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Dynamical connections and higher-order Lagrangian systems

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. We show that if ξ is a $(2r)$-order differential equation (semispray) on the $(2r-1)$-jet bundle $\mathcal{J}^{2r-1}Q$ whose paths are solutions of the non-autonomous Lagrange equations, then there is a connection Γ on $\mathcal{J}^{2r-1}Q$ whose paths are also solutions of the same equations. Moreover, Γ is a connection whose associated semispray is precisely ξ. This is an extension to higher-order Lagrangian dynamics of a previous result given by M. de Leon and P. Rodrigues [3].

1. Preliminaries

Throughout the text we shall keep in mind the results, definitions and notations previously introduced in [1], [2]. All structures and functions are assumed to be smooth. Let M be an m-dimensional manifold, called configuration manifold and Γ an $(r+1)$-order differential equation field on M. We recall here that Γ generates on T^rM two projectors: $A : T(T^rM) \to \text{Hor }(T^rM)$ and $B : T(T^rM) \to \text{Ver }(T^rM)$ such that $T(T^rM) = \text{Hor }(T^rM) \oplus \text{Ver}(T^rM)$ [1]. If $\bar{\xi}$ is an arbitrary semispray, i.e. an $(r+1)$-order differential equation field, then $\xi = A(\bar{\xi})$ is a semispray on T^rM which does not depend on the choice of $\bar{\xi}$. We call ξ the associated semispray of Γ. In the non-autonomous situation the relation between connections and semisprays becomes much more simple, as we will show below.

Let $\pi : Q \to X$ be a fibered manifold. In the following we assume that X is a connected real 1-dimensional manifold (i.e. $X = R$ or $X = S^1$) and Q is a real $(n+1)$-dimensional manifold. The r-order jet-prolongation is denoted by $\pi^r : \mathcal{J}^rQ \to X$. We denote by VQ the vertical bundle

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of Q, i.e. the vector subbundle of TQ defined as $VQ = \mathrm{Ker}(\pi')$ and by $V^0\mathcal{J}^rQ$ the vertical bundle of $\mathcal{J}^rQ$, defined as $V^0\mathcal{J}^rQ = \mathrm{Ker}(\pi_0^{r\prime})$, where $\pi_0^r : \mathcal{J}^rQ \to Q$ is the canonical projection.

Let $\pi_r^{r+1} : \mathcal{J}^{r+1}Q \to \mathcal{J}^rQ$ be the canonical projection, and $\eta_r : V\mathcal{J}^rQ \to \mathcal{J}^rQ$, the usual projection of the vertical bundle of $\mathcal{J}^rQ$. Then there exists $T : \mathcal{J}^{r+1}Q \to V\mathcal{J}^rQ$ satisfying $\eta_r \cdot \pi = \pi_r^{r+1}$ [1]. We use the map T to construct a differential operator d_T, which maps each function on $\mathcal{J}^rQ$ to a function on $\mathcal{J}^{r+1}Q$, and is called the partial time derivative. It follows that d_T is represented in coordinates $(t, q^i_{(k)})$, $1 \le i \le n$, $0 \le k \le r+1$ by the operator:

$$d_T = \sum_{h=0}^{r} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}}.$$

The operator d_T is a derivation in the sense that

$$d_T(f_1 \cdot f_2) = (d_T f_1)(\pi_r^{r+1*} f_2) + (\pi_r^{r+1*} f_1) d_T f_2; \quad f_1, f_2 \in \mathcal{F}(\mathcal{J}^rQ).$$

The extended operator, which we also denote d_T, bears the same relation to the operator on functions as a Lie derivative operator does to the action of a vector field on functions. Thus so far as its action on 1-forms is concerned (and this will be sufficient for our purposes), d_T satisfies the following rules:

1) for any 1-form α on $\mathcal{J}^rQ$, $d_T\alpha$ is an 1-form on $\mathcal{J}^{r+1}Q$
2) $d_T \cdot d = d \cdot d_T$
3) for any function f on $\mathcal{J}^rQ$,

$$d_T(f\alpha) = (d_T f)(\pi_r^{r+1*}\alpha) + (\pi_r^{r+1*} f) d_T(\alpha).$$

In particular, the coordinate 1-forms satisfy:

$$d_T(dq^i_{(k)}) = dq^i_{(k+1)}; \quad 0 \le k \le r.$$

We shall now define the lifts of a function f in Q to $\mathcal{J}^rQ$. For $k = 0, 1, \dots, r$ we define the $k+1$ lift f_{k+1} of a function f to $\mathcal{J}^rQ$, by $f_{k+1} = d(f_k)$, $k = 0, 1, \dots, r$ where $f_0 = f$ is a function on Q.

Let $\xi \in VQ_0$ be a vector field. The vector field $\xi^v \in V_0\mathcal{J}^rQ$ given by

$$\xi^v(f_r) = \pi_{r-1}^{r*}\xi(f_{r-1})$$

for any $f \in \mathcal{F}(Q)$, is called the vertical lift of ξ.

Using the vertical lift and the map $T : \mathcal{J}^rQ \to V\mathcal{J}^{r-1}Q$ we construct a canonical vector field on $\mathcal{J}^rQ$ as follows: $C = T^v$.

The coordinate representation of C is:

$$C = \sum_{h=0}^{r-1} (h+1) q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h+1)}}$$

where the vector field C is a generalization of the Liouville field or the dilation field on TQ.

We may also use the vertical lift construction to define an (1,1)-type tensor field S on $\mathcal{J}^rQ$, given by

$$S(\xi) = \left[\left(\pi^r_{r-1}\right)'\xi\right]^v, \qquad \forall \xi \in V\mathcal{J}^rQ.$$

The coordinate representation of S is

$$S = \sum_{h=0}^{r-1}(h+1)\frac{\partial}{\partial q^i_{(h+1)}} \otimes dq^i_{(h)}.$$

Therefore we transport the geometric structures defined on $V\mathcal{J}^rQ$ to $\mathcal{J}^rQ$. We may define a new tensor field $\tilde{S}$ of (1,1)-type $V\mathcal{J}^rQ$, by

$$\tilde{S} = S - C \otimes dt.$$

We define the adjoint $\tilde{S}^*$ of $\tilde{S}$, as the endomorphism of the exterior algebra $\Lambda(\mathcal{J}^rQ)$ of $\mathcal{J}^rQ$, locally given by

$$\tilde{S}^*(dt) = 0, \quad \tilde{S}^*(dq^i_{(0)}) = 0, \quad \tilde{S}^*(dq^i_{(h)}) = h\theta^i_{(h-1)}; \quad h = 1, 2, \dots, r$$

where

$$\theta^i_{(h)} = dq^i_{(h)} - q^i_{(h+1)}dt; \quad h = 0, 1, \dots, (r-1).$$

A vector field ξ on $\mathcal{J}^rQ$ is a semispray iff $S\xi = C$ and $\tilde{S}\xi = 0$.

Remark 1. It is not hard to see that a vector field ξ on $\mathcal{J}^rQ$ is a semispray iff $\theta^i_{(h)}(\xi) = 0$; $h = 0, 1, \dots, (r-1)$; $dt(\xi) = 1$.

In such a case ξ is locally given by

$$\xi = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)}\frac{\partial}{\partial q^i_{(h)}} + \xi^i\frac{\partial}{\partial q^i_{(r)}}. \tag{1.1}$$

Let now c be a global section of the affine bundle $\mathcal{J}^{r+1}Q \to \mathcal{J}^rQ$. One may construct a semispray ξ_c, given by

$$\xi_c = C^* \circ d_T. \tag{1.2}$$

Let $c(t, q^i_{(h)}) = (t, q^i_{(h)}, c^i)$; $0 \le h \le r$ be the local representation of c in a natural fibred chart. The semispray ξ_c is given by

$$\xi_c = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)}\frac{\partial}{\partial q^i_{(h)}} + c^i\frac{\partial}{\partial q^i_{(r)}}. \tag{1.3}$$

It follows then:

Proposition 1.1. *Let c be a section of the affine bundle $\mathcal{J}^{r+1}Q \to \mathcal{J}^r Q$ and ξ_c the spray given by (1.3), then we have:*

$$S\xi_c = C; \quad \tilde{S}\xi_c = 0; \quad \tilde{S} \circ L_{\xi_c}\tilde{S} = -\tilde{S} \circ \tilde{C}$$
$$(L_{\xi_c}\tilde{S} - rI)(L_{\xi_c}\tilde{S} + \tilde{C}) = 0.$$

Here L_{ξ_c} is the Lie derivative.

2. Semisprays and dynamical connections

The tensor fields S and $\tilde{S}$ on $\mathcal{J}^r Q$ permit us to give a characterization of a kind of connections for the fibration $\pi_0^r : \mathcal{J}^r Q \to Q$.

Definition 2.1. By a dynamical connection on $\mathcal{J}^r Q$ we mean a tensor field Γ of (1,1)-type on $\mathcal{J}^r Q$ satisfying

$$S\Gamma = \tilde{S}\Gamma = \tilde{S}; \quad \Gamma\tilde{S} = -\tilde{S}; \quad \Gamma S = -S. \tag{2.1}$$

By a straightforward computation we deduce from (2.1) that the local expression of Γ are:

$$\begin{cases} \Gamma\left(\dfrac{\partial}{\partial t}\right) = -\displaystyle\sum_{h=0}^{r-1} q^i_{(h+1)} \dfrac{\partial}{\partial q^i_{(h)}} + \Gamma^i_{(r)} \dfrac{\partial}{\partial q^i_{(r)}} \\ \Gamma\left(\dfrac{\partial}{\partial q^i_{(m)}}\right) = \dfrac{\partial}{\partial q^i_{(m)}} + \Gamma^{j(m)}_{i(r)} \dfrac{\partial}{\partial q^i_{(r)}} \qquad 0 \le m \le (r-1) \\ \Gamma\left(\dfrac{\partial}{\partial q^i_{(r)}}\right) = -\dfrac{\partial}{\partial q^i_{(r)}}. \end{cases} \tag{2.2}$$

The functions $\Gamma^i_{(r)} = \Gamma^i_{(r)}(t, q^i_{(h)})$; $\Gamma^{j(m)}_{i(r)} = \Gamma^{j(m)}_{i(r)}(t, q^i_{(h)})$ will be called the components of the connection Γ. From (2.2) we easily deduce that

$$\Gamma^3 - \Gamma = 0 \text{ and } \operatorname{rank}(\Gamma) = (r+1)n.$$

This type of polynomial structure is called $f(3,-1)$-structure in the literature [4]. Now, we can associate to Γ two canonical operators ℓ and m given by: $\ell = \Gamma^2$; $m = -\Gamma^2 + I$.
Then we have:

$$\ell^2 = \ell; \quad m^2 = m; \quad \ell m = m\ell = 0; \quad \ell + m = I \tag{2.3}$$

where ℓ and m are complementary projectors. From (2.3) we deduce that ℓ and m are locally given by:
(2.4)

$$\ell\left(\frac{\partial}{\partial t}\right) = -\sum_{h=0}^{r-1} q^i_{(h+1)}\frac{\partial}{\partial q^i_{(h)}} - \left(\Gamma^i_{(h)} + \sum_{h=0}^{r-1} q^i_{(h+1)}\Gamma^{j(h)}_{i(r)}\right)\frac{\partial}{\partial q^j_{(r)}}$$

$$\ell\left(\frac{\partial}{\partial q^i_{(k)}}\right) = \frac{\partial}{\partial q^i_{(k)}}; \quad m\left(\frac{\partial}{\partial q^i_{(k)}}\right) = \frac{\partial}{\partial q^i_{(k)}}; \qquad k = 0, 1, \dots, r$$

$$m\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)}\frac{\partial}{\partial q^i_{(h)}} + \left(\Gamma^j_{(r)} + \sum_{h=0}^{r-1} q^i_{(h+1)}\Gamma^{j(h)}_{i(r)}\right)\frac{\partial}{\partial q^j_{(r)}}.$$

If we put $\mathcal{L} = \mathrm{Im}\ell$, $\mathcal{M} = \mathrm{Im}m$, then we have that $\mathcal{L}$ and $\mathcal{M}$ are complementary distributions on $\mathcal{J}^rQ$, that is

$$T(\mathcal{J}^rQ) = \mathcal{M} \oplus \mathcal{L}.$$

From (2.4) we deduce that $\mathcal{L}$ is $(r+1)n$-dimensional and is locally spanned by $\left\{\frac{\partial}{\partial q^i_{(k)}}\right\}$, $k = 0, 1, \dots, r$; $\mathcal{M}$ is one-dimensional, and globally spanned by the vector field $\xi = m\left(\frac{\partial}{\partial t}\right)$. Taking into account the local expression of ξ we deduce that ξ is a semispray which will be called the canonical semispray associated to the dynamical connection Γ. Furthermore, we have $\Gamma^2\ell = \ell$ and $\Gamma m = 0$. Thus Γ acts on $\mathcal{L}$ as an almost product structure and trivially on $\mathcal{M}$. Since $\mathcal{M} = \mathrm{Ker}\,\Gamma$, Γ is said to be an $f(3,-1)$-structure on $\mathcal{J}^rQ$ of rank $(r+1)n$ and with parallelizable kernel. Moreover $\Gamma/\mathcal{L}$ has the eigenvalues $+1$ and -1. From (2.2) the eigenspaces corresponding to the eigenvalue $+1$ are the vertical subspaces V^0_z, $z \in \mathcal{J}^rQ$. Thus V is a distribution given by $z \to V^0_z$. The eigenspace at $z \in \mathcal{J}^rQ$ corresponding to the eigenvalue $+1$ will be denoted by H_z and called the strong-horizontal subspace at z. We have a canonical decomposition

$$T_z(\mathcal{J}^rQ) = \mathcal{M}_z \oplus H_z \oplus V^0_z$$

and obviously

$$T(\mathcal{J}^rQ) = \mathcal{M} \oplus H \oplus V^0$$

where H is the distribution $z \to H_z$.
Let us put $H'_z = \mathcal{M}_z \oplus H_z$; H'_z will be called the weak horizontal subspace at z. Then we have the following decompositions:

$$T_z(\mathcal{J}^rQ) = H'_z \oplus V^0_z, \qquad z \in \mathcal{J}^rQ$$

and

$$T(\mathcal{J}^rQ) = H' \oplus V^0 \tag{2.5}$$

where $H' : z \to H'_z$ is the corresponding distribution.

We notice that $\mathcal{L}$, $\mathcal{M}$, H and H' may be considered as vector bundles over $\mathcal{J}^r Q$; the bundles H and H' will be called strong and weak-horizontal bundles, respectively.

A vector field X on $\mathcal{J}^r Q$ which belongs to H (resp. H') will be called a strong (resp. weak) horizontal vector field. From (2.5) we have that the projection $\pi^r_{r-1} : \mathcal{J}^r Q \to \mathcal{J}^{r-1} Q$ induces an isomorphism

$$\pi^r_{0*} : H'_z \to T_{\pi^r_{r-1}(z)}(Q), \qquad z \in \mathcal{J}^r Q.$$

Then, if X is a vector field on $\mathcal{J}^{r-1}Q$, there exists an unique vector field $X^{H'}$ on $\mathcal{J}^r Q$ which is weak-horizontal and projects to X. The projection of $X^{H'}$ to H will be denoted by X^H.

From (2.2), by a straightforward computation we obtain

$$(2.6)\quad \begin{cases} \left(\dfrac{\partial}{\partial t}\right)^{H'} = \dfrac{\partial}{\partial t} + \left(\Gamma^j_{(r)} + \dfrac{1}{2}\displaystyle\sum_{h=0}^{r-1} q^i_{(h+1)} \Gamma^{j(h)}_{i(r)}\right) \dfrac{\partial}{\partial q^j_{(r)}} \\ \left(\dfrac{\partial}{\partial q^i_{(k)}}\right)^{H'} = \dfrac{\partial}{\partial q^i_{(k)}} + \dfrac{1}{2}\Gamma^{j(k)}_{i(r)} \dfrac{\partial}{\partial q^j_{(r)}} \qquad k = 0, 1, \dots, (r-1). \end{cases}$$

Then, if we put $H^{(k)}_i = \left(\dfrac{\partial}{\partial q^i_{(k)}}\right)^{H'}$ and $V^{(r)}_i = \dfrac{\partial}{\partial q^i_{(r)}}$, one deduces that $\left\{\xi, H^{(k)}_i, V^{(r)}_i\right\}$ is a local basis of vector fields on $\mathcal{J}^r Q$. In fact $\mathcal{M} = \langle \xi \rangle$, $H = \langle H^{(k)}_i \rangle$, $V = \langle V^{(r)}_i \rangle$ and $\left\{\xi, H^{(k)}_i, V^{(r)}_i\right\}$ is called an adapted basis of the $f(3,-1)$-structure Γ. In terms of $\left\{\xi, H^{(k)}_i, V^{(r)}_i\right\}$ (2.6) becomes

$$\left(\frac{\partial}{\partial t}\right)^{H'} = \xi - \sum_{k=0}^{r-1} q^i_{(k+1)} H^{(k)}_i; \quad \left(\frac{\partial}{\partial q^i_{(k)}}\right)^{H'} = H^{(k)}_i; \qquad k = 0, 1, \dots, (r-1).$$

Therefore, we obtain

$$\left(\frac{\partial}{\partial t}\right)^{H} = -\sum_{k=0}^{r-1} q^i_{(k+1)} H^{(k)}_i; \quad \left(\frac{\partial}{\partial q^i_{(k)}}\right)^{H} = H^{(k)}_i; \qquad k = 0, 1, \dots, (r-1).$$

If $X = \eta \dfrac{\partial}{\partial t} + \sum_{h=0}^{r-1} X^i_{(h)} \dfrac{\partial}{\partial q^i_{(h)}}$ is a vector field on $\mathcal{J}^{r-1}Q$ we have

$$X^H = \sum_{h=0}^{r-1} (X^i_{(h)} - \eta q^i_{()}) H^{(h)}_i .$$

Finally, we notice that the dual local basis of 1-forms of the adapted basis is given by $(dt, \theta^i_{(h)}, \psi^i)$, where

$$\theta^i_{(h)} = dq^i_{(h)} - q^i_{(h+1)} dt; \qquad h = 0, 1, \dots, (r-1) \text{ and}$$
$$\psi^i = -\left(\Gamma^i_{(r)} + \frac{1}{2} \sum_{h=0}^{r-1} q^i_{(h+1)} \Gamma^{j(h)}_{i(r)} \right) dt - \frac{1}{2} \sum_{h=0}^{r-1} \Gamma^{i(h)}_{j(r)} dq^j_{(h)} + dq^i_{(r)} .$$

Let ξ be a semispray of $\mathcal{J}^r Q$ and we suppose that ξ is locally expressed by (1.1). Then a simple computation in local coordinates shows that we have:

$$(2.7) \qquad \begin{cases} \left[\xi, \dfrac{\partial}{\partial t} \right] = -\dfrac{\partial \xi^i}{\partial t} \dfrac{\partial}{\partial q^i_{(r)}} \\ \left[\xi, \dfrac{\partial}{\partial q^i_{(h)}} \right] = -\dfrac{\partial \xi^j}{\partial q^i_{(h)}} \dfrac{\partial}{\partial q^j_{(r)}} - \dfrac{\partial}{\partial q^i_{(h-1)}} \qquad h = 1, 2, \dots, r \\ \left[\xi, \dfrac{\partial}{\partial q^i_{(0)}} \right] = -\dfrac{\partial \xi^j}{\partial q^i_{(0)}} \dfrac{\partial}{\partial q^j_{(r)}} . \end{cases}$$

Proposition 2.1. *Let $\Gamma = -L_\xi \tilde{S}$. Then Γ is a dynamical connection on $\mathcal{J}^r Q$, whose associated semispray is precisely ξ.*

Proof. In fact from (2.7) we have:

$$(2.8) \qquad \begin{cases} \Gamma \left(\dfrac{\partial}{\partial t} \right) = -\displaystyle\sum_{h=0}^{r-1} q^i_{(h+1)} \dfrac{\partial}{\partial q^i_{(h)}} \\ \qquad\qquad - \left(\displaystyle\sum_{h=0}^{r-1} (h+1) q^i_{(h+1)} \dfrac{\partial \xi^j}{\partial q^i_{(h+1)}} - r\xi^i \right) \dfrac{\partial}{\partial q^i_{(r)}} \\ \Gamma \left(\dfrac{\partial}{\partial q^i_{(k)}} \right) = \dfrac{\partial}{\partial q^i_{(k)}} + \dfrac{\partial \xi^j}{\partial q^i_{(k+1)}} \dfrac{\partial}{\partial q^j_{(r)}}; \quad k = 0, 1, 2, \dots, (r-1) \\ \Gamma \left(\dfrac{\partial}{\partial q^i_{(r)}} \right) = -r \dfrac{\partial}{\partial q^i_{(r)}} . \end{cases}$$

Now, from (2.8) we easily deduce that Γ is a dynamical connection on $\mathcal{J}^rQ$. Furthermore, taking into account (2.4), we have that the associated semispray to Γ is precisely ξ.

Let Γ be a dynamical connection on $\mathcal{J}^rQ$. A curve $s : X \to Q$ is called a path of Γ if the canonical prolongation j^rs of s to $\mathcal{J}^rQ$ is a weak-horizontal curve.

If $s : X \to Q$ is locally given by $t \to (t, q^i(t))$, then we have $j^rs(t) = (t, q^i_{(h)}(t))$; $0 \leq h \leq r$.

Hence

$$\widehat{\dot{j^r s(t)}} = \frac{\partial}{\partial t} + \sum_{h=1}^{r+1} \frac{d^h q^i}{dt^h} \frac{\partial}{\partial q^i_{(h-1)}}.$$

Therefore s is a path of Γ if and only if $\psi^i \left(\widehat{\dot{j^r s(t)}} \right) = 0$; $i = 1, 2, \ldots, n$, that is, s satisfies the following system of differential equations:

$$\frac{d^{r+1}q^i}{dt^{r+1}} = \Gamma^i_{(r)} + \sum_{h=0}^{r-1} \Gamma^{i(h)}_{j(r)} \frac{d^{h+1}q^j}{dt^{h+1}}. \tag{2.9}$$

Let ξ be the associated semispray of Γ. Then ξ is locally given by:

$$\xi = \frac{\partial}{\partial t} + \sum_{h=0}^{r-1} q^i_{(h+1)} \frac{\partial}{\partial q^i_{(h)}} + \xi^i \frac{\partial}{\partial q^i_{(r)}}$$

where:

$$\xi^i = \Gamma^i_{(r)} + \sum_{h=0}^{r-1} q^j_{(h+1)} \Gamma^{i(h)}_{j(r)}; \qquad 1 \leq i \leq n.$$

From (2.9) it is clear that the paths of Γ and ξ satisfy the same system of differential equations. Then we have:

Proposition 2.2. *A dynamical connection and its associated semispray on $\mathcal{J}^rQ$ have the same paths.*

3. The generalized Euler-Lagrange operator

Let $c : \mathcal{J}^rQ \to \mathcal{J}^{r+1}Q$ be a global section and ξ_c given by (1.2). The generalized Euler Lagrange operator associated to ξ_c is the R-linear operator on 1-forms $\mathcal{E}_{\xi_c}$ defined by

$$\mathcal{E}_{\xi_c} = -\xi_c \otimes dt + \sum_{h=0}^{r} (-1)^h \frac{1}{h!} L^h_{\xi_c} \cdot \tilde{S}^h. \tag{3.1}$$

Proposition 3.1. *The Euler-Lagrange operator satisfies*

$$\tilde{S} \circ \mathcal{E}_{\xi_c} = 0.$$

If $\omega \in \Lambda^1(\mathcal{J}^r Q)$ *is given by:*

$$\omega = \alpha dt + \sum_{h=0}^{r} \omega_i^{(h)} dq^i \tag{3.2}$$

then

$$\mathcal{E}_\Gamma(\omega) = \left(\sum_{h=0}^{r} (-1)^r \xi_c^h (\omega_i^{(h)}) \right) \theta_{(0)}^i. \tag{3.3}$$

To each semispray ξ_c we now associate a set of 1-forms $X_{\xi_c}^* = Ker\mathcal{E}_{\xi_0^*}$. The set $X_{\xi_c}^*$ is in fact a vector space over R, by the R-linearity of $\mathcal{E}_{\xi_c}$. Its elements satisfy the relation:

$$\omega_i^{(0)} - \xi_c(\omega_i^{(1)}) + \xi_c^2(\omega_i^{(2)}) - \ldots + (-1)^r \xi_c^r(\omega_i^{(r)}) = 0. \tag{3.4}$$

Furthermore, we define an R-linear operator σ_{ξ_c} of 1-forms on $\mathcal{J}^r Q$, called the generalized Cartan operator, by:

$$\sigma_{\xi_c} = \sum_{h=0}^{r-1} (-1)^h \frac{1}{(h+1)!} L_{\xi_c}^h \circ \tilde{S}^{(h+1)}. \tag{3.5}$$

It follows from this definition that:

$$\mathcal{E}_{\xi_c}\omega = \omega - i_{\xi_c}\omega dt - L_{\xi_c} \circ \sigma_{\xi_c}\omega. \tag{3.6}$$

Let ξ_c be a semispray given by (1.2) and $L \in \mathcal{F}(\mathcal{J}^r Q)$. The Poincaré-Cartan 1-form θ_{L,ξ_c} is given by:

$$\theta_{L,\xi_c} = Ldt + \sigma_{\xi_c}(dL). \tag{3.7}$$

We say that a semispray ξ_c is Lagrangian if there exists $L \in \mathcal{F}(\mathcal{J}^r Q)$ such that $dL \in X_{\xi_c}^*$.

Proposition 3.2. ξ_c *is Lagrangian iff there exists* $L \in \mathcal{F}(\mathcal{J}^r Q)$ *such that:*

$$i_{\xi_c}\omega_L = 0 \tag{3.8}$$

where $\omega_L = -d\theta_{L,\xi_c}$. *We call* ξ_c *the Lagrange vector for* L.

We now describe how the usual formulation of higher-order dynamics fits into to framework described above. It will be recalled that the Lagrangian function of $\mathcal{J}^r Q$ leads to Euler-Lagrange equations which are

$2r$-order differential equations. We must therefore consider the $(2r-1)$-order jet-prolongation $\mathcal{J}^{2r-1}Q$ and functions on it of the form $\pi_r^{2r-1*}L$, where L is a function on $\mathcal{J}^rQ$.

It follows that, if ξ_c is a $2r$-order differential equation field which is Lagrangian, with Lagrangian function L on $\mathcal{J}^rQ$, then :

$$\frac{\partial L}{\partial q^i_{(0)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^i_{(0)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q^i_{(0)}}\right) + \ldots + (-1)^r \frac{d^r}{dt^r}\left(\frac{\partial L}{\partial q^i_{(0)}}\right) = 0 \tag{3.9}$$

along any integral curve of ξ_c.

Proposition 3.3. *Let L be a non-autonomous regular Lagrangian on $\mathcal{J}^rQ$, and let ξ_c be a Lagrange vector field for L. Then there exists a dynamical connection Γ on $\mathcal{J}^{2r-1}Q$ whose paths are the solutions of the equations. This connection is given by $\Gamma = -L_{\xi_c}\tilde{S}$.*

References

[1] M. Crampin and W. Sarlet and F. Cantrijn, Higher-order differential equations and higher-order Lagrangian mechanics, *Math. Proc. Camb. Phil. Soc.* **99** (1986), 565.

[2] M. Ferraris and M. Francaviglia, On the globalization of Lagrangian and Hamiltonian formalisms in higher order mechanics, *Atti della Accademia delle Science di Torino* **117** (1983).

[3] M. De Leon and P. Rodrigues, Dynamical connections and non-autonomous Lagrangian systems, *Annales Faculté des Sciences de Toulouse* **IX,** no. 2 (1988).

[4] K. Yano and S. Ishihara, Tangent and cotangent bundles, *Pure and Appl. Math. Ser.* **16.** Marcel Dekker N. Y. (1973).

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