Publ. Math. Debrecen
56 / 3-4 (2000), 451–455

Note on a result of I. Nemes and A. Pethő concerning polynomial values in linear recurrences

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Dedicated to Professor Kálmán Győry on his 60th birthday

Abstract. Let G_n $(n = 0, 1, 2, \dots)$ be a linear recurrence of order s $(s \geq 2)$ and let $F(x)$ be a polynomial of degree q. In the paper, under some conditions, we prove that the equation $G_n = F(x)$ can have integer solutions only if $q < c$, where the constant c is effectively computable. Similar result was proved by I. Nemes and A. Pethő with another methods and stronger conditions.

Let G_n $(n = 0, 1, 2, \dots)$ be a linear recurrence sequence of rational integers of order s (≥ 2) satisfying the recurrence relation

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_s G_{n-s} \quad (n \geq s),$$

where $A_1, \dots, A_s$ and the initial terms $G_0, \dots, G_{s-1}$ are integers with $A_s \neq 0$ and $|G_0| + \cdots + |G_{s-1}| > 0$. Denote by $\alpha = \alpha_1, \alpha_2, \dots, \alpha_r$ the distinct roots of the polynomial

$$g(x) = x^s - A_1 x^{s-1} - A_2 x^{s-2} - \cdots - A_s.$$

In the followings we suppose that $\alpha = \alpha_1, \alpha_2, \dots, \alpha_r$ have multiplicity $m_1 = 1, m_2, \dots, m_r$, respectively and that $|\alpha| > |\alpha_2| \geq |\alpha_3| \geq \cdots \geq |\alpha_r|$.

Mathematics Subject Classification: 11B39.
Key words and phrases: linear recurrences, polynomial value.
Research (partially) was supported by the Hungarian OTKA foundation grant No. T 29 330 and 020295.

It is known that in this case the terms of the sequence can be expressed as

$$G_n = a\alpha^n + p_2(n)\alpha_2^n + \cdots + p_r(n)\alpha_r^n \quad (n \geq 0), \tag{1}$$

where p_i $(2 \leq i \leq r)$ is a polynomial of degree $m_i - 1$, furthermore a $(\neq 0)$ and the coefficients of the polynomials are algebraic numbers from the field $\mathbb{Q}(\alpha, \alpha_2, \dots, \alpha_r)$.

Let

$$F(x) = bx^q + b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0 \tag{2}$$

be a polynomial with integer coefficients supposing that $b \neq 0$, $q \geq 2$ and $k < q$. The Diophantine equation

$$G_n = F(x) \tag{3}$$

was investigated by several authors. It is known that if G_n is a nondegenerate second order linear recurrence ($s = 2$ and α_1/α_2 is not a root of unity) and $F(x) = bx^q$, then (3) has only finitely many integer solutions in variables $n \geq 0$, x and $q \geq 2$, furthermore, in the case $q = 2$, the solutions were exactly determined for some special sequences. Recently W. L. McDaniel [2], [3] has proved that the solutions of the equation

$$G_n = x^2 + x$$

are $(n, x) = (0, 0)$ and $(3, 1)$ if G_n is the Fibonacci sequence and $(n, x) = (0, 1)$ if G_n is the Lucas sequence. For general linear recurrences, with some restrictions, we know that the equality $G_n = bx^q$, with $G_n \neq a\alpha^n$, can be satisfied only if $q < c$, where c is an effectively computable constant depending on the sequence G and the constant b (e.g. see [4] and [6]).

A more general result was proved by I. Nemes and A. Pethő [5]. They proved the following: Let G_n be a linear recurrence defined by (1) and let $F(x)$ be a polynomial defined by (2). Suppose that $\alpha_2 \neq 1$, $|\alpha| = |\alpha_1| > |\alpha_2| > |\alpha_i|$ for $3 \leq i \leq r$, $G_n \neq a\alpha^n$ for $n > c_1$ and $k \leq qc_2$. Then all integer solutions n, $|x| > 1$, $q \geq 2$ of equation (3) satisfy $q < c_3$, where c_1, c_2 and c_3 are effectively computable positive constants depending on the parameters of the sequence G_n and the polynomial $F(x)$.

The purpose of this note is to show that the restrictions $\alpha_2 \neq 1$ and $|\alpha_2| > |\alpha_i|$ $(3 \leq i \leq r)$ are not necessary in the above result. Furthermore we shall show that c_2 can be arbitrary in the interval $0 < c_2 < 1$ except when $|\alpha|^k = |\alpha_2|^q$. We prove a theorem which extends the above result for some more general sequences using another method. In the theorem and in its proof $c_4, c_5, \dots$ will denote effectively computable positive constants which depend only on the sequence G_n and polynomial $F(x)$.

Theorem. *Let G_n be a linear recurrence defined by (1) and let $F(x)$ be a polynomial defined by (2). Suppose that $k < \gamma q$ for a fixed real number γ with $0 < \gamma < 1$ and that $|\alpha|^k \neq |\alpha_2|^q$. If $G_n \neq a\alpha^n$ or $F(x) \neq bx^q$ and equation (3) is satisfied by integers $n \geq 0$, x ($|x| > 1$) and $q \geq 2$ then $q < c_4$, where c_4 is an effectively computable positive constant depending on the sequence G_n, the polynomial $F(x)$ and γ.*

In the proof we shall use a result due to A. Baker [1].

Lemma. *Let*

$$\lambda = |\gamma_1 \log \omega_1 + \gamma_2 \log \omega_2 + \cdots + \gamma_t \log \omega_t|,$$

where ω_i's ($i = 1, 2, \ldots, t$) are algebraic integers different from zero and one and γ_i's are rational integers not all zero. We suppose that the logarithms mean their principal values and assume that ω_i have heights at most M_i (≥ 4), $\max(|\gamma_1|, |\gamma_2|, \ldots, |\gamma_{t-1}|) \leq B$ ($B \geq 4$) and $|\gamma_t| \leq B'$. If $\lambda \neq 0$, then for any δ with $0 < \delta < \frac{1}{2}$

$$\lambda > (\delta/B')^{C \cdot \log M_t} \cdot e^{-\delta B},$$

where $C > 0$ is an effectively computable constant depending only on t, $M_1, \ldots, M_{t-1}$ and on the degree of the field generated by ω_i's over the rational numbers.

Proof of the Theorem. Let G_n be a linear recurrence and let $F(x)$ be a polynomial defined by (1) and (2), respectively. Suppose that (3) and the conditions of the Theorem hold for some integers n and x. We suppose that $b_k \neq 0$ since in the case $F(x) = bx^q$ the Theorem was proved (see above). If $G_n \neq a\alpha^n$, i.e. $p_2(n), \ldots, p_r(n)$ are not all zero, then we can suppose that $p_2(n) \neq 0$. So (3) can be written in the form

$$\frac{a\alpha^n}{bx^q} = \left(1 + \sum_{i=0}^{k} \frac{b_i}{b \cdot x^{q-i}}\right) \cdot \left(1 + \sum_{i=2}^{r} \frac{p_i(n)}{a} \left(\frac{\alpha_i}{\alpha}\right)^n\right)^{-1} = (1 + \varepsilon_1)(1 + \varepsilon_2)^{-1}, \tag{4}$$

where

$$|\varepsilon_1| = \left|\frac{b_k}{b}\left(\frac{1}{x}\right)^{q-k}\right| \cdot \left|1 + \frac{b_{k-1}}{b_k}\left(\frac{1}{x}\right) + \ldots\right| \tag{5}$$

and $\varepsilon_2 = 0$ or

$$|\varepsilon_2| = \left|\frac{p_2(n)}{a} \cdot \left(\frac{\alpha_2}{\alpha}\right)^n\right| \cdot \left|1 + \frac{p_3(n)}{p_2(n)}\left(\frac{\alpha_3}{\alpha_2}\right)^n + \dots\right|. \tag{6}$$

Since by (3) or (4)

$$|\alpha|^{n-c_5} < |x|^q < |\alpha|^{n+c_6}, \tag{7}$$

therefore by (5) and (6), using the conditions and supposing that $\varepsilon_1, \varepsilon_2 \neq 0$,

$$c_7\left|\frac{1}{\alpha}\right|^{\frac{q-k}{q}(n+c_6)} < |\varepsilon_1| < c_8\left|\frac{1}{\alpha}\right|^{\frac{q-k}{q}(n-c_5)} \tag{8}$$

and

$$c_9 n^j\left|\frac{\alpha_2}{\alpha}\right|^n < |\varepsilon_2| < c_{10} n^s\left|\frac{\alpha_2}{\alpha}\right|^n \tag{9}$$

follows with some $j \geq 0$. But $|\frac{1}{\alpha}|^{(q-k)/q} \neq |\alpha_2/\alpha|$ by the condition $|\alpha|^k \neq |\alpha_2|^q$, so $|\varepsilon_1| \neq |\varepsilon_2|$ for $n > c_{11}$ and

$$\left|\frac{a\alpha^n}{bx^q}\right| \neq 1.$$

So, using the Lemma with $t = 4$ and $\omega_t = x$, we have

$$\begin{aligned} \lambda &= \left|\log a + n\log\alpha - \log b - q\log x\right| \\ &> \left(\frac{\delta}{q}\right)^{c_{12}\log x} \cdot e^{-\delta n} = e^{-c_{12}(\log q - \log\delta)\log x - \delta n}. \end{aligned} \tag{10}$$

On the other hand by (4), (5), (6), (8) and (9), using the condition for k, we get

$$\lambda < 2|\varepsilon_1| + 2|\varepsilon_2| < e^{-c_{13}(1-\gamma)n} + e^{-c_{14}n} < e^{-c_{15}n}. \tag{11}$$

From (10) and (11) we obtain the inequality

$$c_{15}n < c_{12}(\log q - \log\delta)\ \log x + \delta n.$$

We can choose δ such that $c_{15} - \delta > 0$ and so from the above inequality, using (7),

$$c_{16} < c_{17} \log q \cdot \frac{\log x}{n} < c_{18} \frac{\log q}{q}$$

follows for any $q > c_{19}$. But it can be satisfied only by finitely many positive integers q and so our Theorem is proved. □

References

[1] A. Baker, A sharpening of the bounds for linear forms in logarithms II, *Acta Arithm.* **24** (1973), 33–36.

[2] W. L. McDaniel, Pronic Fibonacci numbers, *Fibonacci Quart.* **36** (1998), 56–59.

[3] W. L. McDaniel, Pronic Lucas numbers, *Fibonacci Quart.* **36** (1998), 60–62.

[4] P. Kiss, Differences of the terms of linear recurrences, *Studia Sci. Math. Hungar.* **20** (1985), 285–293.

[5] I. Nemes and A. Pethő, Polynomial values in linear recurrences, *Publ. Math. Debrecen* **31** (1984), 229–233.

[6] T. N. Shorey and C. L. Stewart, On the Diophantine equation $ax^{2t} + bx^t y + cy^2 = d$ and pure powers in recurrence sequences, *Math. Scand.* **52** (1983), 24–36.

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(Received August 16, 1999; revised January 10, 2000)