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Combinatorial diophantine equations

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To Professor K. Győry on his 60th birthday

Abstract. In this paper some diophantine equations concerning binomial coefficients, power sums and product of consecutive integers are resolved. The equations are reduced to elliptic equations and then the program package SIMATH is used to determine the solutions.

1. Introduction

Many diophantine equations possess combinatorial background. Some special cases have been intensively investigated by several authors (see Table 1). These problems lead to equations of the type

$$f(x) = g(y) \quad \text{in integers} \quad x, y,$$

where f and g are polynomials with rational coefficients of degree three and two, respectively. The purpose of this note is to solve the unsolved equations from Table 1 by using the program package SIMATH [S]. The algorithm is based upon a theorem obtained by Gebel, Pethő and Zimmer [GPZ] and Stroeker and Tzanakis [ST], independently. Our results are

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summarized in Table 2 through Table 14. We remark that the function *faintp* of SIMATH gives all the integer points on the corresponding elliptic curves within a reasonable amount of CPU-time, however, as we will see, not necessarily all of them provide an integral solution to the original equation (see "other points" below the Tables).

For a positive integer k write

$$P_k(X) = X(X+1)\dots(X+k-1)$$

and

$$S_k(X) = 1^k + 2^k + \dots + X^k.$$

For general results on the equality of these combinatorial numbers we refer to [BP1] and [BP2], respectively.

We consider the equations of Table 1.

The solutions to equations (1) through (9) are known; they can be found in [M], [BK], [TW], [PW], [MB], [A1], [dW2], [A2] and [U], and [BK], respectively. To solve equations (10) through (24), we use the program package SIMATH, but as an example, in the case of equation (17) we follow the arguments of [GPZ] directly. In fact, equations (10) and (24) of Table 1 can be treated in an elementary way, and we will deal with these equations separately. We mention that equations (17) and (18) are independently resolved in [SdW].

Our notation $(x, y) = (a_1, \dots, a_n; b_1, \dots, b_m)$ will mean that (x, y) can be any of the pairs (a_i, b_j), $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$.

Theorem 1. a) *All the solutions of equation* (10) *are* $(x, y) = (-7, 2; -10, 7)$ *and* $(-5, -4, -3, -2, -1, 0; -3, -2, -1, 0)$.
b) *The only solution of equation* (24) *with* $x \geq 1$, $y \geq 0$ *is* $(x, y) = (1, 4)$.

Theorem 2. *All the solutions of the unsolved equations of Table 1 are just those which are summarized in the following Table 2 through Table 14.*

2. Proofs and Tables

Proof of Theorem 1. a) Let $f(x) = (x-2)(x-1)x(x+1)(x+2)(x+3)$. Now the equation $f(x) = P_4(y)$ is equivalent to $256x^6 + 768x^5 - 1280x^4 - 3840x^3 + 1024x^2 + 3072x + 256 = a^2$, where $a = 16y^2 + 48y$. Suppose now that $x > 0$ and put $g(x) = 256x^6 + 768x^5 - 1280x^4 - 3840x^3 + 1024x^2 + 3072x + 256$, $h(x) = 16x^3 + 24x^2 - 58x - 33$. It is easy to verify that for $x \geq 24$ we have $(h(x) - 1)^2 < g(x) < h(x)^2$, which is impossible. Checking

No.	Equation	Transformed equation
1	$P_3(x) = P_2(y)$	$s^3 - 4s + 2 = 2t^2$
2	$P_3(x) = P_4(y)$	$s^3 - s + 1 = t^2$
3	$P_3(x) = \binom{y}{2}$	$s^3 - 4s + 1 = t^2$
4	$P_3(x) = \binom{y}{4}$	$s^3 - 36s + 9 = t^2$
5	$P_6(x) = P_2(y)$	$s^3 - 10s^2 - 4s + 42 = 2t^2$
6	$\binom{x}{3} = \binom{y}{2}$	$s^3 - 4s + 6 = 6t^2$
7	$\binom{x}{3} = \binom{y}{4}$	$s^3 - 4s + 2 = 2t^2$
8	$S_2(x) = \binom{y}{2}$	$s^3 - s + 3 = 3t^2$
9	$S_2(x) = \binom{y}{4}$	$s^3 - s + 1 = t^2$
10	$P_6(x) = P_4(y)$	$s^3 - 5s^2 - s + 6 = t^2$
11	$P_6(x) = \binom{y}{2}$	$s^3 - 10s^2 - 4s + 41 = t^2$
12	$P_6(x) = \binom{y}{4}$	$s^3 - 30s^2 - 36s + 1089 = t^2$
13	$\binom{x}{3} = P_2(y)$	$s^3 - 4s + 12 = 3t^2$
14	$\binom{x}{3} = P_4(y)$	$s^3 - s + 6 = 6t^2$
15	$\binom{x}{6} = P_2(y)$	$s^3 - 5s^2 - s + 185 = 5t^2$
16	$\binom{x}{6} = P_4(y)$	$s^3 - 5s^2 - s + 725 = 5t^2$
17	$\binom{x}{6} = \binom{y}{2}$	$s^3 - 5s^2 - s + 95 = 10t^2$
18	$\binom{x}{6} = \binom{y}{4}$	$s^3 - 5s^2 - s + 35 = 35t^2$
19	$S_2(x) = P_2(y)$	$s^3 - s + 6 = 6t^2$
20	$S_2(x) = P_4(y)$	$s^3 - s + 24 = 6t^2$
21	$S_5(x) = P_2(y)$	$s^3 - s^2 + 12 = 3t^2$
22	$S_5(x) = P_4(y)$	$s^3 - s^2 + 48 = 3t^2$
23	$S_5(x) = \binom{y}{2}$	$s^3 - s^2 + 6 = 6t^2$
24	$S_5(x) = \binom{y}{4}$	$s^3 - s^2 + 2 = 2t^2$

Table 1.

the remaining cases, we obtain just the solutions stated in the first part of Theorem 1.

The proof of part b) is similar. Since $S_5(x) = (2x^6+6x^5+5x^4-x^2)/12$, the equation $S_5(x) = \binom{y}{4}$ is equivalent to $64x^6+192x^5+160x^4-32x^2+16 = a^2$, where $a = 4y^2 - 12y + 4$. We assume again that $x > 0$ and put $g(x) = 64x^6 + 192x^5 + 160x^4 - 32x^2 + 16$, $h(x) = 8x^3 + 12x^2 + x - 1$. Now one can verify easily that if $x \geq 2$, then $(h(x) - 1)^2 < g(x) < h(x)^2$, which is impossible. Hence the second part of Theorem 2 follows. $\square$

In the following Tables 2 through 14 we summarize the solutions of the remaining equations. We note that equations (12) and (19) lead to the same elliptic equation.

(11)	$P_6(x) = \binom{y}{2}$
Transformation:	$u = 18x^2 + 90x + 60$ $v = 54y - 27$
Elliptic curve:	E_{11}: $u^3 - 3024u - 33831 = v^2$
Integer points on E_{11}: $(u, \pm v) =$	Corresponding solutions:$(x, y) =$
$(-12, 27)$	$(-4, -1; 0, 1)$
$(2328, 112293)$	$(-14, 9; -2079, 2080)$
$(-48, 27)$	$(-3, -2; 0, 1)$
$(60, 27)$	$(-5, 0; 0, 1)$

Table 2.

Other points on E_{11}: $(u, \pm v) = (5280, 383643), (20616, 2960091)$.

(12)	$P_6(x) = \binom{y}{4}$
Transformation:	$u = 6x^2 + 30x + 20$ $v = 3y^2 - 9y + 3$
Elliptic curve:	E_{12}: $u^3 - 336u - 1271 = v^2$
Integer points on E_{12}: $(u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(-4, 3)$	$(-4, -1; 0, 1, 2, 3)$
$(-16, 3)$	$(-3, -2; 0, 1, 2, 3)$
$(20, 3)$	$(-5, 0; 0, 1, 2, 3)$

Table 3.

Other points on E_{12}: $(u, \pm v) = (442124, 293978883), (62, 465), (50, 327), (2312, 111165), (-10, 33), (39, 212), (4244, 276477), (-9, 32), (51, 338), (5216, 376707), (20696, 2977347), (150, 1823), (464, 9987), (-13, 30), (26, 87), (24, 67), (110, 1137), (-6, 23), (179, 2382), (212, 3075), (5879, 450768)$.

(13)	$\binom{x}{3} = P_2(y)$
Transformation:	$u = 6x - 6$ $v = 36y + 18$
Elliptic curve:	E_{13}: $u^3 - 36u + 324 = v^2$
Integer points on E_{13}: $(u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(6, 18)$	$(2; -1, 0)$
$(210, 3042)$	$(36; -85, 84)$
$(-6, 18)$	$(0; -1, 0)$
$(0, 18)$	$(1; -1, 0)$
$(30, 162)$	$(6; -5, 4)$
$(42, 270)$	$(8; -8, 7)$
$(1224, 42822)$	$(205; -1190, 1189)$

Table 4.

Other points on E_{13}: $(u, \pm v) = (-8, 10), (9, 27), (16, 62), (1, 17)$.

(14)	$\binom{x}{3} = P_4(y)$
Transformation:	$u = 6x - 6$ $v = 36y^2 + 108y + 36$
Elliptic curve:	E_{14}: $u^3 - 36u + 1296 = v^2$
Integer points on E_{14}: $(u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(54, 396)$	$(10; -5, 2)$
$(-6, 36)$	$(0; -3, -2, -1, 0)$
$(0, 36)$	$(1; -3, -2, -1, 0)$
$(6, 36)$	$(2; -3, -2, -1, 0)$
$(384, 7524)$	$(65; -16, 13)$

Table 5.

Other points on E_{14}: $(u, \pm v) = (-12, 0), (13, 55), (150, 1836), (21, 99), (10, 44), (36, 216), (1066, 34804), (138, 1620), (-11, 19), (82656, 23763564)$.

(15)	$\binom{x}{6} = P_2(y)$
Transformation:	$u = 45x^2 - 225x + 150$ $v = 8100y + 4050$
Elliptic curve:	$E_{15}:\ u^3 - 18900u + 15862500 = v^2$
Integer points on $E_{15}:\ (u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(150, 4050)$	$(0, 5; -1, 0)$
$(-120, 4050)$	$(2, 3; -1, 0)$
$(-30, 4050)$	$(1, 4; -1, 0)$
$(2400, 117450)$	$(-5, 10; -15, 14)$
$(3120, 174150)$	$(-5, 10; -22), (-6, 11; 21)$

Table 6.

Other points on E_{15}: $(u, \pm v) = (870, 25650), (-264, 1566),$
$(1905, 83025), (249, 5157), (8250, 749250), (64, 3862), (366, 7614),$
$(330, 6750), (-255, 2025), (130, 3950), (159720, 63832050), (25, 3925),$
$(-174, 3726), (600, 14850), (1014, 32238), (21030, 3049650),$
$(158505, 63105075), (5470, 404450), (10914707400, 1140297432700050).$

(16)	$\binom{x}{6} = P_4(y)$
Transformation:	$u = 45x^2 - 225x + 150$ $v = 8100y^2 + 24300y + 8100$
Elliptic curve:	$E_{16}:\ u^3 - 18900u + 65070000 = v^2$
Integer points on $E_{16}:\ (u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(150, 8100)$	$(0, 5; -3, -2, -1, 0)$
$(-120, 8100)$	$(2, 3; -3, -2, -1, 0)$
$(-30, 8100)$	$(1, 4; -3, -2, -1, 0)$

Table 7.

Other points on E_{16}: $(u, \pm v) = (-291, 6777), (3570, 213300),$
$(7980, 712800), (32550, 5872500), (61, 8009).$

(17)	$\binom{x}{6}=\binom{y}{2}$
Transformation:	$u=90x^2-450x+300$ $v=16200y-8100$
Elliptic curve:	$E_{17}\colon\ u^3-75600u+61290000=v^2$
Integer points on $E_{17}\colon\ (u,\pm v)=$	Corresponding solutions: $(x,y)=$
$(300,8100)$	$(0,5;0,1)$
$(-240,8100)$	$(2,3;0,1)$
$(-60,-8100)$	$(1,4;0,1)$
$(840,-24300)$	$(-1,6;-1,2)$
$(2460,121500)$	$(-3,8;-7,8)$
$(4800,-332100)$	$(-5,10;-20,21)$
$(11640,1255500)$	$(-9,14;-77,78)$

Table 8.

Other points on E_{17}: $(u,\pm v)=(-456,972),(80436,-22812516),$
$(516,-12636),(370585,225596125),(1785,-74925),(8400,769500),$
$(136,7316),(-375,6075),(84,-7452),\ (160,7300).$

(18)	$\binom{x}{6}=\binom{y}{4}$
Transformation:	$u=30x^2-150x+100$ $v=900y^2-2700y+900$
Elliptic curve:	$E_{18}\colon\ u^3-8400u+650000=v^2$
Integer points on $E_{18}\colon\ (u,\pm v)=$	Corresponding solutions: $(x,y)=$
$(100,900)$	$(0,5;0,1,2,3)$
$(-80,900)$	$(2,3;0,1,2,3)$
$(-20,900)$	$(1,4;0,1,2,3)$
$(280,4500)$	$(-1,6;-1,4)$
$(1600,63900)$	$(-5,10;-7,10)$

Table 9.

Other points on E_{18}: $(u,\pm v)=(80,700),(6220,490500),(-56,972),$
$(145,1575),(2780,146500),(196,2556),(1000,31500),(56,596),$
$(1220,42500),(-116,252),(25,675),(316,5436),(520,11700),(20,700),$

$$(1,801),(-100,700),(64,612),(6580,533700),(8081225,22972898725),$$
$$(2261,3415779),(75580,20778300).$$

(19)	$S_2(x)=P_2(y)$
Transformation:	$u=12x+6$ $v=72y+36$
Elliptic curve:	$E_{19}:\ u^3-36u+1296=v^2$
Integer points on $E_{19}:\ (u,\pm v)=$	Corresponding solutions: $(x,y)=$
$(54,396)$	$(4;-6,5)$
$(-6,36)$	$(-1;-1,0)$
$(150,1836)$	$(12;-26,25)$
$(6,36)$	$(0;-1,0)$
$(138,1620)$	$(11;-23,22)$

Table 10.

Other points on E_{19}: $(u,\pm v)=(-12,0),(13,55),(21,99),(0,36),(10,44),$
$(36,216),(-11,19),(384,7524),(1066,34804),(82656,23763564).$

(20)	$S_2(x)=P_4(y)$
Transformation:	$u=12x+6$ $v=72y^2+216y+72$
Elliptic curve:	$E_{20}:\ u^3-36u+5184=v^2$
Integer points on $E_{20}:\ (u,\pm v)=$	Corresponding solutions: $(x,y)=$
$(6,72)$	$(0;-3,-2,-1,0)$
$(-6,72)$	$(-1;-3,-2,-1,0)$

Table 11.

Other points on E_{20}: $(u,\pm v)=(-18,0),\ (21,117),(144,1728),(60,468),$
$(0,72),(34,208),(570,13608),(582,14040).$

(21)	$S_5(x) = P_2(y)$
Transformation:	$u = 6x^2 + 6x - 1$ $v = 36y + 18$
Elliptic curve:	$E_{21}:\ u^3 - 3u + 322 = v^2$
Integer points on $E_{21}:\ (u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(-1, 18)$	$(-1, 0; -1, 0)$

Table 12.

Other points on E_{21}: $(u, \pm v)$=$(17, 72), (2, 18), (9, 32), (-7, 0), (143, 1710)$.

(22)	$S_5(x) = P_4(y)$
Transformation:	$u = 6x^2 + 6x - 1$ $v = 36y^2 + 108y + 36$
Elliptic curve:	E_{22}: $u^3 - 3u + 1294 = v^2$
Integer points on E_{22}: $(u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(-1, 36)$	$(-1, 0; -3, -2, -1, 0)$

Table 13.

Other points on E_{22}: $(u, \pm v) = (2, 36), (47, 324), (-10, 18), (15, 68)$.

(23)	$S_5(x) = \binom{y}{2}$
Transformation:	$u = 12x^2 + 12x - 2$ $v = 72y - 36$
Elliptic curve:	E_{23}: $u^3 - 12u + 1280 = v^2$
Integer points on E_{23}: $(u, \pm v) =$	Corresponding solutions: $(x, y) =$
$(-2, 36)$	$(-1, 0; 0, 1)$
$(22, 108)$	$(-2, 1; -1, 2)$
$(862, 25308)$	$(-9, 8; -351, 352)$
$(142, 1692)$	$(-4, 3; -23, 24)$

Table 14.

Other points on E_{23}: $(u, \pm v) = (61, 477), (4, 36), (-10, 20), (16, 72),$
$(-11, 9),\ (38, 236)$.

3. Description of the algorithm

As an example, we illustrate the algorithm for finding the integer solutions in case of equation (17) of Table 1, and follow the discussion and terminology of [GPZ]. However, instead of the Theorem in [GPZ], which contains some errors, we use the corrected version of it, given in [PZGH].

Let, as above,

$$E_{17} = \{(u,v) \in \mathbb{Q}^2 \mid u^3 - 75600u + 61290000 = v^2\} \cup \{\mathcal{O}\},$$

where $\mathcal{O}$ denotes the point at infinity. In the sequel, we determine some parameters of E_{17} using SIMATH.

The modular invariant of E_{17} is

$$j = \frac{j_1}{j_2} = \frac{-1404928}{46899},$$

and the height of E_{17} is

$$\mu_\infty = 5.97704241\ldots .$$

To use the algorithm of [GPZ], one has to know a basis as well as the torsion group of E_{17}. Using SIMATH, it turns out that the only torsion point of E_{17} is $\mathcal{O}$, and the rank of E_{17} is $r = 2$. At this point we have to mention that to determine the rank of elliptic curves, SIMATH uses an algorithm which depends upon the validity of the famous conjecture of Birch and Swinnetorn–Dyer. However, using the program *mwrank* of Cremona, one can calculate the rank of E_{17}, as well as the ranks of the other elliptic curves occuring in Tables 11 through 23, independently of any conjecture. The ranks obtained by SIMATH and *mwrank* were identical in every case. For the algorithm used by *mwrank*, we refer to [C].

Using SIMATH again, we obtain that a basis of the Mordell–Weil group of E_{17} is $\{P_1 = (300, 8100),\ P_2 = (-240, 8100)\}$ with

$$\hat{h}(P_1) = 0.42722736\ldots, \quad \hat{h}(P_2) = 0.44856058\ldots ,$$

where $\hat{h}(\,.\,)$ is the Néron–Tate height. It is well known that $\hat{h}(\,.\,)$ is a positive semidefinite quadratic form. For the smallest eigenvalue λ_1 of $\hat{h}(\,.\,)$ we obtain $\lambda_1 = 0.24519249\ldots .$

The real and complex periods of E_{17} are

$$\omega_1 = 1.33636708\dots$$

and

$$\omega_2 = 0.66818354\dots + i\cdot 0.34901212\dots,$$

respectively. Put $\tau = \omega_2/\omega_1$; thus

$$\mathrm{Im}(\tau) = 0.26116486\dots.$$

Set $c_1 = \max\{(\log(2^{13/6}/\omega_1))/\lambda_1, 1\}$; we have $c_1 < 4.94254024$. Moreover, with the notation $h = \log(11497758840000)$ we have

$$\max\left\{h, \frac{3\pi}{\mathrm{Im}(\tau)}\right\} \le 36.09$$

and

$$\max\left\{\hat{h}(P_i), h, \frac{3\pi|u_i|^2}{\omega_1^2\,\mathrm{Im}(\tau)}\right\} = h \quad \text{for } i = 1, 2.$$

Let

$$V_0 = \exp(36.09)$$

and

$$V_i = e^h = 11497758840000 \quad \text{for } i = 1, 2.$$

We define

$$c_2 := \max\left\{\frac{C}{\lambda_1}, 10^9\right\}\left(\frac{h}{2}\right)^4 \prod_{i=0}^{2} \log V_i$$

with the constant $C = 2.9\cdot 10^{6(r+2)}\cdot 4^{2(r+1)^2}\cdot (r+2)^{2r^2+13r+23.3} < 6.28\cdot 10^{69}$. Substituting the values of the parameters into the above formula, we obtain

$$c_2 < 4.28\cdot 10^{79}.$$

Let now $P = n_1P_1 + n_2P_2$ be an integer point on E_{17} with $n_1, n_2 \in \mathbb{Z}$ and put $N = \max\{|n_1|, |n_2|\}$. Using the above parameters, we have the estimate

$$N \le 2^{r+3}\sqrt{c_1c_2}\Big(\log\left(c_2(r+3)^{r+3}\right)\Big)^{(r+3)/2}$$

(see [PZGH]). That is, we obtain the initial bound

$$N < 2.36\cdot 10^{47}.$$

Now we will use B. M. M. de Weger's method (see [dW1]) to reduce this bound. We will outline de Weger's method in our special case only.

Consider the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ [C_0 u_1] & [C_0 u_2] & C_0 \end{pmatrix}$$

where u_1 and u_2 are the elliptic logarithms of P_1 and P_2, respectively. C_0 will be chosen a bit later. Using SIMATH again, we obtain

$$u_1 = 0.24604231\ldots \quad \text{and } u_2 = 0.40147506\ldots.$$

Let b_1, b_2, b_3 be the LLL-reduced basis of the lattice spanned by the coloumns of the above matrix. We set

$$N' = \frac{1}{2\sqrt{18}}\ \|b_1\|.$$

Now if $N' \geq N$, then we have the new estimate

$$N \leq \sqrt{\frac{1}{\lambda_1} \log \frac{2^{\frac{7}{6}} C_0}{\omega_1 N'}}.$$

We start with $C_0 = 10^{150}$, and we get $N' \geq 1.08 \cdot 10^{48}$, whence $N \leq 30.98$. Now we repeat the whole process with $C_0 = 10^{10}$ to obtain $N' \geq 72.87$ and $N \leq 8.87$. In the third iteration we set $C_0 = 10^8$ which yields $N' \geq 38.47$ and $N \leq 7.89$. A fourth application of de Weger's method yields the same bound for N. We now only have to test the integrality of the points

$$n_1 P_1 + n_2 P_2 \quad \text{with } |n_1|, |n_2| \leq 7.$$

We have to check the pairs

$$(u, v) \in \mathbb{Z}^2 \quad \text{with } \log |u| \leq \mu_\infty = 5.97704241\ldots$$

as well as the case $u < 0$ (cf. [GPZ]). After all, we obtain that the only integral points on E_{17} are those given in Table 8. $\square$

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