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Interpolation determinants of exponential polynomials

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To Kálmán Győry for his sixtieth birthday

1. Introduction

Let N be an integer ≥ 1. We call *interpolation determinant* any $N\times N$ determinant of the type

$$\Delta = \det\left(\frac{1}{t_j!}\left(\frac{\partial}{\partial z}\right)^{t_j}\varphi_i(\zeta_j)\right)_{\substack{1\leq i\leq N\\ 1\leq j\leq N}},$$

where $\varphi_1,\dots,\varphi_N$ denotes a sequence of analytic functions in one complex variable z, $\zeta_1,\dots,\zeta_N$ is a sequence of points located in a disk $B(0,R)\subset\mathbb{C}$ centered at the origin in which all the functions φ_i are holomorphic, and where $t_1,\dots,t_N$ are natural integers. The name *alternant* is also employed when all the $t_j=0$ (see for instance Chapter XI from [8]). We are specifically interested with the case of functions φ_i which are *exponential polynomials*, that is to say linear combinations of entire functions of the form $z^k e^{\omega z}$ where $k\in\mathbb{N}$ and $\omega\in\mathbb{C}$.

For any pair of integral N-tuples $k=(k_1,\dots,k_N)$ and $t=(t_1,\dots,t_N)$, any pair of N-tuples of complex numbers $X=(X_1,\dots,X_N)$ and $Y=(Y_1,\dots,Y_N)$, denote by $\Delta_{k,t}(X,Y)$ (or sometimes Δ in brief) the

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determinant

$$\Delta_{k,t}(X,Y) := \det\left(\sum_{l=0}^{\min(k_i,t_j)} \frac{1}{l!(k_i-l)!(t_j-l)!} X_i^{t_j-l} Y_j^{k_i-l} e^{X_i Y_j}\right)_{\substack{1\le i\le N\\ 1\le j\le N}}.$$

Leibniz's formula shows immediately that

$$\begin{aligned}&\sum_{l=0}^{\min(k,t)} \frac{1}{l!(k-l)!(t-l)!} X^{t-l} Y^{k-l} e^{XY}\\ &\qquad = \frac{1}{t!}\left(\frac{\partial}{\partial z}\right)^t \left(\frac{z^k}{k!} e^{Xz}\right)(Y) = \frac{1}{k!}\left(\frac{\partial}{\partial z}\right)^k \left(\frac{z^t}{t!} e^{Yz}\right)(X),\end{aligned}$$

so that $\Delta_{k,t}(X,Y)$ can be viewed as an interpolation determinant of exponential polynomials in two different ways:

$$\begin{aligned}\Delta &= \det\left(\frac{1}{t_j!}\left(\frac{\partial}{\partial z}\right)^{t_j}\left(\frac{z^{k_i}}{k_i!} e^{X_i z}\right)(Y_j)\right)\\ &= \det\left(\frac{1}{k_j!}\left(\frac{\partial}{\partial z}\right)^{k_j}\left(\frac{z^{t_i}}{t_i!} e^{Y_i z}\right)(X_j)\right).\end{aligned}$$

Any interpolation determinant of exponential polynomials can clearly be written as a linear combination $\sum_{k,X} c_{k,X}\Delta_{k,t}(X,Y)$ of determinants of the type Δ. The above duality formula implies that these determinants Δ satisfy the relation

$$\Delta_{k,t}(X,Y) = \Delta_{t,k}(Y,X)$$

in which are simultaneously interchanged the sequences of frequencies X with that of points Y and the sequence of order of derivations t with that of exponents k. We refer to [11] for an extension in several variables of this relation which is the basis of the duality between the transcendence methods of Gel'fond and of Schneider. From the point of view of interpolation determinants, the two methods coincide since they lead to the same determinants by the duality formula.

We plan to achieve an analytical study of the determinants $\Delta_{k,t}(X,Y)$ reflecting this symmetry at the level of the estimations. First we expand each interpolation determinant as a Taylor series and deduce from this general formula (quoted as Theorem 1 below) a precise upper bound for the absolute value $|\Delta_{k,t}(X,Y)|$. Next we show that, in the real case, the signs of all terms appearing in the Taylor expansion of $\Delta_{k,t}(X,Y)$ are the same under rather weak conditions, incidentally proving the nonvanishing of $\Delta_{k,t}(X,Y)$. This property follows easily from some standard facts about Schur polynomials. Finally, combining these results, we obtain a new simple proof of the Gel'fond–Schneider Theorem in the real case.

2. Expansion in Taylor series of interpolation determinants

Let

$$\Delta = \det\left(\frac{1}{t_j!}\left(\frac{\partial}{\partial z}\right)^{t_j}\varphi_i(\zeta_j)\right)_{\substack{1\le i\le N\\ 1\le j\le N}}$$

be any interpolation determinant. By expanding each function

$$\varphi_i(z) = \sum_{\nu\ge 0} p_{i,\nu} z^\nu \qquad (1\le i\le N)$$

in a Taylor series around the origin, we obtain the following formula:

Theorem 1. *The interpolation determinant Δ is equal to the sum*

$$\Delta = \sum_{0\le\nu_1<\cdots<\nu_N} \det\left(p_{i,\nu_j}\right) \times \det\left(\binom{\nu_i}{t_j}\zeta_j^{\nu_i-t_j}\right)$$

where the summation index $(\nu_1,\ldots,\nu_N)$ ranges along the set of N-tuples of increasing natural integers.

Proof. For any N-tuple $\underline{z}=(z_1,\ldots,z_N)$ of complex numbers, set

$$\Phi(\underline{z}) = \det\left(\varphi_i(z_j)\right)_{\substack{1\le i\le N\\ 1\le j\le N}} = \det\begin{pmatrix}\mathcal{R}_1\\ \ldots\\ \mathcal{R}_N\end{pmatrix},$$

where $\mathcal{R}_i = (\varphi_i(z_1),\ldots,\varphi_i(z_N))$ denotes the i-th row in the matrix $(\varphi_i(z_j))$. From the Taylor series of the functions φ_i we deduce an expansion of $\mathcal{R}_i$ as an infinite sum of rows

$$\mathcal{R}_i = \sum_{\nu\ge 0} p_{i\nu}\mathcal{L}_\nu \quad \text{with} \quad \mathcal{L}_\nu = (z_1^\nu,\ldots,z_N^\nu).$$

By multilinearity on the rows of the determinant $\Phi(\underline{z})$ we find

$$\begin{aligned}\Phi(\underline{z}) &= \sum_{(\nu_1,\dots,\nu_N)\in\mathbb{N}^N} \left(\prod_{i=1}^N p_{i,\nu_i}\right) \times \det\begin{pmatrix}\mathcal{L}_{\nu_1}\\ \dots\\ \mathcal{L}_{\nu_N}\end{pmatrix}\\ &= \sum_{0\le\nu_1<\dots<\nu_N} \det(p_{i,\nu_j}) \times \det\begin{pmatrix}\mathcal{L}_{\nu_1}\\ \dots\\ \mathcal{L}_{\nu_N}\end{pmatrix}\\ &= \sum_{0\le\nu_1<\dots<\nu_N} \det(p_{i,\nu_j}) \times \det(z_j^{\nu_i}),\end{aligned}$$

noting that each integral N-tuple has a unic representative with $0 \le \nu_1 \le \dots \le \nu_N$ under the permutation's action of the symmetric group $\mathfrak{S}_N$ on $\mathbb{N}^N$, and remarking that in the first sum we can restrict to integral N-tuples having pairwise distinct components ν_j, since otherwise the corresponding term vanishes. Derivating now the function Φ, we find the formula

$$\begin{aligned}&\frac{1}{t_1!\dots t_N!}\left(\frac{\partial}{\partial z_1}\right)^{t_1}\dots\left(\frac{\partial}{\partial z_N}\right)^{t_N}\Phi(\underline{z})\\ &\quad= \sum_{0\le\nu_1<\dots<\nu_N} \det\left(p_{i,\nu_j}\right) \times \det\left(\binom{\nu_i}{t_j} z_i^{\nu_i-t_j}\right).\end{aligned}$$

from which follows the expansion

$$\begin{aligned}\Delta &= \frac{1}{t_1!\dots t_N!}\left(\frac{\partial}{\partial z_1}\right)^{t_1}\dots\left(\frac{\partial}{\partial z_N}\right)^{t_N}\Phi(\zeta_1,\dots,\zeta_N)\\ &= \sum_{0\le\nu_1<\dots<\nu_N} \det(p_{i,\nu_j}) \times \det\left(\binom{\nu_i}{t_j} \zeta_i^{\nu_i-t_j}\right). \qquad \square\end{aligned}$$

Corollary 2. *Let k and t be integral N-tuples, X and Y be N-tuples of complex numbers. Then*

$$\Delta_{k,t}(X,Y) = \sum_{0\le\nu_1<\dots<\nu_N} \frac{\det\left(\binom{\nu_j}{k_i} X_i^{\nu_j-k_i}\right) \times \det\left(\binom{\nu_i}{t_j} Y_j^{\nu_i-t_j}\right)}{\nu_1! \times \dots \times \nu_N!}.$$

Proof. Using for instance the first expression of $\Delta_{k,t}(X,Y)$ as an interpolation determinant with

$$\varphi_i(z) = \frac{1}{k_i!} z^{k_i} e^{X_i z} = \sum_{\nu \geq 0} \frac{\binom{\nu}{k_i} X_i^{\nu - k_i}}{\nu!} z^\nu \quad (1 \leq i \leq N)$$

we obtain the formula. □

3. An upper bound of $|\Delta|$

We shall bound $|\Delta_{k,t}(X,Y)|$ by estimating the absolute value of each term in the sum of Theorem 1. This approach was already used in [4] for the Six Exponentials Theorem, and in [1] to obtain sharp lower bounds for p-adic linear forms in two logarithms. Of course it is also possible to use the classical Schwarz Lemma as was done in [5]–[6]. In the Archimedian case, which is our context here, some ratio ρ of radii then plays an important role, especially for the numerical value of the constants occurring in the theory of linear forms in logarithms. In Theorem 3 below, we recover an analogue of ρ whose definition is however different. It should be interesting to test numerically the following type of estimation in the context of [5]–[6].

Theorem 3. *Let R and S be positive real numbers such that*

$$\max_{1 \leq j \leq N}(|Y_j|) \leq R \quad \text{and} \quad \max_{1 \leq i \leq N}(|X_i|) \leq S.$$

Suppose that the ratio $\rho := \frac{N}{(R+1)(S+1)}$ is ≥ 1. Then we have an upper bound of the shape

$$\log|\Delta_{k,t}(X,Y)| \leq \left(-\frac{\log \rho}{2} + \frac{3}{4} + \frac{1}{4\rho^2} \right) N^2 + cN \log N$$

for some universal constant c when N is large enough.

Proof. From Corollary 2 let us write

$$\Delta = \sum_{0 \leq \nu_1 < \cdots < \nu_N} \frac{\det\left(\binom{\nu_j}{k_i} X_i^{\nu_j - k_i}\right) \times \det\left(\binom{\nu_i}{t_j} Y_j^{\nu_i - t_j}\right)}{\nu_1! \times \cdots \times \nu_N!}.$$

The idea is simple. Since the summation's indices ν satisfy the condition

$$\nu_1 \geq 0,\ \nu_2 \geq 1, \ldots,\ \nu_N \geq N-1,$$

which implies

$$\nu_1! \times \cdots \times \nu_N! \geq 0! \times 1! \times \cdots \times (N-1)! \geq N^{(1/2)N^2 - o(N^2)},$$

the denominator in each term of the above formula for Δ is much larger than the numerator provided that N is much greater than $\Omega{:=}(R{+}1)(S{+}1)$. More precisely, we expand the two determinants of the numerator and bound

$$\left| \binom{\nu_j}{k_i} X_i^{\nu_j - k_i} \right| \leq (|X_i| + 1)^{\nu_j} \leq (S+1)^{\nu_j}$$

$$\left| \binom{\nu_i}{t_j} Y_j^{\nu_i - t_j} \right| \leq (|Y_j| + 1)^{\nu_i} \leq (R+1)^{\nu_i}.$$

We obtain

$$\begin{aligned} |\Delta| &\leq (N!)^2 \sum_{0 \leq \nu_1 < \cdots < \nu_N} \frac{\Omega^{\nu_1 + \cdots + \nu_N}}{\nu_1! \times \cdots \times \nu_N!} \\ &\leq (N!)^2 \sum_{\nu_1 \geq 0, \nu_2 \geq 1, \ldots, \nu_N \geq N-1} \frac{\Omega^{\nu_1 + \cdots + \nu_N}}{\nu_1! \times \cdots \times \nu_N!} = (N!)^2 \prod_{n=0}^{N-1} \left(\sum_{\nu \geq n} \frac{\Omega^\nu}{\nu!} \right). \end{aligned}$$

Let M be the smallest integer which is $\geq \Omega$. For $0 \leq n < M \leq N$, we bound trivially

$$\sum_{\nu \geq n} \frac{\Omega^\nu}{\nu!} \leq \sum_{\nu \geq 0} \frac{\Omega^\nu}{\nu!} = e^\Omega,$$

and

$$\prod_{n=0}^{M-1} \left(\sum_{\nu \geq n} \frac{\Omega^\nu}{\nu!} \right) \leq e^{M\Omega} \leq e^{\Omega^2 + \Omega}.$$

If $M = N$, then $\rho \leq N/(N-1)$ and Theorem 3 is satisfied for some constant c. When $n \geq M$, the sum $\sum_{\nu \geq n} \frac{\Omega^\nu}{\nu!}$ has approximately the same magnitude as its first term:

$$\sum_{\nu\geq n}\frac{\Omega^\nu}{\nu!}=\frac{\Omega^n}{n!}\left(1+\frac{\Omega}{n+1}+\frac{\Omega^2}{(n+1)(n+2)}+\cdots\right)$$
$$\leq\frac{\Omega^n}{n!}\left(1+\frac{\Omega}{n+1}+\left(\frac{\Omega}{n+1}\right)^2+\cdots\right)=\frac{\Omega^n}{n!}\frac{n+1}{n+1-\Omega}.$$

Notice that

$$\prod_{n=M}^{N-1}\frac{n+1}{n+1-\Omega}\leq\prod_{n=M}^{N-1}\frac{n+1}{n+1-M}=\prod_{n=1}^{N-M}\frac{n+M}{n}=\binom{N}{N-M}\leq 2^N.$$

We deduce the upper bound

$$\prod_{n=M}^{N-1}\left(\sum_{\nu\geq n}\frac{\Omega^\nu}{\nu!}\right)\leq 2^N\prod_{n=M}^{N-1}\frac{\Omega^\nu}{\nu!}=2^N\times\Omega^{(N-M)(N+M-1)/2}\times\frac{\prod_{\nu=1}^{M-1}\nu!}{\prod_{\nu=1}^{N-1}\nu!}.$$

Then the Euler–MacLaurin summation formula, together with the elementary estimate

$$\nu^\nu e^{-\nu}\leq\nu!\leq 3\nu^{\nu+(1/2)}e^{-\nu},$$

which is valid for any integer $\nu\geq 1$, implies that

$$\sum_{\nu=1}^{n-1}\log\nu!=\frac{1}{2}n^2\log n-\frac{3}{4}n^2+\mathcal{O}(n\log n).$$

Combining the above estimates, we obtain the upper bound

$$\begin{aligned}\log|\Delta|&\leq-\frac{1}{2}N^2\log\frac{N}{\Omega}+\frac{1}{2}M^2\log\frac{M}{\Omega}+\frac{3}{4}N^2-\frac{3}{4}M^2+\Omega^2+\mathcal{O}(N\log N)\\&\leq-\frac{1}{2}N^2\log\rho+\frac{3}{4}N^2+\frac{1}{4}\Omega^2+\mathcal{O}(N\log N)\\&\leq\left(-\frac{\log\rho}{2}+\frac{3}{4}+\frac{1}{4\rho^2}\right)N^2+\mathcal{O}(N\log N).\end{aligned}$$ □

Remark 1. In the above proof, we have bounded $\binom{n}{k}|x|^{n-k}$ by $(1+|x|)^n$ for any integer $k\geq 0$ by grace of the binomial formula. Obviously, this argument is not always the most efficient. For example, we can replace in Theorem 3 the ratio ρ by $N/(R+1)S$ whenever all the $k_i=0$, by $N/R(S+1)$ if all the $t_j=0$, and by N/RS when $k_i=t_j=0$ for all i,j.

Remark 2. Let us introduce the function Φ in one complex variable z defined by

$$\begin{aligned}\Phi(z) &= \det\left(\frac{1}{t_j!}\left(\frac{\partial}{\partial z}\right)^{t_j}\varphi_i(Y_j z)\right)\\ &= \det\left(\sum_{l=0}^{\min(k_i,t_j)}\frac{1}{l!(k_i-l)!(t_j-l)!}(Y_j z)^{k_i-l}X_i^{t_j-l}e^{X_iY_jz}\right)\end{aligned}$$

with $\varphi_i(z) = (k_i!)^{-1}z^{k_i}e^{X_iz}$, in such a way that $\Delta = \Phi(1)$. The function Φ has a zero at the origin with multiplicity $\geq \frac{N^2-N}{2} - \sum t_j$. If we bound

$$\left|\sum_{l=0}^{\min(k_i,t_j)}\frac{1}{l!(k_i-l)!(t_j-l)!}(Y_j z)^{k_i-l}X_i^{t_j-l}e^{X_iY_jz}\right| \\ \leq \frac{(1+|X_i|)^{t_j}}{t_j!}e^{|Y_j|\,|z|+|X_i|\,|Y_j|\,|z|} \leq e^{S+1+R|z|+RS|z|} \leq e^{\Omega|z|}$$

for any complex number z with modulus ≥ 1, the usual Schwarz Lemma applied to the function Φ in the disk $|z| \leq N/(2\Omega)$ gives the slightly weaker upper bound

$$\log|\Delta| \leq \left(-\frac{1}{2}\log\rho + \frac{1}{2} + \frac{\log 2}{2}\right)N^2 + \mathcal{O}\Big((N + \sum t_j)\log N\Big).$$

That is the method used in [5]–[6]–[9] to estimate analytically interpolation determinants.

4. About positivity of interpolation determinants

In this section we are concerned with the special case of N-tuples X and Y which are real. It turns out that under the rather weak conditions (i) or (ii) of Theorem 6 below, all the terms occurring in the sum from Corollary 2 have the same sign. Therefore $\Delta_{k,t}(X,Y)$ is nonzero. On the other hand, this property shows that the absolute estimations for summing series that we achieved in §3 are essentially optimal, at least in the real case.

Let us begin by a digression on Schur polynomials S_ν. For each integral N-tuple $\nu = (\nu_1, \ldots, \nu_N)$ satisfying $0 \le \nu_1 < \cdots < \nu_N$, set

$$S_\nu(X_1, \ldots, X_N) = \frac{\det\left(X_j^{\nu_i}\right)}{\prod\limits_{1 \le i < j \le N} (X_j - X_i)}.$$

We shall use the following basic facts concerning Schur's polynomials

Theorem 4. *The ratio S_ν is a symmetric polynomial with positive integral coefficients.*

Proof. We shall only give some hints about this assertion; a complete proof can be found in Section I.3 of [7]. First, it is clear that S_ν is a symmetric polynomial with integral coefficients since the numerator $\det(X_j^{\nu_i})$ is an alternating polynomial which is therefore divisible by the discriminant $\prod_{1 \le i < j \le N}(X_j - X_i)$. The only non trivial point is in the positivity of the coefficients of S_ν. They have the following combinatorial interpretation. Since $0 \le \nu_1 < \cdots < \nu_N$, define λ_i by the relation

$$\nu_i = i - 1 + \lambda_{N+1-i} \quad (1 \le i \le N)$$

so that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N \ge 0.$$

The polynomial S_ν is most often denoted s_λ in connection with the N-tuple λ viewed as a partition of the integer $\lambda_1 + \cdots + \lambda_N$. We associate to the partition λ its *Ferrers diagram* which contains λ_1 boxes on its first row, ... , λ_N boxes on its N-th row:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

A *tableau* of shape λ is made with a Ferrers diagram of the partition λ whose boxes are filled with integers between 1 and N in such a way that in each row the sequence of integers read from left to right is non decreasing while in each column the sequence increases from top to bottom:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Let $\mu = (\mu_1, \ldots, \mu_N)$ be an N-tuple of integers ≥ 0. Let $K_{\lambda,\mu}$ be the number of tableaux of shape λ in which the integer n appears exactly μ_n times in the tableau for each $n = 1, \ldots, N$. Then we have the explicit formula

$$S_\nu(X_1, \ldots, X_N) = \sum_{\substack{\mu_1 \geq 0, \ldots, \mu_N \geq 0 \\ \mu_1 + \cdots + \mu_N = \lambda_1 + \cdots + \lambda_N}} K_{\lambda,\mu} X_1^{\mu_1} \cdots X_N^{\mu_N}. \qquad \square$$

By differentiating the defining relation

$$\det\left(X_i^{\nu_j}\right) = S_\nu(X_1, \ldots, X_N) \times \prod_{1 \leq i < j \leq N} (X_j - X_i),$$

we can obviously express any interpolation determinant of monomial functions

$$\det\left(\binom{\nu_i}{t_j} \zeta_j^{\nu_i - t_j}\right)$$

in term of the polynomial S_ν and its partial derivatives. This relation is quite simple when all the successive derivatives of order $< T_k$ occur at n given points $\xi_k \in \mathbb{C}$ for $k = 1, \ldots, n$.

Theorem 5. *Let n be a positive integer, $\xi_1, \ldots, \xi_n$ be complex numbers and $T_1, \ldots, T_n$ be positive integers such that $T_1 + \cdots + T_n = N$. Denote by ζ and t the N-tuples*

$$\zeta = \Big(\underbrace{\xi_1, \ldots, \xi_1}_{T_1 \text{ times}}, \ldots\ldots, \underbrace{\xi_n, \ldots, \xi_n}_{T_n \text{ times}}\Big)$$
$$t = \Big(0, 1, \ldots, T_1 - 1, \ldots\ldots, 0, 1, \ldots, T_n - 1\Big).$$

For each integral N-tuple ν with $0 \leq \nu_1 < \cdots < \nu_N$, we have the equality

$$\det\left(\binom{\nu_i}{t_j} \zeta_j^{\nu_i - t_j}\right) = S_\nu(\zeta) \prod_{1 \leq k < l \leq n} (\xi_l - \xi_k)^{T_k T_l}.$$

Proof. Let $X_1, \ldots, X_N$ be independent variables. Denote by

$$V(X) = \prod_{1 \leq i < j \leq N} (X_j - X_i)$$

the discriminant in these variables, and by $\partial^{[\tau]}$ the differential operators

$$\partial^{[\tau]} = \frac{1}{\tau_1!}\left(\frac{\partial}{\partial X_1}\right)^{\tau_1} \cdots \frac{1}{\tau_N!}\left(\frac{\partial}{\partial X_N}\right)^{\tau_N}$$

for each $\tau = (\tau_1, \ldots, \tau_N) \in \mathbb{N}^N$. Applying the operator $\partial^{[t]}$ to both members of the equality $\det(X_i^{\nu_j}) = VS_\nu$ we obtain the formula

$$\det\left(\binom{\nu_i}{t_j} X_i^{\nu_i - t_j}\right) = \sum_{\tau_1=0}^{t_1} \cdots \sum_{\tau_N=0}^{t_N} \partial^{[\tau]} V \cdot \partial^{[t-\tau]} S_\nu.$$

Let us check the formula

$$\partial^{[\tau]} V(\zeta) = \begin{cases} 0 & \text{if } \tau \le t \text{ and } \tau \ne t \\ \displaystyle\prod_{1 \le k < l \le n} (\xi_l - \xi_k)^{T_k T_l} & \text{if } \tau = t. \end{cases}$$

which obviously implies Theorem 5. To that purpose, decompose the discriminant in a product

$$V = \prod_{1 \le k \le n} V_k \prod_{1 \le k < l \le n} R_{k,l}$$

where

$$V_k = \prod_{S_{k-1}+1 \le i < j \le S_k} (X_j - X_i) \quad \text{and} \quad R_{k,l} = \prod_{i=S_{k-1}+1}^{S_k} \prod_{j=S_{l-1}+1}^{S_l} (X_j - X_i)$$

with $S_k = T_1 + \cdots + T_k$. It is clear that

$$R_{k,l}(\zeta) = (\xi_l - \xi_k)^{T_k T_l} \qquad (1 \le k < l \le n).$$

On the other hand V_k, now viewed as a polynomial in the variables $X_{S_{k-1}+1}, \ldots, X_{S_k}$, vanishes at the point $(\xi_k, \ldots, \xi_k) \in \mathbb{C}^{T_k}$ with multiplicity $(T_k^2 - T_k)/2$. It follows that

$$\partial^{[\tau]}\left(\prod_{k=1}^{n} V_k\right)(\zeta) = 0$$

for any $\tau \in \mathbb{N}^N$ with order $\tau_1 + \cdots + \tau_N < \sum_k (T_k^2 - T_k)/2 = t_1 + \cdots + t_N$. Moreover

$$\partial^{[t]} \left(\prod_{k=1}^{n} V_k \right) (\zeta) = \prod_{k=1}^{n} \partial^{[(0,1,\ldots,T_k-1)]} V_k(\xi_k, \ldots, \xi_k) = 1$$

as easily seen using for instance Vandermonde's formula. □

Theorem 6. *Let m and n be two positive integers. Let $K_1, \ldots, K_m$ (resp. $T_1, \ldots, T_n$) be positive integers whose sum is equal to N. Let $x_1, \ldots, x_m$ (resp. $y_1, \ldots, y_n$) be real numbers which are pairwise distinct. Finally, let a and b be two integers ≥ 0. Set*

$$k = (a, a+1, \ldots, a+K_1-1, \ldots\ldots, a, a+1, \ldots, a+K_m-1)$$

$$t = (b, b+1, \ldots, b+T_1-1, \ldots\ldots, b, b+1, \ldots, b+T_n-1)$$

$$X = \Big(\underbrace{x_1, \ldots, x_1}_{K_1 \text{ times}}, \ldots\ldots, \underbrace{x_m, \ldots, x_m}_{K_m \text{ times}} \Big)$$

$$Y = \Big(\underbrace{y_1, \ldots, y_1}_{T_1 \text{ times}}, \ldots\ldots, \underbrace{y_n, \ldots, y_n}_{T_n \text{ times}} \Big).$$

Then the determinant $\Delta_{k,t}(X, Y)$ is nonzero in both of the following two cases:

(i) *$a = b = 0$,*

(ii) *the real numbers $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are all positive.*

Proof. Let us begin with the assumptions (ii). Theorem 1 gives us the expansion

$$\Delta = \sum_{0 \leq \nu_1 < \cdots < \nu_N} \frac{\det\left(\binom{\nu_j}{k_i} X_i^{\nu_j - k_i}\right) \times \det\left(\binom{\nu_i}{t_j} Y_j^{\nu_i - t_j}\right)}{\nu_1! \times \cdots \times \nu_N!}.$$

First notice that we can restrict the above sum to N-tuples ν satisfying

$$\max(a, b) \leq \nu_1 < \cdots < \nu_N.$$

Suppose otherwise that $\nu_1 < \max(a, b)$. Then at least one of the two determinants occurring in the numerator of the term indexed by ν vanishes, because the first row or the first column of the corresponding matrix is identically zero since all the k_i are $\geq a$ and all the t_j are $\geq b$. Now remark that

$$\binom{\nu_j}{k_i}=\frac{\nu_j\ldots(\nu_j-a+1)}{k_i\ldots(k_i-a+1)}\binom{\nu_j-a}{k_i-a},\qquad \binom{\nu_i}{t_j}=\frac{\nu_i\ldots(\nu_i-b+1)}{t_j\ldots(t_j-b+1)}\binom{\nu_i-b}{t_j-b}.$$

We deduce from Theorem 5 the formula

$$\begin{aligned}\Delta = &\prod_{1\leq k<l\leq m}(x_l-x_k)^{K_kK_l}\prod_{1\leq k<l\leq n}(y_l-y_k)^{T_kT_l}\\ &\times\left(\sum_{\max(a,b)\leq\nu_1<\cdots<\nu_N}p_\nu S_{\nu-a\underline{1}}(X)S_{\nu-b\underline{1}}(Y)\right),\end{aligned}$$

where we have set

$$\underline{1}=(1,\ldots,1),\quad p_\nu=\prod_{l=1}^{N}\frac{\binom{\nu_l}{a}\binom{\nu_l}{b}}{\nu_l!\binom{k_l}{a}\binom{t_l}{b}}.$$

All the terms $p_\nu S_{\nu-a\underline{1}}(X)S_{\nu-b\underline{1}}(Y)$ in the above sum are positive since the coefficients of the polynomials $S_{\nu-a\underline{1}}$ and $S_{\nu-b\underline{1}}$ are ≥ 0 and since the coordinates of the N-tuples X and Y are positive. It follows that the sum itself is positive.

Let us now consider (i). We shall reduce this case to (ii) by translating the components of X and Y. Let k and t stand for the above N-tuples with $a=b=0$. Then we have the following translation formula:

$$\Delta_{k,t}(X+\xi\underline{1},Y+\zeta\underline{1})=\Delta_{k,t}(X,Y)\exp\Big(N\xi\zeta+\xi\big(\textstyle\sum Y_i\big)+\big(\sum X_i\big)\zeta\Big),$$

which is valid for any complex numbers ξ and ζ. Let us prove this formula for $\zeta=0$; the general case follows by symmetry. From the initial definition, $\Delta_{k,t}(X+\xi\underline{1},Y)=\det\Big(c_{i,j}(\xi)\Big)$ with

$$\begin{aligned}c_{i,j}(\xi)=&\sum_{l=0}^{\min(k_i,t_j)}\frac{1}{l!(k_i-l)!(t_j-l)!}(X_i+\xi)^{t_j-l}Y_j^{k_i-i}e^{(X_i+\xi)Y_j}\\ =&e^{\xi Y_j}\left(\sum_{l=0}^{\min(k_i,t_j)}\sum_{\tau=0}^{t_j-l}\frac{1}{l!\tau!(k_i-l)!(t_j-l-\tau)!}\xi^\tau X_i^{t_j-l-\tau}Y_j^{k_i-l}e^{X_iY_j}\right).\end{aligned}$$

Viewing the summation index τ appearing in the above double sum as attached to the column labeled by j, we obtain the formula

$$c_{i,j}(\xi)=\sum_{\tau_j=0}^{t_j}\frac{e^{\xi Y_j}\xi^{\tau_j}}{\tau_j!}\left(\sum_{l=0}^{\min(k_i,t_j-\tau_j)}\frac{1}{l!(k_i-l)!(t_j-\tau_j-l)!}X_i^{t_j-\tau_j-l}Y_j^{k_i-l}e^{X_iY_j}\right),$$

which makes appear the j-th column of the matrix $\Big(c_{i,j}(\xi)\Big)$ as a linear combination of the j-th, $(j-1)$-th, $\dots,(j-t_j)$-th columns of the matrix $\Big(c_{i,j}(0)\Big)$. Remind that the sequence

$$t=(0,1,\dots,T_1-1,\dots\dots,0,1,\dots,T_n-1)$$

is composed by the disjoint union of n segments of consecutive integers beginning with 0. By multilinearity, we immediately see that the only nonzero contribution in the determinant $\det\Big(c_{i,j}(\xi)\Big)$ comes from $\tau_1=\cdots=\tau_N=0$. It follows that

$$\Delta_{k,t}(X+\xi\underline{1},Y)=\Delta_{k,t}(X,Y)\exp\Big(\xi(\sum Y_j)\Big).$$

After adding some sufficiently large real numbers ξ and ζ to the components of X and Y respectively, we may assume without loss of generality that all the numbers $x_1,\dots,x_m,y_1,\dots,y_n$ are positive. □

Remarks. When $a=b=0$, Theorem 6 is a reformulation of a well-known result, due to Pólya, on the number of real zeroes of exponential polynomials whose frequencies are real. The vanishing of $\Delta_{k,t}(X,Y)$ is equivalent to the existence of a non trivial relation between the rows of the interpolation matrix, which means that there exists some nonzero exponential polynomial of the shape

$$\sum_{\mu=0}^{m}\sum_{k=0}^{K_\mu-1}p_{k,\mu}z^k e^{x_\mu z}.$$

vanishing with multiplicity $\geq T_\nu$ at the points y_ν for $\nu=1,\dots,n$. Pólya's Theorem then asserts the upper bound

$$\sum_{\nu=1}^{n}T_\nu<\sum_{\mu=1}^{m}K_\mu.$$

Notice that the initial proof of Pólya's Theorem is based on the classical Rolle's Lemma; see for instance Chapter 6 from [10]. Our approach is related to explicit formulas for $\Delta_{k,t}(X, Y)$. We remark also that the Wronskian formula from [2] can be easily deduced from our results, using the expansion of $\Delta_{k,t}(X, Y)$ for $n = 1$ and $y_1 = 0$, and next the translation formula.

5. The real Gel'fond–Schneider Theorem

As an example, we sketch in this section an alternative proof of Gel'fond–Schneider Theorem in the real case. One can find in [3] a detailed proof of this result, which is again based on Rolle's Lemma. Our argument may be viewed as its translation in terms of interpolation determinants.

Real Gel'fond–Schneider Theorem. *Let α be a positive real algebraic number distinct from 1, and let $\log\alpha$ be the real determination of its logarithm. Let β be an algebraic number which is real and irrational. The the number $\alpha^\beta = e^{\beta\log\alpha}$ is transcendental.*

Proof. We construct a family of interpolation determinants depending upon four integral parameters K, L, R, S satisfying $KL = RS$. First order, in any way, the two sequences of $N := KL = RS$ pairs

$$(k, \ell) \in \mathbb{N}^2; \qquad 0 \le k < K, \quad 0 \le \ell < L$$
$$(r, s) \in \mathbb{N}^2; \qquad 0 \le r < R, \quad 0 \le s < S.$$

Then we consider the N functions and the N points

$$\{\varphi_1, \ldots, \varphi_N\} = \left\{ \frac{z^k}{k!} e^{\ell(\log\alpha)z}; \quad 0 \le k < K,\ 0 \le \ell < L \right\}$$
$$\{\zeta_1, \ldots, \zeta_N\} = \Big\{ r + s\beta; \quad 0 \le r < R,\ 0 \le s < S \Big\},$$

together with the associated interpolation determinant

$$\Delta = \det\left(\varphi_i(\zeta_j)\right) = \det\left(\frac{(r+s\beta)^k \alpha^{\ell r} (\alpha^\beta)^{\ell s}}{k!} \right)_{\substack{(k,\ell)\\(r,s)}}.$$

Now suppose that

$$N \geq 10\Big(1 + (L-1)|\log \alpha|\Big)\Big(R - 1 + (S-1)|\beta|\Big).$$

Theorem 3 (or more precisely Remark 1 that follows Theorem 3) gives the upper bound

$$\log|\Delta| \leq -\frac{N^2}{3} + \mathcal{O}(N \log N).$$

On the other hand, Theorem 6 implies that Δ is nonzero. Assume to the contrary that α^β is an algebraic number. Then Liouville's inequality furnishes a lower bound of the type

$$\log|\Delta| \geq -cN\Big(K \log K + K \log \max(R,S) + L \max(R,S)\Big),$$

for some constant c depending only upon α and β. The upper bound and the lower bound contradict one another as soon as

$$K \log K + K \log \max(R,S) + L \max(R,S) \ll N.$$

Choose for instance $K = L^3$, $R = S = L^2$ with L large enough to find a final contradiction to the algebraicity of α^β. $\square$

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