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Integer points on a family of elliptic curves

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Dedicated to Professor Kálmán Győry
on the occasion of his 60th birthday

Abstract. Let the sequence (c_k) be given by the recursion

$$c_0 = 0,\ c_1 = 8,\ c_{k+2} = 14c_{k+1} - c_k + 8, \quad k \geq 0.$$

Let the elliptic curve E_k be defined by the equation $y^2 = (x+1)(3x+1)(c_k x+1)$. We prove in this paper that if the rank of $E_k(\mathbb{Q})$ is equal to two, or $k \leq 40$, with the possible exceptions $k = 23$ and $k = 37$, then all integer points on E_k are given by

$$(x,y) \in \{(-1,0), \quad (0,\pm 1), \quad (c_{k-1}, \pm s_{k-1}t_{k-1}(2c_k - s_k t_k)),$$
$$(c_{k+1}, \pm s_{k+1}t_{k+1}(2c_k + s_k t_k))\}.$$

where $c_k + 1 = s_k^2$ and $3c_k + 1 = t_k^2$.

1. Introduction

A set D of m positive integers is called a *Diophantine m-tuple* if the product of any two distinct elements of D increased by 1 is a perfect square. The first example of a Diophantine quadruple – $\{1,3,8,120\}$ – was found by Fermat (see [6, p. 517]). In 1969, Baker and Davenport [2] proved that if d is a positive integer such that $\{1,3,8,d\}$ is a Diophantine quadruple, then d has to be 120.

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Recently, in [9], we generalized this result to all Diophantine triples of the form $\{1,3,c\}$. The fact that $\{1,3,c\}$ is a Diophantine triple implies that $c=c_k$ for some positive integer k, where the sequence (c_k) is given by

$$c_0=0,\quad c_1=8,\quad c_{k+2}=14c_{k+1}-c_k+8,\quad k\geq 0.$$

Let $c_k+1=s_k^2$, $3c_k+1=t_k^2$ with positive integers s_k,t_k. It is easy to check that

$$c_{k\pm1}c_k+1=(2c_k\pm s_kt_k)^2.$$

The main result of [9] is the following theorem.

Theorem 1. *Let k be a positive integer. If d is an integer which satisfies the system of equations*

$$d+1=x_1^2,\quad 3d+1=x_2^2,\quad c_kd+1=x_3^2, \tag{1}$$

then $d\in\{0,c_{k-1},c_{k+1}\}$.

Eliminating d from the system (1) we obtain the following system of Pellian equations

$$x_3^2-c_kx_1^2=1-c_k \tag{2}$$

$$3x_3^2-c_kx_2^2=3-c_k. \tag{3}$$

We used the theory of Pellian equations and some congruence relations to reformulate the system (2) and (3) to four equations of the form $v_m=w_n$, where (v_m) and (w_n) are binary recursive sequences. After that, a comparison of the upper bound for the solutions obtained from the theorem of Baker and Wüstholz [3] with the lower bound obtained from the congruence condition modulo c_k^2 finished the proof for $k\geq 76$. The statement for $1\leq k\leq 75$ was proved by a variant of the reduction procedure due to Baker and Davenport [2].

Similar results are proved in [7] and [8] for Diophantine triples of the form $\{k-1,k+1,4k\}$ and $\{F_{2k},F_{2k+2},F_{2k+4}\}$. In the second triple F_n denotes the n-th Fibonacci number.

It is clear that every solution $(d,x_1,x_2,x_3)\in\mathbb{Z}^4$ of (1) induce an integer point on the elliptic curve

$$E_k:\qquad y^2=(x+1)(3x+1)(c_kx+1), \tag{4}$$

with $y=x_1x_2x_3$ and $x=d$. The purpose of the present paper is to prove that the converse of this statement is true, provided the rank of $E_k(\mathbb{Q})$ is equal to 2. As we will see in Proposition 2, for all $k\geq 2$ the rank of $E_k(\mathbb{Q})$ is always ≥ 2. Our main result is

Theorem 2. *Let k be a positive integer. If* $\operatorname{rank}(E_k(\mathbb{Q})) = 2$ *or* $k \leq 40$, *with the possible exceptions $k = 23$ and $k = 37$, then all integer points on E_k are given by*

$$(x, y) \in \{(-1, 0),\ (0, \pm 1),\ (c_{k-1}, \pm s_{k-1}t_{k-1}(2c_k - s_k t_k)),\\ (c_{k+1}, \pm s_{k+1}t_{k+1}(2c_k + s_k t_k))\}.$$

2. Torsion group

Under the substitution $x \leftrightarrow 3c_k x$, $y \leftrightarrow 3c_k y$ the curve E_k is transformed into the following Weierstraß form

$$E'_k: \qquad y^2 = x^3 + (4c_k + 3)x^2 + (3c_k^2 + 12c_k)x + 9c_k^2 = (x + 3c_k)(x + c_k)(x + 3).$$

There are three rational points on E'_k of order 2, namely

$$A_k = (-3c_k, 0), \quad B_k = (-c_k, 0), \quad C_k = (-3, 0),$$

and also other two, more or less obvious, rational points on E'_k, namely

$$P_k = (0, 3c_k), \quad R_k = (s_k t_k + 2s_k + 2t_k + 1, (s_k + t_k)(s_k + 2)(t_k + 2)).$$

Note that if $k = 1$, then $R_1 = C_1 - P_1$.

Lemma 1. $E'_k(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. From [17, Main Theorem 1] it follows immediately that $E'_k(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $E'_k(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, and the later is possible iff there exist integers α and β such that $\frac{\alpha}{\beta} \notin \{-2, -1, -\frac{1}{2}, 0, 1\}$ and

$$c_k - 3 = \alpha^4 + 2\alpha^3\beta, \quad 3c_k - 3 = 2\alpha\beta^3 + \beta^4.$$

Now, we have

$$4c_k - 6 = (\alpha^2 + \alpha\beta + \beta^2)^2 - 3\alpha^2\beta^2. \tag{5}$$

Since c_k is even, the left hand side of (5) is $\equiv 2 \pmod 8$. If α and β are both even then the right hand side of (5) is divisible by 8, and if α and β are both odd then the right hand side of (5) is $\equiv 6 \pmod 8$, a contradiction. Hence, $E'_k(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

3. The independence of P_k and R_k

In this section we will often use the following 2-descent proposition (see [12, 4.1, p. 37]).

Proposition 1. *Let $P = (x', y')$ be a $\mathbb{Q}$-rational point on E, an elliptic curve over $\mathbb{Q}$ given by the equation*

$$y^2 = (x-\alpha)(x-\beta)(x-\gamma),$$

where $\alpha, \beta, \gamma \in \mathbb{Q}$. Then there exists a $\mathbb{Q}$-rational point $Q = (x, y)$ on E such that $2Q = P$ iff $x' - \alpha$, $x' - \beta$, $x' - \gamma$ are all $\mathbb{Q}$-rational squares.

Lemma 2. $P_k,\ P_k + A_k,\ P_k + B_k,\ P_k + C_k \notin 2E'_k(\mathbb{Q})$.

Proof. We have:

$$\begin{aligned} P_k + A_k &= (-c_k - 2, -2c_k + 2), \\ P_k + B_k &= (-3c_k + 6, 6c_k - 18), \\ P_k + C_k &= (c_k^2 - 4c_k, -c_k^3 + 4c_k^2 - 3c_k). \end{aligned}$$

It follows immediately from Proposition 1 that $P_k,\ P_k + A_k,\ P_k + B_k \notin 2E'_k(\mathbb{Q})$. If $P_k + C_k \in 2E'_k(\mathbb{Q})$, then $c_k^2 - c_k = \square$, which is impossible. $\square$

Lemma 3. $R_k,\ R_k + A_k,\ R_k + B_k,\ R_k + C_k \notin 2E'_k(\mathbb{Q})$.

Proof. We have:

$$\begin{aligned} R_k &= (s_k t_k + 2s_k + 2t_k + 1, (t_k + s_k)(s_k + 2)(t_k + 2)), \\ R_k + A_k &= (2s_k - 2t_k - s_k t_k + 1, (s_k - t_k)(s_k + 2)(t_k - 2)), \\ R_k + B_k &= (2t_k - 2s_k - s_k t_k + 1, (t_k - s_k)(s_k - 2)(t_k + 2)), \\ R_k + C_k &= (s_k t_k - 2s_k - 2t_k + 1, (t_k + s_k)(2 - s_k)(t_k - 2)). \end{aligned}$$

Since $2s_k - 2t_k - s_k t_k + 4 = (s_k + 2)(2 - t_k) < 0$ and $2t_k - 2s_k - s_k t_k + 4 = (t_k + 2)(2 - s_k) < 0$, we have $R_k + A_k, R_k + B_k \notin 2E'_k(\mathbb{Q})$.

If $R_k \in 2E'_k(\mathbb{Q})$, then $(t_k + s_k)(t_k + 2) = \square$ and $(t_k + s_k)(s_k + 2) = \square$. Let $d = \gcd(t_k + s_k, t_k + 2, s_k + 2)$. Then d divides $(t_k + 2) + (s_k + 2) - (t_k + s_k) = 4$, and since s_k and t_k are odd, we conclude that $d = 1$. Hence, we have

$$t_k + s_k = \square, \quad t_k + 2 = \square, \quad s_k + 2 = \square. \tag{6}$$

Consider the sequence $(t_k+s_k)_{k\in\mathbb{N}}$. It follows easily by induction that $t_k+s_k=2a_{k+1}$, where

$$a_0=0,\quad a_1=1,\quad a_{k+2}=4a_{k+1}-a_k,\quad k\geq 0. \tag{7}$$

Thus, (6) implies $a_{k+1}=2\square$, and this is impossible by a theorem of MIGNOTTE and PETHŐ [14] (see also [16]) which says that $a_k=\square$, $2\square$, $3\square$ or $6\square$ implies $k\leq 3$.

If $R_k+C_k\in 2E'_k(\mathbb{Q})$, then $(t_k+s_k)(t_k-2)=\square$ and $(t_k+s_k)(s_k-2)=\square$. This implies $t_k+s_k=\square$ and we obtain a contradiction as above. $\square$

Lemma 4. *If $k\geq 2$, then R_k+P_k, $R_k+P_k+A_k$, $R_k+P_k+B_k$, $R_k+P_k+C_k\notin 2E'_k(\mathbb{Q})$.*

PROOF. As in the proof of Lemmas 2 and 3, we use Proposition 1.

If $R_k+P_k+A_k\in 2E'_k(\mathbb{Q})$ then $0>c_k(s_k+2)(s_k-t_k)=\square$, and if $R_k+P_k+B_k\in 2E'_k(\mathbb{Q})$ then $0>c_k(s_k-2)(s_k-t_k)=\square$. Hence, $R_k+P_k+A_k$, $R_k+P_k+B_k\notin 2E'_k(\mathbb{Q})$.

If $R_k+P_k\in 2E'_k(\mathbb{Q})$ then

$$\begin{aligned} 3c_k(t_k+s_k)(t_k+2)&=\square,\\ c_k(t_k+s_k)(s_k+2)&=\square,\\ 3(s_k+2)(t_k+2)&=\square. \end{aligned} \tag{8}$$

Substituting $2c_k=(t_k+s_k)(t_k-s_k)$ in (8) we obtain

$$\begin{aligned} (t_k-s_k)(t_k+2)&=6\square,\\ (t_k-s_k)(s_k+2)&=2\square,\\ (s_k+2)(t_k+2)&=3\square. \end{aligned}$$

Let $d=\gcd(s_k+2,t_k+2)$. Then the relation $t_k^2-3s_k^2=-2$ implies $d|6$. Since t_k+2 is odd, we have $d\in\{1,3\}$. Hence we obtain

$$t_k-s_k=6\square\quad\text{or}\quad t_k-s_k=2\square. \tag{9}$$

But $t_k-s_k=2a_k$, where (a_k) is defined by (7). Thus (9) implies $a_k=\square$ or $3\square$. According to [14], this is possible only if $k=2$. But $(s_2,t_2)=(11,19)$ and $(s_2+2)(t_2+2)\neq 3\square$.

If $R_k + P_k + C_k \in 2E'_k(\mathbb{Q})$ then

$$3c_k(t_k + s_k)(t_k - 2) = \square,$$
$$c_k(t_k + s_k)(s_k - 2) = \square,$$
$$3(s_k - 2)(t_k - 2) = \square.$$

Arguing as before, we obtain

$$(t_k - s_k)(t_k - 2) = 6\square,$$
$$(t_k - s_k)(s_k - 2) = 2\square,$$
$$(s_k - 2)(t_k - 2) = 3\square,$$

and conclude that

$$t_k - s_k = 6\square \quad \text{or} \quad t_k - s_k = 2\square.$$

As we have already seen, it is possible only for $(s_2, t_2) = (11, 19)$, but then $(s_2 - 2)(t_2 - 2) \neq 3\square$. $\square$

Proposition 2. *If $k \geq 2$, then the points P_k and R_k generate a subgroup of rank 2 in $E'_k(\mathbb{Q})/E'_k(\mathbb{Q})_{\text{tors}}$.*

Proof. We have to prove that $mP_k + nR_k \in E'_k(\mathbb{Q})_{\text{tors}}$, $m, n \in \mathbb{Z}$, implies $m = n = 0$.

Assume $mP_k + nR_k = T \in E'_k(\mathbb{Q})_{\text{tors}} = \{\mathcal{O}, A_k, B_k, C_k\}$ with $(m, n) \neq (0, 0)$. If m and n are not both even, then $T \equiv P_k, R_k$ or $P_k + R_k$ $\pmod{2E'_k(\mathbb{Q})}$, which is impossible by Lemmas 2, 3 and 4. Hence, m and n are even, say $m = 2m_1$, $n = 2n_1$, and since by Lemma 1 $A_k, B_k, C_k \notin 2E'_k(\mathbb{Q})$,

$$2m_1P_k + 2n_1Q_k = \mathcal{O}.$$

Thus we obtain $m_1P_k + n_1R_k \in E'_k(\mathbb{Q})_{\text{tors}}$. Arguing as above, we obtain that m_1 and n_1 are even, and continuing this process we finally conclude that $m = n = 0$. $\square$

4. Proof of Theorem 2 (rank$(E_k(\mathbb{Q}))=2$)

Let $E'_k(\mathbb{Q})/E'_k(\mathbb{Q})_{\mathrm{tors}}=\langle U,V\rangle$ and $X\in E'_k(\mathbb{Q})$. Then there exist integers m,n and a torsion point T such that $X=mU+nV+T$. Also $P_k=m_PU+n_PV+T_P$, $R_k=m_RU+n_RV+T_R$ with integers m_P, n_P, m_R, n_R and with $T_P,T_R\in E'_k(\mathbb{Q})_{\mathrm{tors}}$. Let $\mathcal{U}=\{\mathcal{O},U,V,U+V\}$. There exist $U_1,U_2\in\mathcal{U}$, $T_1,T_2\in E'_k(\mathbb{Q})_{\mathrm{tors}}$ such that $P_k\equiv U_1+T_1 \pmod{2E'_k(\mathbb{Q})}$, $R_k\equiv U_2+T_2 \pmod{2E'_k(\mathbb{Q})}$. Let $U_3\in\mathcal{U}$ such that $U_3\equiv U_1+U_2 \pmod{2E'_k(\mathbb{Q})}$. Then $P_k+R_k\equiv U_3+(T_1+T_2) \pmod{2E'_k(\mathbb{Q})}$. Now Lemmas 2, 3 and 4 imply that $U_1,U_2,U_3\neq\mathcal{O}$ and accordingly $\{U_1,U_2,U_3\}=\{U,V,U+V\}$. Therefore $X\equiv X_1 \pmod{2E'_k(\mathbb{Q})}$, where

$$\begin{aligned}X_1\in\mathcal{S}=\{&\mathcal{O},A_k,B_k,C_k,P_k,P_k+A_k,P_k+B_k,P_k+C_k,\\&R_k,R_k+A_k,R_k+B_k,R_k+C_k,R_k+P_k,\\&R_k+P_k+A_k,R_k+P_k+B_k,R_k+P_k+C_k\}.\end{aligned}$$

Let $\{a,b,c\}=\{3,c_k,3c_k\}$. By [13, 4.6, p. 89], the function $\varphi: E'_k(\mathbb{Q})\to\mathbb{Q}^*/\mathbb{Q}^{*2}$ defined by

$$\varphi(X)=\begin{cases}(x+a)\mathbb{Q}^{*2}, & \text{if } X=(x,y)\neq\mathcal{O},(-a,0),\\ (b-a)(c-a)\mathbb{Q}^{*2}, & \text{if } X=(-a,0),\\ \mathbb{Q}^{*2}, & \text{if } X=\mathcal{O}\end{cases}$$

is a group homomorphism.

This fact and Theorem 1 imply that it is sufficient to prove that for all $X_1\in\mathcal{S}\setminus P_k$, $X_1=(3c_ku,3c_kv)$, the system

$$x+1=\alpha\square,\quad 3x+1=\beta\square,\quad c_kx+1=\gamma\square \tag{11}$$

has no integer solution, where $\square$ denotes a square of a rational number, and α, β, γ are defined by $u+1=\alpha$, $3u+1=\beta$, $c_ku+1=\gamma$ if all those numbers are $\neq 0$, and if e.g. $u+1=0$ then we choose $\alpha=\beta\gamma$ (so that $\alpha\beta\gamma=\square$). Note that for $X_1=P_k$ we obtain the system $x+1=\square$, $3x+1=\square$, $c_kx+1=\square$, which is completely solved in Theorem 1.

For $X_1\in\{A_k,B_k,P_k+A_k,P_k+B_k,R_k+A_k,R_k+B_k,R_k+P_k+A_k,R_k+P_k+B_k\}$ exactly two of the numbers α, β, γ are negative and thus the system (11) has no integer solution.

The rest of the proof falls naturally into 7 parts. By a' we will denote the square free part of an integer a.

1) $X_1 = \mathcal{O}$:
We have

$$x+1 = 3c_k\Box, \quad 3x+1 = c_k\Box, \quad c_kx+1 = 3\Box. \tag{12}$$

From the second equation in (12) we see that $3 \nmid c_k'$ and thus the first and second equations imply that c_k' divides $3x+1$ and $x+1$. Accordingly, $c_k' \mid 3(x+1)-(3x+1) = 2$ and we conclude that $c_k' = 1$ or 2. Hence,

$$c_k = \Box, \quad \text{or} \quad c_k = 2\Box.$$

However, $c_k = s_k^2 - 1 = \Box$ is obviously impossible, while $c_k = 2w^2$ leads to the system of Pellian equations

$$s_k^2 - 2w^2 = 1, \quad t_k^2 - 6w^2 = 1.$$

This system is solved by Anglin [1], and the only positive solution is $(s_k, t_k, w) = (3, 5, 2)$ which corresponds to $c_k = c_1 = 8$, contradicting our assumption that $k \geq 2$. (Note that for $c_1 = 8$ there is also no solution because in this case the first and the third equations in (12) imply $3 \mid 7$.)

2) $X_1 = C_k$:
We have

$$\begin{aligned} x+1 &= c_k(c_k-1)\Box, \\ 3x+1 &= c_k(c_k-3)\Box, \\ c_kx+1 &= (c_k-1)(c_k-3)\Box. \end{aligned}$$

If $3 \nmid c_k$ then, as in **1)**, we obtain $c_k' = 1$ or 2, and $c_k = \Box$ or $2\Box$, which is impossible.

If $c_k = 3e_k$ then e_k' divides $3x+1$ and $3x+3$ and thus $e_k' = 1$ or 2. Hence,

$$c_k = 3\Box, \quad \text{or} \quad c_k = 6\Box.$$

The relation $c_k = 3\Box$ is impossible since it implies $t_k^2 - 1 = 9\Box$, while $c_k = 6w^2$ leads to the system of Pellian equations

$$s_k^2 - 6w^2 = 1, \quad t_k^2 - 18w^2 = 1$$

which has no positive solution according to [1].

3) $X_1 = P_k + C_k$:
We have

$$x + 1 = 3(c_k - 1)\square,$$

$$3x + 1 = (c_k - 3)\square,$$

$$c_k x + 1 = 3(c_k - 1)(c_k - 3)\square.$$

Since $c_k = s_k^2 - 1$, we see that $c_k \not\equiv 1 \pmod 3$, and thus $x \equiv -1 \pmod 3$. From the second equation we have that $(c_k - 3)'$ is not divisible by 3, and then the third equation gives $c_k x + 1 \equiv 0 \pmod 3$. This implies $c_k \equiv 1 \pmod 3$, a contradiction.

4) $X_1 = R_k$:
We have

$$x + 1 = 6(t_k - s_k)(t_k + 2)\square,$$

$$3x + 1 = 2(t_k - s_k)(s_k + 2)\square,$$

$$c_k x + 1 = 3(s_k + 2)(t_k + 2)\square.$$

From the relation $t_k^2 - 3s_k^2 = -2$ it follows that $\gcd(t_k - s_k, s_k + 2) = \gcd(t_k - s_k, t_k + 2) = 1$ or 3.

If $3 \nmid t_k - s_k$ then $[2(t_k - s_k)]'$ divides $x + 1$ and $3x + 1$, and thus $[2(t_k - s_k)]' = 1$ or 2. Accordingly,

$$t_k - s_k = 2\square \quad \text{or} \quad t_k - s_k = \square.$$

As we have already seen in the proof of Lemma 4, this implies

$$a_k = \square \quad \text{or} \quad a_k = 2\square,$$

and [14] implies again that $k = 2$. Now we obtain $120x + 1 = 91\square$, which is impossible modulo 4.

If $t_k - s_k = 3z_k$ then $(2z_k)'$ divides $x + 1$ and $9x + 3$. Hence $(2z_k)'$ divides 6, which implies $a_k = \square, 2\square, 3\square$ or $6\square$, and this is possible only if $k = 2$. But for $k = 2$, $t_k - s_k = 8 \not\equiv 0 \pmod 3$.

5) $X_1 = R_k + C_k$:
We have

$$x + 1 = 6(t_k - s_k)(t_k - 2)\square, \quad 3x + 1 = 2(t_k - s_k)(s_k - 2)\square,$$
$$c_k x + 1 = 3(s_k - 2)(t_k - 2)\square.$$

This case is completely analogous to the case 4).

6) $X_1 = R_k + P_k$:
We have

$$x + 1 = (t_k + s_k)(t_k + 2)\square, \quad 3x + 1 = (t_k + s_k)(s_k + 2)\square,$$
$$c_k x + 1 = (s_k + 2)(t_k + 2)\square.$$

As in 4), we obtain that if $3 \nmid t_k + s_k$ then $(t_k + s_k)'$ divides 2, and if $t_k + s_k = 3z_k$ then z_k' divides 6. Hence, we have $a_{k+1} = \square, 2\square, 3\square$ or $6\square$, which is impossible for $k \geq 2$.

7) $X_1 = R_k + P_k + C_k$:
We have

$$x + 1 = (t_k + s_k)(t_k - 2)\square, \quad 3x + 1 = (t_k + s_k)(s_k - 2)\square,$$
$$c_k x + 1 = (s_k - 2)(t_k - 2)\square.$$

This case is completely analogous to the case 6). $\square$

Remark 1. It is easy to check that $\operatorname{rank}(E_1(\mathbb{Q})) = 1$, and from the proof of the first statement of Theorem 2 (parts 1), 2) and 3)) it is clear that all integer points on E_1 are given by $(x, y) \in \{(-1, 0), (0, \pm 1), (120, \pm 6479)\}$. Hence Theorem 2 is true for $k = 1$.

Remark 2. As the coefficients of E_k grow exponentially, the computation of the rank of E_k for large k is difficult. The following values of $\operatorname{rank}(E_k(\mathbb{Q}))$ were computed using the programs SIMATH ([18]) and *mwrank* ([5]):

k	1	2	3	4	5	7	8^*	9	10^*
$\operatorname{rank}(E_k(\mathbb{Q}))$	1	2	3	3	2	4	4	3	3

In the cases $k = 8, 10$, the rank is computed assuming the Parity Conjecture. For $k = 6, 11, 12$, under the same conjecture, we obtained that $\operatorname{rank}(E_k(\mathbb{Q}))$ is equal to 2 or 4. We also verified by SIMATH that for $k = 3$ and $k = 4$ (when $\operatorname{rank}(E_k(\mathbb{Q})) > 2$) all integer points on E_k are given by the values from Theorem 2.

Remark 3. Let us mention that Bremner, Stroeker and Tzanakis [4] proved recently a similar result as the first statement of our Theorem 2 for the family of elliptic curves

$$C_k: \quad y^2 = \frac{1}{3}x^3 + \left(k - \frac{1}{2}\right)x^2 + \left(k^2 - k + \frac{1}{6}\right)x,$$

under the assumptions $\operatorname{rank}(C_k(\mathbb{Q})) = 1$ and $C_k(\mathbb{Q})/C_k(\mathbb{Q})_{\text{tors}} = \langle(1,k)\rangle$.

5. Proof of Theorem 2 ($3 \leq k \leq 40$)

We pointed out in Remark 2 that the coefficients of E_k are growing very fast. Therefore, using SIMATH[1] we were able to compute the integer points of $E_k(\mathbb{Q})$ only for $k \leq 4$. However, the following elementary argument gives us the proof of the second statement of Theorem 2.

Notice the following relations

$$c_0 = 0, \quad c_1 = 8, \quad c_{k+2} = 14c_{k+1} - c_k + 8, \qquad \text{if } k \geq 0, \tag{13}$$

$$t_0 = 1, \quad t_1 = 5, \quad t_{k+2} = 4t_{k+1} - t_k, \qquad \text{if } k \geq 0, \tag{14}$$

$$s_0 = 1, \quad s_1 = 3, \quad s_{k+2} = 4s_{k+1} - s_k, \qquad \text{if } k \geq 0, \tag{15}$$

$$c_k + 1 = s_k^2 \quad \Longrightarrow \quad c_k = (s_k+1)(s_k-1), \tag{16}$$

$$3c_k + 1 = t_k^2 \quad \Longrightarrow \quad 3c_k = (t_k+1)(t_k-1), \tag{17}$$

$$3(c_k - 1) = (t_k+2)(t_k-2), \tag{18}$$

$$c_k - 3 = (s_k+2)(s_k-2). \tag{19}$$

We have $8 \mid c_k$ for any $k \geq 0$ by (13). Hence s_k and t_k are odd. We have further $3 \nmid c_k - 1$ by (16).

Assume that $(x,y) \in \boldsymbol{Z}^2$ is a solution of (4). Put $D_1 = \gcd(x+1, 3x+1)$, $D_2 = \gcd(x+1, c_kx+1)$ and $D_3 = \gcd(3x+1, c_kx+1)$. As $D_1 = \gcd(x+1, 3x+1) = \gcd(x+1, 2)$, we have $D_1 = 1$ if $x+1$ is odd, and $D_1 = 2$ if $x+1$ is even. We have further $D_2 = \gcd(x+1, c_kx+1) =$

[1]SIMATH is presently the only available computer algebra system which is capable to compute all integer points of elliptic curves. There is implemented the algorithm of Gebel, Pethő and Zimmer [10].

$\gcd(x+1, c_k-1)$ and $D_3 = \gcd(3x+1, c_kx+1) = \gcd(3x+1, c_k-3)$. Hence D_1, D_2 and D_3 are pairwise relatively prime.

Assume first $D_1 = 1$. Then there exist $x_1, x_2, x_3 \in \boldsymbol{Z}$ such that

$$\begin{aligned} x+1 &= D_2x_1^2 \\ 3x+1 &= D_3x_2^2 \\ c_kx+1 &= D_2D_3x_3^2. \end{aligned}$$

Eliminating x we obtain the following system of equations

$$\begin{aligned} 3D_2x_1^2 - D_3x_2^2 &= 2 \\ c_kx_1^2 - D_3x_3^2 &= \frac{c_k-1}{D_2}. \end{aligned}$$

Similarly, if $D_1 = 2$, then (4) implies

$$\begin{aligned} x+1 &= 2D_2x_1^2 \\ 3x+1 &= 2D_3x_2^2 \\ c_kx+1 &= D_2D_3x_3^2, \end{aligned}$$

from which we obtain

$$\begin{aligned} 3D_2x_1^2 - D_3x_2^2 &= 1 \\ 2c_kx_1^2 - D_3x_3^2 &= \frac{c_k-1}{D_2}. \end{aligned}$$

Hence, to find all integer solutions of (4), it is enough to find all integer solutions of the systems of equations

$$d_1x_1^2 - d_2x_2^2 = j_1, \tag{20}$$

$$d_3x_1^2 - d_2x_3^2 = j_2, \tag{21}$$

where

- $d_1 = 3D_2$, D_2 is a square-free divisor of $c_k - 1 = (t_k+2)(t_k-2)/3$,
- $d_2 = D_3$, D_3 is a square-free divisor of $c_k - 3 = (s_k+2)(s_k-2)$, which is not divisible by 3,

- $(d_3, j_1, j_2) = (c_k, 2, \frac{c_k-1}{D_2})$ or $(d_3, j_1, j_2) = (2c_k, 1, \frac{c_k-1}{D_2})$.

We expect that most of the systems (20)–(21) are not solvable. To exclude as early as possible the unsolvable systems we considered the equations (20) and (21) separately modulo appropriate prime powers.

As $8 \mid c_k$ and $c_k \mid d_3$, and d_2 and j_2 are odd, the equation (21) is solvable modulo 8 only if $-d_2 j_2 \equiv 1 \pmod 8$.

Assume that equation (20) is solvable. Let p be an odd prime divisor of d_2. Then (20) implies

$$d_1 x_1^2 \equiv j_1 \pmod p,$$

hence

$$(d_1 x_1)^2 \equiv j_1 d_1 \pmod p,$$

i.e. $\left(\frac{j_1 d_1}{p}\right) = 1$, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Similarly, (21) implies $\left(\frac{j_2 d_3}{p}\right) = 1$. If q and r are odd prime divisors of d_1 and d_3 respectively, then we obtain the following conditions for the solvability of (20) and (21): $\left(\frac{-j_1 d_2}{q}\right) = 1$ and $\left(\frac{-j_2 d_2}{r}\right) = 1$.

Let finally p_1 be an odd prime divisor of j_2, such that $\mathrm{ord}_{p_1}(j_2)$ is odd. Then a necessary condition for solvability of equation (21) is: $\left(\frac{d_2 d_3}{p_1}\right) = 1$.

We performed this test for $3 \le k \le 40$ and we found that, apart from the systems listed in the following table, all are unsolvable except those of the form

$$3x_1^2 - x_2^2 = 2,$$
$$c_k x_1^2 - x_3^2 = c_k - 1,$$

and this system is equivalent to the system (2) and (3) which is completely solved by Theorem 1.

We considered in the case $k = 19$ equations (20) and (21), with the values of d_1, d_2, d_3, j_1, j_2 given in the table, modulo 5. We obtained

$$x_1^2 - 4x_2^2 \equiv 2 \pmod 5,$$
$$3x_1^2 - 4x_3^2 \equiv 1 \pmod 5.$$

k	$d_1,\ d_2,\ d_3,\ j_1,\ j_2$
19	$251210975091, 44809, 33713442698726470914 08, 2, 40261110431$
23/1	$3806315104884143835 27682077, 11263976658479,$ $25375434032560958901 8454720, 1, 1$
23/2	$19509779867757, 11263976658479, 2537543032560958901 8454720,$ $1, 19509779867761$
23/3	$58529339603283, 1, 1268771701628047945 09227360, 2,$ 6503259955919
35	$20288310329233162249058888791445649852717,$ $22542567032481291387843209768272944 28079,$ $13525540219488774832705925860963766568480, 1, 1$
37	$187060083, 1489467623820555129,$ $131194254072438972350592900266788017500 5208, 2,$ $210404462515563471150485216453 34887$

The first congruence implies $x_1^2 \equiv 1,\ 2$ or $3 \pmod 5$, and the second congrunce implies $x_1^2 \equiv 0$, 2 or $4 \pmod 5$. Hence, $x_1^2 \equiv 2 \pmod 5$, which is a contradiction.

In the cases $k = 23/3$ and $k = 35$ we used arithmetical properties of some real quadratic number fields.

In the case $k = 23/3$ we have $d_3 = 1268771701628047945 09227360$. The fundamental unit of the order $\mathbb{Z}[\sqrt{d_3}\,] = \mathbb{Z}[\sqrt{d_2 d_3}\,]$ is $\varepsilon = 11263976658481 + \sqrt{d_3}$. By a theorem of Nagell [15, Theorem 108a] the base solution of the equation

$$x_3^2 - 12687717012628047945 09227360 x_1^2 = -6503259955919$$

satisfies $0 < x_1^{(0)} < 1$, which is impossible.

In the case $k = 35$ the fundamental unit of the order $\mathbb{Z}[\sqrt{d_1 d_2}\,]$ is $u + \sqrt{d_1 d_2}$, where $u = 67627701097443874163529629304818832842 38$. A necessary condition for the solvability of the equation $d_1 x_1^2 - d_2 x_2^2 = 1$ is that $2d_1 | (u+1)$ (see [11]). But $\frac{u+1}{2d_1} = \frac{1}{6}$, and hence the last equation has no solution. $\square$

In the remaining three cases $k = 23/1$, $23/2$ and 37 all our methods fail to work.

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