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Pethő's cubics

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This paper is dedicated to Kálmán Győry,
for the occasion of his 60th birthday

Abstract. We compute all the solutions of the family of cubic Thue equations

$$\Phi_n(x,y) = x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$$

for all rational integers n.

1. Introduction

We continue the study of a non-Galois family of cubic Thue equations $\Phi_n(x,y) = 1$ which was initiated in a joint paper with N. Tzanakis [MT]. The associated fields $Q(\theta_n)$, where $\Phi_n(\theta_n, 1) = 0$, are totally real.

The family of cubics we consider is

$$\Phi_n(x,y) = x^3 - nx^2y - (n+1)xy^2 - y^3. \tag{1}$$

Notice that the transformation $(x,y) \mapsto (-y,-x)$ defines a one-to-one correspondence between the solutions of the equations $\Phi_n(X,Y) = 1$ and $\Phi_{-n-1}(X,Y) = 1$, thus we consider only the case $n \geq 0$.

Note also that each equation $\Phi_n(x,y) = 1$ has the solutions $(x,y) = (1,0)$, $(0,-1)$, $(1,-1)$, $(-n-1,-1)$, $(1,-n)$. This gives five "trivial solutions" for $n \neq 0$, 1 and four ones otherwise. To simplify we solve the

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equations for $0 \le n \le 19$ using Kant, this shows that in this range the equation has only trivial solutions except for $n = 0$ where there is the extra solution $(x, y) = (4, 3)$, for $n = 3$ (then the discriminant is 49) where there are the four non-trivial solutions $(-5, 14)$, $(-2, 3)$, $(-1, 2)$ and $(9, -13)$ and for $n = 4$ (then the discriminant is 257) where there is the non-trivial solution $(7, -9)$. From now on we always suppose $n \ge 20$, without refering explicitly to this assumption.

According to a conjecture of A. Pethő [P] based on extensive computations, for any irreducible cubic form $\Phi_n(x, y) \in Z[x, y]$ with positive discriminant $\neq 49, 81, 148, 257, 361$, the equation $\Phi_n(x, y) = 1$ has at most five solutions. In [MT], it is proved that, indeed, the above mentioned five solutions are the only solutions of the equation

$$x^3 - nx^2y - (n+1)xy^2 - y^3 = 1, \tag{2}$$

if $n \ge 3.67 \times 10^{32}$, in accordance to Pethő's conjecture. (We chose the title of this paper because this family gives the maximum number of solutions known for a family of cubics.) Here we prove this result for all $n \ge 5$:

Theorem. *If $n \ge 5$, then the only solutions of the diophantine equation*

$$x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$$

are

$$(x, y) = (1, 0),\ (0, -1),\ (1, -1),\ (-n-1, -1),\ (1, -n).$$

We give a sketch of the method, which contains several steps. We work in number fields K attached to the Thue equation, depending on the parameter n. We know explicitly a fundamental system $\{\xi, 1+\xi\}$, for the units of K; and we notice that a solution (x, y) of the Thue equation satisfies $x + y\xi = \xi^a(1+\xi)^b$.

It is understood that all estimates and bounds refered to below are explicit and contain the parameter n, except if they are explicitly characterized as "numerical". The plan is the following.

1. Estimate the regulator R of K
2. Find an upper bound for $A := \max\{|a|, |b|\}$, in terms of R and $\log|y|$.
3. Obtain an upper bound for the linear form $|\Lambda|$ in three logarithms obtained by Siegel's formula, of the form $|\Lambda| = O(|y|^{-3})$.

Combine the results of steps 1, 2, 3 to find an upper bound for $|\Lambda|$ in terms of A.

5. Find a lower bound for A: this is a fundamental step, and there is no systematic way to get it.
6. Combine the results of steps 4 and 5 to obtain a negative upper bound for $\log|\Lambda|$.
7. Transform Λ into a homogeneous linear form in two logarithms in order that the sharp result of Laurent–Mignotte–Nesterenko can be applied to give a good negative lower bound for $\log|\Lambda|$.
8. Combine the results of steps 6, 7 to obtain a *numerical* upper bound for n, say $n \le N$.
9. View Λ, again, as a homogeneous linear form in three logarithms and apply Waldschmidt's result in order to obtain a negative lower bound for $\log|\Lambda|$, containing A.
10. Combine the results of steps 4, 9 to obtain a *numerical* upper bound for A.
11. Apply a lemma *à la* Baker-Davenport, in which the bound for A, obtained in step 10, is necessary, to treat the values of $n \le N$, the bound found in step 8.

2. Preliminaries

We work in the field $K = Q(\xi)$, where $\xi^3 - n\xi^2 - (n+1)\xi - 1 = 0$ (clearly $\xi = \xi_n$ and $K = K_n$ depend on n). The equation $x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$ implies that $x - y\xi$ is a unit of K.

The discriminant of ξ is $n^4 + 2n^3 - 5n^2 - 6n - 23 = (n^2+n-3)^2 - 32$, hence it is positive for $n \ge 3$ and it is a square only if $n = 3$, hence K is not Galois for $n > 3$. For $n \ge 4$ we know two fundamental units in K: Put $\xi = \lambda^{-1} - 1$. Then $K = Q(\lambda)$ and $\lambda^3 - (n+2)\lambda^2 + (n+3)\lambda - 1 = 0$, therefore, by E. Thomas' paper [T1], a pair of fundamental units is λ, $\lambda - 1$, i.e. $1/(1+\xi)$ and $(-\xi)/(1+\xi)$. From this it follows that ξ, $\xi + 1$ is a pair of fundamental units of K. Then, $x - y\xi = \pm\xi^a(1+\xi)^b$ for some a, $b \in Z$. Since the norms of ξ and $1+\xi$ are $+1$, the minus sign is excluded and

$$x - y\xi = \xi^a(1+\xi)^b.$$

Put

$$F(X) = F_n(X) = X^3 - nX^2 - (n+1)X - 1.$$

We can have good estimates of the roots of F by appropriate substitutions. Since $F(n+1) = -1$ and also $F(n+1+n^{-2}) = 3n^{-1} + 2n^{-2} + 2n^{-3} + 3n^{-4} + n^{-6} > 0$, the polynomial F has a root, say ξ_1, with

$$(3)_1 \qquad n+1 < \xi_1 < n+1+n^{-2}.$$

Similarly, sign changes of the polynomial F show that

$$(3)_2 \qquad -1 + \frac{1}{n+1} < \xi_2 < -1 + \frac{1}{n+1} + \frac{1}{(n+1)^2}$$

and

$$(3)_3 \qquad -\frac{1}{n} - \frac{1}{n^3} < \xi_3 < -\frac{1}{n}.$$

We shall often use the simpler following estimates: the roots of ξ_1, ξ_2, ξ_3 of F satisfy:

$$(3) \qquad n+1 < \xi_1 < n+1+\frac{1}{n^2},$$

$$-\frac{n}{n+1} < \xi_2 < -\frac{n-1}{n}, \quad -\frac{1}{n-1} < \xi_3 < -\frac{1}{n},$$

But, more precise estimates will also be necessary. We use the Lagrange's method to compute the beginning of the continued fraction expansion of the ξ's.

R1) *Approximate value of* ξ_1

By the change of variable $X = n+1+Y^{-1}$, the polynomial F is transformed into $g(Y) = -Y^3 + (n^2+3n+2)Y^2 + (2n+3)Y + 1$. Since $g(n^2+3n+2) = 2n^3 + 9n^2 + 13n + 7 > 0$, and $g(n^2+3n+3) = -n^4 - 4n^3 - 6n^2 - 3n + 1 < 0$, we have

$$n+1+\frac{1}{n^2+3n+3} < \xi_1 < n+1+\frac{1}{n^2+3n+2},$$

thus the beginning of the continued fraction expansion of ξ_1 is

$$\xi_1 = [n+1; n^2+3n+2, \dots].$$

R2) *Approximate value of* ξ_2

By the successive changes of variables $X = -1 + Y^{-1}$, $Y = n + Z^{-1}$ and $Z = 1 + T^{-1}$ we get the continued fraction expansion

$$\xi_2 = [-1; n, 1, \lfloor (n-2)/2 \rfloor, \dots].$$

Which shows that

$$-1 + \frac{1}{n+1-\frac{2}{n-3}} < \xi_2 < -1 + \frac{1}{n+1-\frac{2}{n}}.$$

R3) *Approximate value of* ξ_3

By a similar study we see that

$$\xi_3 = -[0; n-1, 1, n^2-n-2, \dots],$$

hence

$$-\frac{1}{n-\frac{1}{n^2-n-1}} < \xi_3 < -\frac{1}{n-\frac{1}{n^2-n}}.$$

Notice also the formulae

$$\Phi_n(x, n-1) = x^3 - (n^2-n)x^2 - (n^3-n^2-n+1)x - (n^3-3n^2+3n-1),$$

$$\Phi_n(x, n) \quad = x^3 - n^2x^2 - (n^3+n^2)x - n^3,$$

$$\Phi_n(x, n+1) = x^3 - (n^2+n)x^2 - (n^3+3n^2+3n+1)x - (n^3+3n^2+3n+1).$$

We make a very elementary study of the solutions of equation (2):

• If $y = 0$ then, clearly, $x = 0$.

• If $|y| = 1$, consider first the case $y = 1$, then $\Phi_n(x, y) = x^3 - nx^2 - (n+1)x - 1 = g(x)$, say. It is easy to verify that $g(x) = -1$ iff $x \in \{-1, 0, n+1\}$ and that $|g(x)| > 1$ for all other $x \in Z$, hence $\Phi_n(x, 1) \neq 1$ for any $x \in Z$. If $y = -1$ then since $\Phi_n(x, -y) = -\Phi(-x, y)$, we have $\Phi_n(x, -y) = 1$ iff $x \in \{1, 0, -(n+1)\}$, showing that in this case solutions (x, y) are the "trivial ones" $(0, -1)$, $(1, -1)$ and $(-n-1, -1)$.

• If $|y| = 2$, consider first the case $y = 2$, then $\Phi_n(x, y) = x^3 - 2nx^2 - 4(n+1)x - 8 = h(x)$, say. And it is easy to verify that $|h(x)| \geq 8$ for $x \neq -1$, whereas $h(-1) = 2n - 5$. Thus $\Phi_n(x, 2) = 1$ only when $n = 3$ and $x = -1$. Moreover, using again the formula $\Phi_n(x, -y) = -\Phi(-x, y)$, we see that the diophantine equation $\Phi_n(x,-2)=1$ has no solution for $n\ge 3$. Thus we may now suppose that $|y|\ge 3$.

From formula (2), we have

$$(x-\xi_1 y)(x-\xi_2 y)(x-\xi_3 y)=1.$$

Let i be the index such that

$$|x-\xi_i y|=\min_{1\le j\le 3}|x-\xi_j y|,$$

then $|x-\xi_i y|<1$ and, by the estimate (3) for the roots of F, $\xi_1-\xi_k>n+1$ for $k\neq 1$, thus

$$(j\neq i)\ \&\ (1\in\{i,j\})$$

$$\Rightarrow |x-\xi_j y|\ge|\xi_j-\xi_i|\,|y|-|x-\xi_i y|\ge(n+1)|y|-1>(n+2/3)|y|.$$

Hence, $|x-\xi_i y|^2\ (n+2/3)|y|<1$, in other words

$$|x-\xi_i y|<\big((n+2/3)|y|\big)^{-1/2}\le 1/\sqrt{62},$$

in particular i is indeed unique.

For $n\ge 20$, by (3),

$$|\xi_2-\xi_3|>1-\frac{1}{n+1}-\frac{1}{n-1}=1-\frac{2n}{n^2-1}$$

and by a previous computation $|x-\xi_i y|<1/\sqrt{62}$, thus

$$\prod_{j\neq i}|x-\xi_j y|>n\left(1-\frac{40}{399}-\frac{1}{2\sqrt{62}}\right)y^2>0.836\,n\,y^2>16y^2$$

and

$$\left|\frac{x}{y}-\xi_i\right|<\frac{1}{0.836n|y|^3}<\frac{1}{16|y|^3}.$$

This short study proves that the rational number x/y is a principal convergent of ξ_i. Now, we have to consider the three cases $i=1,2,3$.

$\boxed{i=1}$

Then $\xi_1=[n+1;n^2+3n+2,\dots]$ thus $|y|\ge n^2+3n+2$.

$\boxed{i=2}$

Then $\xi_2 = [-1; n, 1, \lfloor (n-2)/2 \rfloor, \dots]$ thus $|y| = n$, $n+1$ or $|y| \geq (n^2-3)/2$.

$\boxed{i=3}$

Then $\xi_3 = -[0; n-1, 1, n^2-n-2, \dots]$ thus $|y| = n-1$, n or $|y| > n^2$.

If $y = \pm(n-1)$ then $x = \pm 1$ and it is easy to verify that no solution (x,y) with $|y| = n-1$ exists.

If $y = \pm n$ then $x = \pm(n-1)$ or $x = \mp 1$. In the second case we have $\Phi_n(1,-n) = 1$, finding the last "trivial solution" $(x,y) = (1,-n)$. While, in the first case a direct computation shows that there is no solution (x,y) with $|y| = n$.

If $y = \pm(n+1)$ then $x = \pm n$ and by direct computation we see that there is no solution (x,y) with $|y| = n+1$.

This elementary study shows that a non trivial solution (x,y) must satisfy $|y| \geq (n^2-3)/2$.

Put $\varepsilon_h = 1 + \xi_h$ for $h = 1, 2, 3$. From the formula $x - \xi y = \xi^a(1+\xi)^b$, we get

$$x - \xi_j y = \xi_j^a \varepsilon_j^b, \quad 1 \leq j \leq 3,$$

which implies

$$y(\xi_{j+1} - \xi_j) = \xi_j^a \varepsilon_j^b - \xi_{j+1}^a \varepsilon_{j+1}^b, \tag{4}$$

where $\xi_{j+1} = \xi_{(j+1) \bmod 3}$, with some abuse of notation.

Now we want to estimate the exponents a and b in terms of y. Put $l_h = \log|\xi_h|$, $l'_h = \log|\varepsilon_h|$ and $\mu_h = \log|x - \xi_h y|$ for $h = 1, 2, 3$. Then the relations $x - \xi_h y = \xi_h^a \varepsilon_h^b$ can be written as $l_h a + l'_h b = \mu_h$, from which we get, for $j \neq k$,

$$a = \frac{l'_k \mu_j - l'_j \mu_k}{l'_k l_j - l'_j l_k}, \qquad b = -\frac{l_k \mu_j - l_j \mu_k}{l'_k l_j - l'_j l_k}. \tag{5}$$

Put $R = l'_2 l_3 - l'_3 l_2$; using the obvious relations $l_1 + l_2 + l_3 = 0$ and $l'_1 + l'_2 + l'_3 = 0$ it is easy to verify that $R = l'_3 l_1 - l'_1 l_3 = l'_1 l_2 - l'_2 l_1$ and we shall see that R is positive.

From the estimates (3) one easily deduces that

$$\log(n+1) < l_1 < \log(n+2), \qquad -\frac{1}{n-1} < l_2 < -\frac{1}{n+1},$$
$$-\log n < l_3 < -\log(n-1),$$

$$\log(n+2) < l_1' < \log(n+3), \qquad -\log(n+1) < l_2' < -\log n,$$
$$-\frac{1}{n-2} < l_3' < -\frac{1}{n}.$$

To estimate the μ_h we can write, for $h \neq i$,

$$|(\xi_i - \xi_h)y| - \frac{1}{16y^2} \le |x - \xi_h y| = |(\xi_i - \xi_h)y + x - \xi_i y| \le |(\xi_i - \xi_h)y| + \frac{1}{16y^2},$$

since $|\xi_i - \xi_h| > 0.8$ we get

$$|(\xi_i - \xi_h)y| \left(1 - \frac{0.08}{|y|^3}\right) \le |x - \xi_h y| \le |(\xi_i - \xi_h)y| \left(1 + \frac{0.08}{|y|^3}\right).$$

This implies

$$\log|\xi_i - \xi_h| - \frac{0.1}{|y|^3} \le \mu_h - \log|y| \le \log|\xi_i - \xi_h| + \frac{0.1}{|y|^3}. \tag{6}$$

From the estimates of the roots of F it is easy to check the following inequalities

$$1 - \frac{2n}{n^2-1} < \xi_3 - \xi_2 < 1 - \frac{2}{n+1},$$
$$n + 2 - \frac{1}{n} < \xi_1 - \xi_2 < n + 2,$$
$$n + 1 + \frac{1}{n} < \xi_1 - \xi_3 < n + 1 + \frac{1}{n-2}.$$

It will be very useful to have estimates for R. From the above estimates of the l_h and l_h', we get

$$\log(n-1) \times \log n - \frac{1}{(n-1)(n-2)} < R < \log n \times \log(n+1).$$

Indeed, we have the simpler estimate

$$\log^2(n-1) < R < \log n \times \log(n+1). \tag{7}$$

Now we want to get estimates of a and b in terms of y. To simplify the notations, put

$$\eta = 0.1|y|^{-3} \quad \text{and} \quad z = \log|y|.$$

We have to distinguish the three cases $i = 1$, 2 and 3.

$\boxed{i = 1}$

By (5),

$$a = \frac{l_2'\mu_3 - l_3'\mu_2}{R}, \qquad b = \frac{l_3\mu_2 - l_2\mu_3}{R}.$$

Here,

$$\log(n+2-1/n) - \eta \le \mu_2 - z \le \log(n+2) + \eta,$$
$$\log(n+1) - \eta \le \mu_3 - z \le \log(n+2) + \eta.$$

Hence, a and b are negative and we have

$$l_3'\mu_2 - l_2'\mu_3 \le |l_2'|\mu_3 \le \log n \times \big(z + \log(n+2) + \eta\big),$$
$$l_2\mu_3 - l_3\mu_2 \le |l_3|\mu_2 \le \log n \times \big(z + \log(n+2) + \eta\big),$$

thus

$$A \le \frac{\log n}{R}\big(z + \log(n+2) + \eta\big) \le \frac{\log n}{\log^2(n-1)}\big(z + \log(n+2) + \eta\big),$$

where we have put

$$A = \max\{|a|, |b|\}.$$

This implies

$$A \le z\frac{\log n}{R} + 3. \tag*{$(8)_1$}$$

$\boxed{i=2}$

By (5),

$$a = \frac{l'_3\mu_1 - l'_1\mu_3}{R}, \qquad b = \frac{l_1\mu_3 - l_3\mu_1}{R}.$$

Here,

$$\log\left(n+2-1/n\right) - \eta \le \mu_1 - z \le \log(n+2) + \eta,$$

$$\log\left(1 - \frac{2n}{n^2-1}\right) - \eta \le \mu_3 - z \le \eta.$$

In this case $a < 0 < b$ and we have

$$0 < l'_1\mu_3 - l'_3\mu_1 \le \log(n+3)\cdot(z+\eta) + \frac{1}{n-2}\big(z + \log(n+2) + \eta\big),$$

$$|l_3\mu_1 - l_1\mu_3| \le \log n \cdot \big(z + \log(n+2) + \eta\big) + \log(n+2)\cdot(z+\eta),$$

thus

$$(8)_2 \qquad A \le \frac{2\log(n+2)}{R}\left(z + \tfrac{1}{2}\log(n+2) + \eta\right).$$

$\boxed{i=3}$

Here,

$$a = \frac{l'_1\mu_2 - l'_2\mu_1}{R}, \qquad b = \frac{l_2\mu_1 - l_1\mu_2}{R}.$$

And,

$$\log\left(n+2-1/n\right) - \eta \le \mu_1 - z \le \log(n+2) + \eta,$$

$$\log\left(1 - \frac{2n}{n^2-1}\right) - \eta \le \mu_2 - z \le \eta.$$

In this case $b < 0 < a$ and we have

$$|l'_2\mu_1 - l'_1\mu_2| \le \log(n+1)\cdot\big(z + \log(n+2) + \eta\big) + \log(n+3)\cdot(z+\eta),$$

$$|l_2\mu_1 - l_1\mu_2| \le \log(n+2)\cdot(z+\eta) + \frac{1}{n-1}\cdot\big(z + \log(n+2) + \eta\big),$$

thus

$$(8)_3 \qquad A \le \frac{2\log(n+3)}{R}\left(z + \tfrac{1}{2}\log(n+2) + \eta\right).$$

Comparing the inequalities $(8)_i$ we get the following conclusion:

$$(8) \qquad A \le \frac{2\log(n+3)}{R}\left(z + \frac{1}{2}\log(n+2) + \eta\right).$$

3. A study of a linear form in three variables

In our case Siegel's identity is

$$(\xi_i - \xi_j)\xi_k^a\varepsilon_k^b + (\xi_j - \xi_k)\xi_i^a\varepsilon_i^b + (\xi_k - \xi_i)\xi_j^a\varepsilon_j^b = 0$$

which leads to the relation

$$\frac{(\xi_j - \xi_i)\xi_k^a\varepsilon_k^b}{(\xi_k - \xi_i)\xi_j^a\varepsilon_j^b} - 1 = \frac{(\xi_k - \xi_j)\xi_i^a\varepsilon_i^b}{(\xi_i - \xi_k)\xi_j^a\varepsilon_j^b}.$$

We choose $j = i+1$ and $k = i+2$, where these values are counted modulo 3, and consider the linear form of three logarithms

$$\Lambda = \log|\delta_i| + a\log|\xi_k/\xi_j| + b\log|\varepsilon_k/\varepsilon_j|,$$

where

$$\delta_i = \frac{\xi_{i+1} - \xi_i}{\xi_{i+2} - \xi_i}.$$

Then elementary computation using estimates of Section 2 show that

$$(9) \qquad |\Lambda| < \frac{|\xi_k - \xi_j|}{|\xi_i - \xi_k|} \times \frac{1}{0.709n|y|^3} \times \frac{1.02}{|\xi_i - \xi_j|} < \frac{5}{n|y|^3}.$$

Using (8) and (7) this implies

$$(10) \qquad \log|\Lambda| < -\frac{3A}{2}\frac{R}{\log(n+2)} + \frac{3}{2}\log(n+3) + 2$$
$$< -\frac{3A}{2}\frac{\log^2(n-1)}{\log(n+3)} + \frac{3}{2}\log(n+3) + 2.$$

Now, we have to find upper bounds for the heights of the algebraic numbers which appear in Λ.

M1) *Measure of* δ

We have $\delta = \frac{\xi_3-\xi_2}{\xi_1-\xi_2}$, this number is reciprocal. The conjugates of modulus >1 correspond to a numerator which contains the largest of the conjugates of ξ, this shows that the measure of δ is given by the formula

$$\begin{aligned}\mathrm{M}(\delta) &= \left|\frac{\xi_1-\xi_3}{\xi_2-\xi_3}\right|\cdot\left|\frac{\xi_2-\xi_1}{\xi_3-\xi_1}\right|\cdot\left|\frac{\xi_1-\xi_2}{\xi_3-\xi_2}\right|\cdot\big((\xi_1-\xi_2)(\xi_2-\xi_3)(\xi_3-\xi_1)\big)^2\\ &= |\xi_1-\xi_2|^4|\xi_1-\xi_3|^2.\end{aligned}$$

Thus

$$\mathrm{M}(\delta) < (n+2)^6.$$

M2) *Measure of* ξ_1/ξ_2

This number is a unit and it is also reciprocal. Its conjugates of modulus >1 correspond to a numerator which contains the largest of the conjugates of ξ, or to the denominator equal to the smallest conjugate. Thus

$$\begin{aligned}\mathrm{M}(\xi_1/\xi_2) &= \big|(\xi_1/\xi_2)\cdot(\xi_1/\xi_3)\cdot(\xi_2/\xi_3)\big|\\ &= (\xi_1/\xi_3)^2 < \left(\frac{n+1+1/n}{1/n}\right)^2 < (n+2)^4.\end{aligned}$$

M3) *Measure of* $\varepsilon_1/\varepsilon_2$

The same arguments than for the study of δ apply and show that the measure of $\varepsilon_1/\varepsilon_2$ satisfies

$$\mathrm{M}(\varepsilon_1/\varepsilon_2) = \left|\frac{\varepsilon_1}{\varepsilon_2}\right|\cdot\left|\frac{\varepsilon_1}{\varepsilon_3}\right|\cdot\left|\frac{\varepsilon_3}{\varepsilon_2}\right|\cdot(\varepsilon_1\varepsilon_2\varepsilon_3)^2 = \varepsilon_1^4\varepsilon_3^2.$$

This easily leads to the estimate

$$\mathrm{M}(\varepsilon_1/\varepsilon_2) < (n+2)^4.$$

We quote the result of [LMN] that we shall use three times.

Proposition 1. *Let* α_1, α_2 *be nonzeroalgebraic numbers, and let* $\log\alpha_1$ *and* $\log\alpha_2$ *be any determinations of their logarithms. Consider the linear form*

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1,$$

where b_1 *and* b_2 *are positive integers. We suppose that* $|\alpha_1|$ *and* $|\alpha_2|$ *are* ≥ 1. *Put*

$$D = [\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}].$$

Let a_1, a_2, h, ρ *be positive real numbers, with* $\rho > 1$. *Put* $\lambda = \log\rho$ *and suppose that*

(i) $$h \geq \max\left\{\frac{D}{2}, 5\lambda, D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+2.1\right)\right\},$$

(ii) $$a_i \geq \max\Big\{2, 2\lambda, \rho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\Big\}, \quad (i=1,2).$$

When α_1 *and* α_2 *are multiplicatively independent, we have*

(iiii) $$\begin{aligned}\log|\Lambda| \geq &-\frac{\lambda a_1a_2}{9}\left(\frac{4h}{\lambda^2}+\frac{4}{\lambda}+\frac{1}{h}\right)^2 - \frac{2\lambda}{3}(a_1+a_2)\left(\frac{4h}{\lambda^2}+\frac{4}{\lambda}+\frac{1}{h}\right)\\ &-\frac{16\sqrt{2a_1a_2}}{3}\left(1+\frac{h}{\lambda}\right)^{3/2} - 2(\lambda+h) - \log\left(a_1a_2\left(1+\frac{h}{\lambda}\right)^2\right)\\ &+\frac{\lambda}{2}+\log\lambda - 0.15.\end{aligned}$$

Now we consider the three cases for i.

$\boxed{i=1}$

We have seen above that in this case $a<0$, $b<0$ and $a\approx b$, for this reason we put $c=a-b$ and rewrite the linear form Λ as

$$\Lambda_1 = \Lambda = \log|\delta_1| - c\log\left|\frac{\varepsilon_3}{\varepsilon_2}\right| + a\log\left|\frac{\xi_3\varepsilon_3}{\xi_2\varepsilon_2}\right| = b_2\log\alpha_2 - b_1\log\alpha_1,$$

with

$$b_1 = |a|,\ \ \alpha_1 = \left|\frac{\xi_3\varepsilon_3}{\xi_2\varepsilon_2}\right|,\ \ b_2 = 1,\ \ \alpha_2 = \left|\delta_1(\varepsilon_2/\varepsilon_3)^c\right|^\sigma \quad \text{where } \sigma\in\{-1,+1\}.$$

Put $\ell_1 = \log \alpha_1$, $\ell_1' = \log \alpha_2$ then

$$\Lambda = |a|\ell_1 - \ell_1'.$$

1. Estimating ℓ_1

One can verify that the minimal polynomial for $\varepsilon\xi = \xi + \xi^2$ is

$$G(X) = X^3 - (n^2 + 3n + 2)X^2 - (2n + 3)X - 1$$

and that

$$\begin{aligned} G\left(-\frac{1}{n} + \frac{1}{n^2}\right) &> 0, \\ G\left(-\frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3}\right) &< 0, \\ G\left(-\frac{1}{n} + \frac{2}{n^2} - \frac{4}{n^3}\right) &< 0, \\ G\left(-\frac{1}{n} + \frac{2}{n^2} - \frac{5}{n^3}\right) &> 0. \end{aligned}$$

Since the function $x \mapsto x(1-x)$ is increasing for $0 < x < 0.5$, we have

$$\left(1 - \frac{1}{n}\right)\frac{1}{n} < |\xi_3\varepsilon_3| < \left(1 - \frac{1}{n-1}\right)\frac{1}{n-1}.$$

For similar reasons,

$$\left(1 - \frac{1}{n+1}\right)\frac{1}{n+1} < |\xi_2\varepsilon_2| < \left(1 - \frac{1}{n}\right)\frac{1}{n}.$$

Which implies $|\xi_3\varepsilon_3|/|\xi_2\varepsilon_2| > 1$.

These remarks show that

$$\begin{aligned} \frac{1}{n} - \frac{1}{n^2} &< |\xi_3\varepsilon_3| < \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} \\ \frac{1}{n} - \frac{2}{n^2} + \frac{4}{n^3} &< |\xi_2\varepsilon_2| < \frac{1}{n} - \frac{2}{n^2} + \frac{5}{n^3} \end{aligned}$$

thus

$$1+\frac{1}{n+5}\le\frac{n^2-n}{n^2-2n+5}<\left|\frac{\xi_3\varepsilon_3}{\xi_2\varepsilon_2}\right|<\frac{n^2-n+1}{n^2-2n+4}<1+\frac{1}{n}$$

and

$$\frac{1}{n+6}<\ell_1<\frac{1}{n}.$$

2. Estimating ℓ_1'

We have

$$1+\frac{1-\frac{2n}{n^2-1}}{n+2}<\delta_1=1+\frac{\xi_3-\xi_2}{\xi_1-\xi_3}<1+\frac{1}{n+1},$$

thus $0<\log\delta_1<\frac{1}{n+1}$, and moreover

$$n-1.5<n-1-\frac{1}{n-1}=\frac{1-\frac{1}{n-1}}{1/n}<\left|\frac{\varepsilon_3}{\varepsilon_2}\right|<\frac{1-\frac{1}{n}}{1/(n+1)}=n-\frac{1}{n}.$$

So that

$$|c|\log(n-1.5)-\frac{1}{n+1}<\ell_1'<|c|\log n+\frac{1}{n+1}.$$

As a consequence of the estimates of ℓ_1, ℓ_1' and $|\Lambda|$, we have

$$n\big(|c|\log(n-1.5)-1/n\big)\le|a|\le(n+6)\big(|c|\log n+1/n\big).$$

3. Estimating measures

We have

$$\mathrm{M}\big(\xi_3\varepsilon_3/(\xi_2\varepsilon_2)\big)=|\xi_1\varepsilon_1|^4|\xi_2\varepsilon_2|^2<(n+2)^4(n+3)^4n^{-2}<(n+4)^6$$

and

$$\mathrm{M}(\xi_3/\xi_2)\le(n+2)^4,\qquad \mathrm{M}(\delta_1)\le(n+2)^6.$$

4. Application of Proposition 1

We have to take

$$h\ge\max\left\{5\lambda,\ D\log\left(\frac{|a|}{a_2}+\frac{1}{a_1}\right)+\log\lambda+1.56\right\},$$

by the upper bound of $|a|$ we choose

$$h = \max\left\{5\lambda,\ 6\log\left(\frac{(n+6)\big(|c|\log n + 1/n\big)}{a_2} + \frac{1}{a_1}\right) + \log\lambda + 1.56\right\},$$

and we can choose

$$a_1 = \max\big\{2\lambda,\ (\rho-1)/n + 12\log(n+4)\big\},$$
$$a_2 = \max\left\{2\lambda,\ (\rho-1)\big(|c|\log n + |, 1/n\big) + 12\left(1+\frac{2|c|}{3}\right)\log(n+4)\right\}.$$

Applying inequality (iii), we get,

$$\log|\Lambda_1| \geq -L_1, \quad \text{(say)}.$$

5. Upper bound on n

In this case, using $(8)_1$ and (7), we get

$$\log|\Lambda_1| \leq -3A\frac{R}{\log(n+3)} + \frac{3}{2}\log(n+3) + 2$$
$$\leq -3A\frac{\log^2(n-1)}{\log(n+3)} + \frac{3}{2}\log(n+3) + 2,$$

where $A \geq |a|$.

We have already seen that

$$|a| \geq n\big(|c|\log(n-1.5) - 1/n\big).$$

When $c \neq 0$, choosing $\rho = 67.1$ and combining the previous inequalities, we get

$$n \leq 150000.$$

6. The case $c = 0$

In the special case $c = 0$, we have

$$\Lambda_1 = \log|\delta_1| + a\log\left|\frac{\xi_3\varepsilon_3}{\xi_2\varepsilon_2}\right|.$$

By the estimates of δ_1, $\frac{\xi_3\varepsilon_3}{\xi_2\varepsilon_2}$ and $|\Lambda|$ we have

$$|a| \le \frac{1/n}{1/(n+6)} = \frac{n+6}{n} < 2.$$

The case $a = b = 1$ gives $x - \xi y = \xi + \xi^2$ which is impossible. Whereas the case $a = b = -1$ gives

$$|y| = \left| \frac{(\xi_1\varepsilon_1)^{-1} - (\xi_2\varepsilon_2)^{-1}}{\xi_2 - \xi_1} \right| < \frac{\left(\left(1 - \frac{1}{n}\right)\frac{1}{n+1}\right) + (n+1)^{-2}}{n+1}$$
$$< \frac{\frac{n(n+1)}{n-1} + \frac{1}{n-1}}{n+1} = \frac{n+1}{n-1},$$

so that $|y| \le 1$, and this has been studied above.

$\boxed{i = 2}$

Here we choose $j = 3$, $k = 1$ and put $b = -2a + c - 1$ (recall that $a < 0$ and $b > 0$) and rewrite Λ as

$$\Lambda_2 = \Lambda = \log \delta_2' + c \log \left| \frac{\varepsilon_1}{\varepsilon_3} \right| + a \log \left| \frac{\xi_1\varepsilon_3^2}{\xi_3\varepsilon_1^2} \right| = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

with

$$b_1 = |a|, \quad \alpha_1 = \left| \frac{\xi_3\varepsilon_1^2}{\xi_1\varepsilon_3^2} \right|, \quad b_2 = 1, \quad \alpha_2 = |\delta_2'(\varepsilon_1/\varepsilon_3)^c|^\sigma \quad \text{where } \sigma \in \{-1, +1\},$$

and where

$$\delta_2' = \frac{\varepsilon_3(\xi_1 - \xi_2)}{\varepsilon_1(\xi_3 - \xi_2)}.$$

Now we put $\ell_2 = \log \alpha_1$, $\ell_2' = \log \alpha_2$ then

$$\Lambda_2 = |a|\ell_2 - \ell_2'.$$

1. Estimating ℓ_2

Using the estimate R3 we get

$$\frac{(n+1)\left(1-\frac{1}{n-1/(n^2-n-1)}\right)^2}{\frac{1}{n-1/(n^2-n)}(n+2+n^{-2})^2} \le \alpha_1 = \left|\frac{\xi_1\varepsilon_3^2}{\xi_3\varepsilon_1^2}\right| \le \frac{(n+1+n^{-2})\left(1-\frac{1}{n-1/(n^2-n)}\right)^2}{\frac{1}{n-1/(n^2-n-1)}(n+2)^2},$$

from which we can deduce

$$1+\frac{5}{n} < \alpha_1 < 1+\frac{5}{n}+\frac{11}{n^2}.$$

Thus

$$\frac{5}{n+3} < \frac{5}{n}-\frac{13}{n^2} < \ell_2 < \frac{5}{n}+\frac{11}{n^2} < \frac{5}{n-3}.$$

2. Estimating ℓ_2'

Here,

$$\delta_2' = \frac{\varepsilon_3(\xi_1-\xi_2)}{\varepsilon_1(\xi_3-\xi_2)} = 1+\frac{\varepsilon_2(\xi_1-\xi_3)}{\varepsilon_1(\xi_3-\xi_2)},$$

which implies

$$1+\frac{1}{n} < 1+\frac{1}{(n+2)\left(1-\frac{2}{n+1}\right)} < \delta_2' < 1+\frac{\frac{n+2}{n}}{(n+1)\left(1-\frac{2n}{n^2-1}\right)}$$

$$=1+\frac{1+2/n}{n-1-2/(n-1)} < 1+\frac{n-1}{(n-1)^2-2}+\frac{2}{(n-1)^2-2} = 1+\frac{n+1}{n^2-2n-1},$$

so that $1+\frac{1}{n} < \delta_2' < 1+\frac{1}{n-3}$ for $n \ge 4$. Also

$$n+3 < \frac{n+2}{1-\frac{1}{n-1}} < \left|\frac{\varepsilon_1}{\varepsilon_3}\right| < \frac{n+2+1/n^2}{1-\frac{1}{n}} < n+4.$$

So that

$$|c|\log(n+3)-\frac{1}{n-3} < \ell_2' < |c|\log(n+4)+\frac{1}{n-3}.$$

As a consequence of the estimates of ℓ_2, ℓ_2' and $|\Lambda|$, we have

$$\frac{n-3}{5}\bigl(|c|\log(n+3)-1/(n-4)\bigr) < |a| < \frac{n+3}{5}\bigl(|c|\log(n+4)+1/(n-4)\bigr).$$

3. Estimating measures

Here

$$\delta_2' = \frac{\varepsilon_1(\xi_2-\xi_3)}{\varepsilon_3(\xi_2-\xi_1)},$$

and

$$\mathrm{h}(\delta_2') \le \frac{5}{3}\log(n+2), \quad \mathrm{h}\left(\frac{\varepsilon_1}{\varepsilon_3}\right) \le \frac{2}{3}\log(n+2).$$

Moreover

$$\mathrm{M}\bigl(\xi_1\varepsilon_3^2/(\xi_3\varepsilon_1^2)\bigr) \le (n+2)^6.$$

[Look at the conjugates of modulus >1 of this number.]

4. Application of Proposition 1

We take

$$h = \max\left\{5\lambda,\ 6\log\left(\frac{(n+3)\bigl(|c|\log(n+4)+1\bigr)}{5a_2}+\frac{1}{a_1}\right)+\log\lambda+1.56\right\}$$

and we can choose

$$a_1 = \max\left\{2\lambda,\ \frac{5}{n-3}(\rho-1)+12\log(n+4)\right\},$$

$$a_2 = \max\left\{2\lambda, (\rho-1)\bigl(|c|\log(n+4)+1/(n-4)\bigr)+12\left(\frac{5}{3}+\frac{2}{3}|c|\right)\log(n+4)\right\}.$$

By Proposition 1,

$$\log|\Lambda_2| \ge -L_2, \quad \text{(say)}.$$

5. Upper bound on n

By (8),

$$\log|\Lambda_2| \le -\frac{3A}{2}\frac{R}{\log(n+3)}+\frac{3}{2}\log(n+3)+2.$$

We have seen that

$$|a| \geq \frac{n-3}{5}\big(|c|\log(n+3) - 1/(n-4)\big),$$

this implies

$$A = |b| \geq |a|\left(2 - \frac{5}{(n-3)\log n}\right) - 2.$$

When $c \neq 0$, choosing $\rho = 81.2$ we get

$$n \leq 810000.$$

6. The special case $c = 0$

If $c = 0$ then the relations $\Lambda_2 = |a|\ell_2 - \log\delta_2'$, $\ell_2 > 5/(n-3)$ and $\log(1+1/n) < \log\delta_2' < 1/(n-3)$ imply

$$|\Lambda_2| \geq \min\left\{\frac{5}{n+3} - \frac{1}{n-3}, \frac{1}{n} - \frac{1}{2n^2}\right\}$$

in contradiction with (9).

$\boxed{i = 3}$

Here $j = 1$ and $k = 2$, put $a = -2b + c + 1$ and rewrite Λ as

$$\Lambda_3 = \Lambda = \log\delta_3' + c\log\left|\frac{\xi_2}{\xi_1}\right| + b\log\left|\frac{\xi_1^2\varepsilon_2}{\xi_2^2\varepsilon_1}\right| = b_2\log\alpha_2 - b_1\log\alpha_1,$$

with

$$b_1 = |b|,\ \ \alpha_1 = \left|\frac{\xi_1^2\varepsilon_2}{\xi_2^2\varepsilon_1}\right|,\ \ b_2 = 1,\ \ \alpha_2 = \left|\delta_3'(\xi_2/\xi_1)^c\right|^\sigma \quad \text{where } \sigma \in \{-1, +1\},$$

and where

$$\delta_3' = \frac{\xi_2(\xi_1 - \xi_3)}{\xi_1(\xi_2 - \xi_3)}.$$

Put $\ell_3 = \log\alpha_1$, $\ell_3' = \log\alpha_2$ then

$$\Lambda_3 = |b|\ell_3 - \ell_3'.$$

1. Estimating ℓ_3

One can prove that

$$\frac{1}{n} < \log\left(1+\frac{1}{n}+\frac{3}{n^2}\right) < \ell_3 < \frac{1}{n}+\frac{4}{n^2}.$$

2. Estimating ℓ_3'

One can also prove that

$$|c|\log(n+1)-\frac{1}{n}-\frac{5}{n^2} < \ell_3' < |c|\log(n+4)+\frac{1}{n}+\frac{5}{n^2}.$$

As a consequence of the estimates of ℓ_3, ℓ_3' and $|\Lambda|$, we have

$$\frac{n}{1+4/n}\left(|c|\log(n+1)-\frac{1}{n}-\frac{5}{n^2}\right) < |b| < n\left(|c|\log(n+4)+\frac{1}{n}+\frac{5}{n^2}\right).$$

3. Estimating measures

One has

$$\mathrm{h}(\delta_3) \le \log(n+2), \quad \mathrm{h}(\xi_2/\xi_1) \le \frac{2}{3}\log(n+2), \quad \mathrm{h}\left(\frac{\xi_1^2\varepsilon_2}{\xi_2^2\varepsilon_1}\right) \le \log n.$$

4. Application of Proposition 1

We take

$$h = \max\left\{5\lambda,\ 6\log\left(\frac{n\big(|c|\log(n+1)+1\big)}{a_2}+\frac{1}{a_1}\right)+\log\lambda+1.56\right\}$$

and we can choose

$$a_1 = \max\left\{2\lambda,\ \frac{1}{n}(\rho-1)+12\log(n+4)\right\},$$

$$a_2 = \max\left\{2\lambda,\ \left(|c|+\tfrac{1}{n}\right)(\rho-1)\log(n+4)+12\left(1+\tfrac{2}{3}|c|\right)\log(n+4)\right\}.$$

By Proposition 1,

$$\log|\Lambda_3| \ge -L_3, \quad \text{(say)}.$$

5. Upper bound on n

We have

$$\log|\Lambda_3| \le \frac{3A}{2}\frac{R}{\log(n+3)} + \frac{3}{2}\log(n+3) + 2.$$

We have seen that

$$|b| \ge \frac{n}{1+4/n}\left(|c|\log(n+1) - \frac{1}{n} - \frac{5}{n^2}\right),$$

this implies

$$A = |a| \ge |b|\left(2 - \frac{1+4/n}{n\log n}\right) - 2.$$

When $c \neq 0$, choosing $\rho = 48.3$ we get

$$n \le 260000.$$

6. The special case $c = 0$

If $c = 0$ then $b = -1$ and $a = 3$, and

$$|y| = \left|\frac{\xi_1^3\varepsilon_1^{-1} - \xi_2^3\varepsilon_2^{-1}}{\xi_2 - \xi_1}\right| < \frac{(n+1+1/n)^2}{n+2-1/n} < n+2,$$

in contradiction with the hypothesis $|y| \ge (n^2-3)/2$.

Application of a theorem of M. Waldschmidt

Let α_i, $1 \le i \le n$ be non-zero algebraic numbers and $b_1, b_2, \ldots, b_n$ be positive rational integers and suppose that the number

$$\Lambda = b_1\log\alpha_1 + \cdots + b_n\log\alpha_n$$

is not zero. We apply a theorem of M. Waldschmidt [W], Corollaire 1.5.

Put $D = [Q(\alpha_1,\ldots,\alpha_n) : Q]$ and $g = [R(\log\alpha_1,\ldots,\log\alpha_n) : R]$. For $1 \le i \le n$, let $A_i > 1$ be real numbers such that $\log A_i \ge \mathrm{h}(\alpha_i)$. Then the quoted result is the following:

Proposition 2. *Let E and f be positive real numbers, $E \geq e$ such that,*

$$E \leq \min\left\{A_1^D, \dots, A_n^D, \frac{nD}{f}\left(\sum_{i=1}^{n} \frac{|\log \alpha_i|}{|\log A_i|}\right)^{-1}\right\}.$$

Put

$$Z_0 = \max\left\{7 + 3\log n, \frac{g}{D}\log E, \log\left(\frac{D}{\log E}\right)\right\},$$

$$M = \max_{1\leq j<n}\left\{\frac{b_n}{\log A_j} + \frac{b_j}{\log A_n}\right\},$$

$$G_0 = \max\{4nZ_0, \log M\},$$

$$U_0 = \max\bigl\{D^2\log A_1, \dots, D^2\log A_n, D^{n+2}G_0Z_0\log A_1\cdots \cdots\log A_n(\log E)^{-n-1}\bigr\}.$$

Then

$$|\Lambda| \geq \exp\bigl\{-1500\, g^{-n-2}\, 2^{2n}\, n^{3n+5}(1+g/f)^n U_0\bigr\}.$$

In the present case we have three logarithms, $D = 6$, $g = 1$ and, for $n \geq 3$ (here n is again the parameter of our cubic equations), we can take

$$\log A_1 = \log(n+2), \quad \log A_2 = \log A_3 = \frac{2}{3}\log(n+2),$$

and

$$E = e, \quad f = 3/e, \quad Z_0 = 7 + 3\log 3, \quad G_0 = \max\{12Z_0, \log M\}.$$

A short computation shows that Proposition 2 implies

$$\log|\Lambda| > \begin{cases} -1.398\times 10^{19}\times \log^3(n+2), & \text{if } \log M < 123.6,\\ -1.132\times 10^{17}\times \log M\times \log^3(n+2), & \text{otherwise.}\end{cases}$$

We can take

$$M = \frac{3}{2\log(n+2)} + \frac{A}{\log(n+2)} < \frac{A+2}{\log(n+2)}.$$

Using the upper bound (10) on $\log|\Lambda|$ proved before (we get

$$\frac{3A}{2}\,\frac{R}{\log(n+3)} \le C\max\left\{123.6, \log\Big(\frac{A+2}{\log(n+2)}\Big)\right\}\times\log^3(n+2)$$
$$+2\log(n+3)+3,$$

where $C = 1.398\times 10^{19}$. Which gives the following upper bound for A in terms of n:

$$A \le \left(\frac{2C}{3}\max\left\{123.6, \log\Big(\frac{A+2}{\log(n+2)}\Big)\right\}\times\ \log^3(n+2)+2\log(n+3)+3\right)$$
$$\times\frac{\log(n+3)}{R}.$$

Using the upper bound on n, we find $A < 1.1\times 10^{23}$.

4. Application of Diophantine approximation

We use the following lemma which is a variant of a result of Baker–Davenport.

Lemma. *Let $\Lambda = u\alpha + v\beta + \gamma$, where α, β and γ are nonzero real numbers and where u and v are rational integers, with $|u|\le A$. Let $Q>0$ be a real number. Suppose that θ_1 and θ_2 satisfy*

$$|\theta_1-\alpha/\beta| < \frac{1}{100Q^2}, \quad \text{and} \quad |\theta_2-\gamma/\beta| < \frac{1}{Q^2}.$$

Let p/q be a rational number with $1\le q\le Q$ and $|\theta_1 - p/q| < 1/q^2$ and suppose that $q\|q\theta_2\| \ge 1.01, A+2$, [where $\|\cdot\|$ denotes the distance to the nearest integer] then

$$|\Lambda| \ge \frac{|\beta|}{Q^2}.$$

Proof. Put $|\Lambda| = \eta$, then

$$\left|q\frac{\Lambda}{\beta}\right| = \left|uq\left(\frac{\alpha}{\beta}-\theta_1\right)+u(q\theta_1-p)+pu+vq+q\left(\frac{\gamma}{\beta}-\theta_2\right)+q\theta_2\right| = \frac{q\eta}{|\beta|}.$$

Hence,

$$q\|q\theta_2\| \le q\left(\frac{q\eta}{|\beta|} + |uq|\left|\frac{\alpha}{\beta} - \theta_1\right| + \frac{|u|}{q} + q\left|\frac{\gamma}{\beta} - \theta_2\right|\right)$$
$$< \frac{q^2\eta}{|\beta|} + \frac{|u|q^2}{100Q^2} + |u| + \frac{q^2}{Q^2} \le \frac{Q^2\eta}{|\beta|} + 1.01A + 1,$$

which leads at once to the result. □

We applied the above lemma for $n \le 150000$, $n \le 810000$ and $n \le 260000$ respectively in the three cases $i = 1$, 2 and 3. We found no non-trivial solution for $n \ge 10$. The verification took less than six hours on a DEC *alpha* Station 1000A.

Thus, we have proved the Theorem stated in the Introduction.

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