related results see [**3**].

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