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Divisor class group and theory of quasi-divisors

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Dedicated to Professor Kálmán Győry on his 60th birthday

Abstract. A divisor class group $\mathcal{C}_h = \Gamma/h(G)$ of a *po*-group G with an *o*-isomorphism h into an l-group Γ is investigated. Some relationships between properties of $\mathcal{C}_h$ and conditions under which h is a strong theory of quasi-divisors of a finite character are derived. This generalizes the results of Skula about the strong theory of divisors. Some heraditary properties of a divisor class group $\mathcal{C}_h$ and a value group Γ are also investigated.

1. Introduction

The notion of a directed partially ordered group (*po*-group) with a theory of quasi-divisors was introduced by K. E. Aubert [2] although P. Jaffard [13] proved several principal characterizations of these groups without mentioning a notion of quasi-divisor theory. A directed *po*-group $(G, .)$ has a *theory of quasi-divisors* if there exists an l-group $(\Gamma, .)$ and a map $h : G \longrightarrow \Gamma$ such that

(i) h is an order isomorphism from G into Γ.

(ii) $(\forall \alpha \in \Gamma_+)(\exists g_1, \dots, g_n \in G_+)\alpha = h(g_1) \wedge \dots \wedge h(g_n)$.

If in this case the l-group Γ is a free abelian group $\mathbb{Z}^{(P)}$ with pointwise ordering then h is called the *theory of divisors*. The notion of a partially

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ordered group with a theory of divisors was introduced by Borevicz and Shafarevicz [4] and extensively investigated by L. Skula [22].

The principal tool for the investigation of the properties of *po*-groups with quasi-divisor theory seem to be the notions of r-system and of t-valuation. By an r-system of ideals in a directed *po*-group G we mean a map $X \mapsto X_r$ (X_r is called an r-ideal) from the set of all lower bounded subsets X of G into the power set of G such that

(1) $X \subseteq X_r$

(2) $X \subseteq Y_r \implies X_r \subseteq Y_r$

(3) $\{a\}_r = a.G^+ = (a)$, for all $a \in G$

(4) $a.X_r = (a.X)_r$, for all $a \in G$.

An important example of an r-system is a t-system of ideals such that

$$X_t = \begin{cases} \{g \in G : \text{if is } s \in G \text{ such that } s \leq X, \text{ then } g \geq s\}, & X \text{ is finite,} \\ X_t = \bigcup \{K_t : K \text{ is a finite subset in } X\}, & X \text{ is lower bounded.} \end{cases}$$

By a *t-valuation* of a directed *po*-group G we mean an order homomorphism w of G onto a totally ordered group G_w (*o*-group) with the property $w(X_t) \subseteq (w(X))_t$ for any lower bounded subset X. Moreover, a t-valuation w is called *essential* if $\ker w$ is a directed subgroup of G and w is an *o*-epimorphism. Using t-valuations it is possible to characterize *po*-groups with special types of a theory of quasi-divisors and to investigate properties of these *po*-groups. Recall that a family W of t-valuations is called a *defining family for* G, if

$$(\forall g \in G) g \geq 1 \Leftrightarrow (\forall w \in W) w(g) \geq 1.$$

We say that W is of finite character, if

$$(\forall g \in G)(\forall' w \in W) w(g) = 1,$$

where $\forall'$ means "for all but a finite number". Then a theory of quasi-divisors of G is said to be of *finite character*, if there exists a defining family of t-valuations of finite character for G. In [7] it was proved that for any *po*-group with quasi-divisor theory there exists a defining family of essential t-valuations (Theorem 3.5).

For defining families of t-valuations of a *po*-group G various types of approximation theorems can be investigated. Let w, v be t-valuations of G with value groups G_w, G_v respectively, and let $[\ker w, \ker v]$ be the smallest convex subgroup of G containing $\ker w$ and $\ker v$. Then the canonical o-homomorphism $G \longrightarrow G/[\ker w, \ker v]$ is a t-valuation and there are o-homomorphisms d_{vw}, d_{wv} such that $d_{vw}.v = d_{wv}.w$. This common t-valuation will be denoted by $v \wedge w$. Now, the elements $(g_1, g_2) \in G_w \times G_v$ are called *compatible*, if $d_{wv}(g_1) = d_{vw}(g_2)$. Moreover, if W is a set of t-valuations, an element $(g_w)_w \in \prod_{w\in W'} G_w$ (where $W' \subseteq W$) is called *compatible* if any pair (g_w, g_v) from this element is compatible. Finally, we say that an element $(g_w)_w \in \prod_{w\in W} G_w$ is W'*-complete* for $W' \subseteq W$, if $\bigcup_{w\in W'} W(g_w) \subseteq W'$, where $W(g_w) = \{v \in W : d_{wv}(g_w) \neq 1\}$. We set $W(1) = \emptyset$. Then we say that G with a defining family W of t-valuations satisfies the Approximation Theorem, if for any finite subset $F \subseteq W$ and any compatible and F-complete system $(g_w)_w \in \prod_{w\in F} G_w$ there exists $g \in G$ such that $w(g) = g_w$ for all $w \in F$ and $w(g) \geq 1$ for all $w \in W \setminus F$. In [16] it was proved that a *po*-group G has a theory of quasi-divisors of a finite character if and only if there exists its defining family of t-valuations of a finite character which satisfies the Approximation Theorem.

In [16] we introduced a stronger version of *po*-groups with theory of quasi-divisors which is a generalization of a strong divisor theory introduced by L. Skula [22]. An order homomorphism h of G into an l-group Γ is called a *strong theory of quasi-divisors*, if

$$(\forall \alpha, \beta \in \Gamma_+)(\exists \gamma \in \Gamma_+)\alpha.\gamma \in h(G),\ \beta \wedge \gamma = 1.$$

It may be proved that any strong theory of quasi-divisors is also a theory of quasi-divisors. We say that a strong theory of quasi-divisors h is of finite character if h is a theory of quasi-divisors of a finite character.

In classical divisor theory of *po*-groups an important role is played by the divisor class group. This notion was introduced by L. Skula [22] as a natural generalization of a class group known from the theory of Krull and Dedekind domains. This notion can naturally be defined for any o-isomorphism $h : G \longrightarrow \Gamma$ of a *po*-group G into another *po*-group Γ, as it was done in [17]. In this case the *divisor class group* $\mathcal{C}_h$ of h is the abstract quotient group $\Gamma/h(G)$. The canonical map $\Gamma \longrightarrow \mathcal{C}_h$ is then denoted by φ_h.

It was again L. Skula [22], who showed that $\mathcal{C}_h$ and φ_h have great importance in deciding whether or not h is a theory of divisors. Unfortunately his proofs are based significantly on the discrete character of the value group $\mathbb{Z}$ and it is almost impossible to apply his technics directly to any *po*-group. Nevertheless, in this paper we want to prove that the divisor class group is of the same importance also for *po*-groups with theory of quasi-divisors as it is for groups with classical divisor theory. Namely, we show that by using some properties of $\mathcal{C}_h$ it is possible to characterize *po*-groups G with strong theory of quasi-divisors of a finite character. These properties are based on the identification of special "dense" subsets $\mathcal{A} \subseteq \Gamma$, i.e. subsets with the property $\varphi_h(\mathcal{A}) = \mathcal{C}_h$. We show that these results also generalize Skula's characterization of strong divisor theory. Further, we investigate conditions under which a quotient group $\mathcal{C}_h/R$ is also the divisor class group of some epimorphic image of the original *po*-group with quasi-divisor theory. Recently in 1998 F. Lucius [25] proved that there exists a proper generalization of quasi-divisor theory (he called it *groups with greatest common divisor theory* (shortly GCD theory)). We show that the existence of this GCD theory follows also from some "density" properties of some subsets.

2. Quasi-divisors and divisor class group

In this section we deal with the divisor class group of a *po*-group with a theory of quasi-divisors and we show that this abstract group is of the same importance as the classical class group for groups with divisor theory, although it is not possible to utilize the discrete character of the group $\mathbb{Z}$.

L. Skula [22] was the first to prove that a theory of divisors can be characterized by some *density* property. For an embedding $G \xrightarrow{h} \Gamma$ of a *po*-group G into an *l*-group Γ ($\Gamma = \mathbb{Z}^{(P)}$ in his approach) he introduced a short exact sequence

$$1 \longrightarrow G \xrightarrow{h} \Gamma \xrightarrow{\varphi_h} \mathcal{C}_h \longrightarrow 1$$

and proved that h is a strong divisor theory if and only if the map φ_h has some algebraic density property. Namely, he proved the following theorem where the set P can be considered as a subset in $\mathbb{Z}^{(P)}$ under the identification map $p \in P \mapsto \delta_p \in \mathbb{Z}^{(P)}$ where $(\delta_p)_q = 1$ if $q = p$ and $(\delta_p)_q = 0$ otherwise, for all $q \in P$.

Theorem (Skula, L. [22]). *Let G be a po-group and let $h : G \to \mathbb{Z}^{(P)}$ be an o-isomorphism into. Then the following conditions are equivalent:*

(1) *h is a strong theory of divisors.*

(2) *For* $p_1, \dots, p_n \in P$ $(n \geq 1)$, *the set* $\varphi_h(P \setminus \{p_1, \dots, p_n\})$ *is a semigroup generator of a divisor class group* $\mathcal{C}_h$.

In this section we show that even the strong theory of quasi-divisors of a finite character can be characterized by some "density property" of the above short exact sequence, i.e. by an identification of appropriate subsets $\mathcal{A} \subseteq \Gamma$ such that $\varphi_h(\mathcal{A}) = \mathcal{C}_h$.

Theorem 2.1. *Let* G *be a directed po-group,* Γ *be an* l*-group and let* $h : G \to \Gamma$ *be an* o*-isomorphism into. Let* $\mathcal{C}_h$ *be the divisor class group of* h *and* φ_h *be the canonical map* $\Gamma \to \mathcal{C}_h$. *Then the following conditions are equivalent:*

(1) h *is a strong theory of quasi-divisors of a finite character.*

(2) *There exists a defining family* W *of* t*-valuations of* Γ *of a finite character such that for any finite subset* $F \subseteq W$, $\varphi_h(\Gamma_+ \setminus \bigcup_{w\in F} M_w) = \mathcal{C}_h$ *holds, where* $M_w = \{\alpha \in \Gamma : w(\alpha) > 1\}$.

(3) *For any defining family* W *of* t*-valuations of* Γ *of a finite character and any finite subset* $F \subseteq W$, $\varphi_h(\Gamma_+ \setminus \bigcup_{w\in F} M_w) = \mathcal{C}_h$ *holds, where* M_w *is the same as in* (2). *Moreover, there exists at least one such defining family.*

(4) *Any element* $\alpha \in \Gamma$, $\alpha \neq 1$, *lies in all but a finite number of prime* l*-ideals of* Γ *and for any finite set* J *of prime* l*-ideals of* Γ, $\varphi_h(\bigcap_{\Delta\in J} \Delta_+) = \mathcal{C}_h$ *holds.*

(5) *Any element* $\alpha \in \Gamma$, $\alpha \neq 1$, *lies in all but a finite number of minimal prime* l*-ideals of* Γ *and for any finite set* K *of minimal prime* l*-ideals of* Γ, $\varphi_h(\bigcap_{\Delta\in K} \Delta_+) = \mathcal{C}_h$ *holds.*

(6) *There exists a system* $\mathcal{L}$ *of prime* l*-ideals of* Γ *such that* $\bigcap_{\Delta\in\mathcal{L}} \Delta = \{1\}$, *any element* $\alpha \in \Gamma, \alpha \neq 1$, *lies in all but a finite number of elements of* $\mathcal{L}$ *and for any finite subset* $K \subseteq \mathcal{L}$, $\varphi_h(\bigcap_{\Delta\in K} \Delta_+) = \mathcal{C}_h$ *holds.*

Proof. (1) $\Longrightarrow$ (3). Let W be a defining family of Γ of a finite character and let $F \subseteq W$ be a finite set. Let $\boldsymbol{a} \in \mathcal{C}_h$. Since h is a theory of quasi-divisors as well, (see [16]), there exists $\alpha \in \Gamma_+$ such that $\boldsymbol{a} = \varphi_h(\alpha)$. If $w(\alpha) > 1$ for some $w \in F$, we put $W_1 = \{w \in W : w(\alpha) > 1\}$. Since

W satisfies the approximation theorem, for a compatible and $(W_1 \cup F)$-complete system $(w(\alpha)^{-1})_w$ there exists $g \in G$ such that

$$w(h(g)) = w(\alpha)^{-1}; \quad w \in W_1 \cup F$$
$$w(h(g)) \geq 1; \quad w \in W \setminus (W_1 \cup F).$$

Then for any $w \in W$ we have $w(\alpha.h(g)) \geq 1$ and it follows that $\alpha.h(g) \in \Gamma_+$ and $w(\alpha.h(g)) = 1$ for all $w \in F$. Therefore, $\boldsymbol{a} = \varphi_h(\alpha.h(g))$ and (3) holds.

(3) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (1). Let W be a defining family of Γ of a finite character which satisfies the property (2) and let $\alpha, \beta \in \Gamma_+$. Then for some $\delta \in \Gamma_+$ we have $\varphi_h(\alpha^{-1}) = \varphi_h(\delta)$ and for some $g \in G$ we obtain $\alpha.\delta = h(g)$. Let $F = \{w \in W : w(\beta) > 1\}$. Then F is a finite set and according to (2) there exists $\gamma \in \Gamma_+$ such that $w(\gamma) = 1$ for any $w \in F$ and $\varphi_h(\gamma) = \varphi_h(\delta)$. Hence, $\gamma.h(g_1) = \delta$ for some $g_1 \in G$ and for any $w \in W$ we have $w(\gamma \wedge \beta) = 1$. Thus, $\gamma \wedge \beta = 1$ in Γ and $\alpha.\gamma \in h(G)$. Therefore, h is a strong theory of quasi-divisors of a finite character.

(1) $\Longrightarrow$ (5). First recall that an l-group is said to satisfy *Conrad's (F)-conditions* if each of its positive element is greater only than a finite number of pairwise disjoint elements. Then from [19]; 2.1, it follows that Γ satisfies the Conrad (F)-condition and according to [5]; p. 3.28, any element from Γ is contained in all but a finite number of minimal prime l-ideals of Γ. Hence,

$$W = \{w_\Delta : \Gamma \xrightarrow{w_\Delta} \Gamma/\Delta,\ \Delta \text{ minimal prime } l\text{-ideal of } \Gamma\}$$

is a defining family of a finite character of Γ. Moreover, $\Gamma_+ \setminus M_{w_\Delta} = \Delta_+$ for any $w_\Delta \in W$ and it may be proved analogously as in the proof of implication (1) $\Longrightarrow$ (3) that $\varphi_h(\bigcap_{\Delta \in J} \Delta_+) = \mathcal{C}_h$ for any finite subset of minimal prime l-ideals.

(5) $\Longrightarrow$ (4). It is clear that any prime l-ideal of Γ contains a minimal prime l-ideal. Hence, any element of Γ lies in all but a finite number of prime l-ideals of Γ. Let J be a finite set of prime l-ideals of Γ. It is clear that in any $\Delta \in J$ only a finite number of minimal prime l-ideals of Γ can be contained. Let K be this finite set of minimal prime l-ideals contained in elements of J. Then we have

$$\mathcal{C}_h = \varphi_h\Big(\bigcap_{\Lambda \in K} \Lambda_+\Big) \subseteq \varphi_h\Big(\bigcap_{\Delta \in J} \Delta_+\Big) \subseteq \mathcal{C}_h.$$

$(4) \Longrightarrow (6)$. It is trivial.

$(6) \Longrightarrow (2)$. Let $\mathcal{L}$ be a set of prime l-ideals of Γ which satisfies the conditions of (6). Let

$$W = \{w_\Delta : \Gamma \xrightarrow{w_\Delta} \Gamma/\Delta,\ \Delta \in \mathcal{L}\}.$$

Then it is clear that W is a defining family of Γ of a finite character and for any $w_\Delta \in W$, $\Gamma_+ \setminus M_{w_\Delta} = \Delta_+$ holds. Hence, the statement (2) holds. □

It should be observed firstly that Skula's result can be derived from Theorem 2.1. In fact the following lemma holds:

Lemma. *Let $h : G \to \mathbb{Z}^{(P)}$ be an o-isomorphism into. For any $p \in P$ let $w_p : G \to \mathbb{Z}$ be the composition of h and the p-th projection map. Then the following statements are equivalent;*

(1) $\forall p_1, \dots, p_n \in P$, $\varphi_h(P \setminus \{p_1, \dots, p_n\})$ *is a strong generator in* $\mathcal{C}_h$,

(2) $\forall p_1, \dots, p_n \in P$, $\varphi_h(\mathbb{Z}_+^{(P)} \setminus \bigcup_i M_{w_{p_i}}) = \mathcal{C}_h$, *where* $M_w = \{a \in \mathbb{Z}_+^{(P)} : w(a) > 0\}$.

Proof. $(1) \Longrightarrow (2)$. Let $\boldsymbol{a} \in \mathcal{C}_h$ and let $p_1, \dots, p_n \in P$. Then there exist $q_1, \dots, q_m \in P \setminus \{p_1, \dots, p_n\}$ and $b_1 > 0, \dots, b_m > 0$ in $\mathbb{Z}$ such that

$$\boldsymbol{a} = b_1\varphi_h(\delta_{q_1}) + \dots + b_m\varphi_h(\delta_{q_m}) = \varphi_h((a_q)_q),$$

where $a_q = b_i$ if $q = q_i$, $i = 1, \dots, m$, and $a_q = 0$, otherwise. It is clear that $a \in \mathbb{Z}_+^{(P)} \setminus \bigcup_i M_{w_{p_i}}$.

$(2) \Longrightarrow (1)$. Let $\boldsymbol{a} \in \mathcal{C}_h$ and let $p_1, \dots, p_n \in P$. Then there exists $a = (a_q)_q \in \mathbb{Z}_+^{(P)}$ such that $\varphi_h(a) = \boldsymbol{a}$. Let $q_1, \dots, q_m \in P$ be all elements $q \in P$ such that $a_{q_i} > 0$. Then $a = a_{q_1}\delta_{q_1} + \dots + a_{q_m}\delta_{q_m}$, $\delta_{q_i} \in P \setminus \{p_1, \dots, p_n\}$ and $\varphi_h(a)$ is strongly generated by $\varphi_h(\delta_{q_1}), \dots, \varphi_h(\delta_{q_m})$. □

The Theorem 2.1 has a lot of various corollaries. First, it enables us to determine the divisor class groups of some special quotient groups G/H.

Let $G \xrightarrow{h} \Gamma \xrightarrow{\varphi_h} \mathcal{C}_h$ be a strong theory of quasi-divisors of a finite character. In [17]; 2.5, it was proved that for any directed convex subgroup H of G (H is called an o-ideal in this case), the quotient po-group G/H has a strong theory of quasi-divisors, as well. We now show that in some nontrivial cases the divisor class group of G/H is trivial. To prove this result we need to mention the relationship between o-ideals of G and some l-ideals of Γ. In [20] we proved that there is a bijection between the set of o-ideals of G and some special l-ideals of Γ. These special l-ideals are called G-dense where an l-ideal Δ of Γ is *G-dense*, if for any $\alpha \in \Delta$ there exists $g \in G$ such that $\alpha \leq h(g) \in \Delta$. In [20]; 3.5, it was proved that if F is a finite set of prime l-ideals of Γ, then $\bigcap_{\Delta \in F} \Delta$ is a G-dense l-ideal (if h is of a finite character). Then, according to the same proposition, $H = h^{-1}(\bigcap_{\Delta \in F} \Delta)$ is an o-ideal of G and we can evaluate the divisor class group of the corresponding quotient group.

Proposition 2.2. *Let $h : G \to \Gamma$ be a strong theory of quasi-divisors of a finite character and let F be a finite set of prime l-ideals of Γ. Then $H = h^{-1}(\bigcap_{\Delta \in F} \Delta)$ is an o-ideal of G and G/H has a strong theory of quasi-divisors of a finite character with trivial divisor class group. Moreover, G/H is an l-group.*

Proof. Let $h_H : G/H \to \Gamma'$ be a strong theory of quasi-divisors which exists according to [17]; 2.5, and [20]; 3.5. Moreover, according to [19]; 2.7 and 2.5, there exists a defining family W_H of t-valuations of a finite character of G/H. Now, according to [17]; 2.6, the relationship between the quasi-divisor theories h and h_H can be described by the following commutative diagram:

$$\begin{array}{ccccc} G & \xrightarrow{h} & \Gamma & \xrightarrow{\varphi_h} & \mathcal{C}_h \\ \varphi\downarrow & & \downarrow\psi & & \downarrow\sigma \\ G/H & \xrightarrow{h_H} & \Gamma' & \xrightarrow{\varphi_{h_H}} & \mathcal{C}_{h_H} \end{array}$$

where vertical maps are epimorphisms. Then $\Delta_F = \bigcap_{\Delta \in F} \Delta \subseteq \ker \psi$. In fact, according to [20]; 3.5, Δ_F is G-dense and for any $\alpha \in \Delta_F^+$ there exists $g \in G$ such that $1 \leq \alpha \leq h(g) \in \Delta_F$. Then $\psi h(g) \geq \psi(\alpha) \geq 1$ and since $\psi h(g) = h_H \varphi(g) = 1$, we have $\alpha \in \ker \psi$. Then, according to 2.1, we have $\mathcal{C}_h = \varphi_h(\bigcap_{\Delta \in F} \Delta_+) = \varphi(\Delta_F^+)$ and it follows that

$$\begin{aligned} \mathcal{C}_{h_H} &= \sigma(\mathcal{C}_h) = \sigma\varphi_h(\Delta_F^+) \subseteq \sigma\varphi_h(\Delta_F) \\ &\subseteq \sigma\varphi_h(\ker \psi) = \varphi_{h_H}\psi(\ker \psi) = \{0\}. \end{aligned}$$ □

From 2.1. a result concerning rings of Krull type can also be derived. Recall that an integral domain R is a ring of Krull type if for R there exists a defining family of essential valuations of a finite character. In [7] it was proved that R is of Krull type if and only if its group of divisibility $G(R)$ has a quasi-divisor theory of a finite character.

Proposition 2.3. *Let R be an integral domain of Krull type and let $\mathcal{P}$ be a finite set of prime ideals of R such that R_P is a valuation ring for any $P \in \mathcal{P}$. Then for $S = R \setminus \bigcup_{P\in\mathcal{P}} P$, $R_S = \bigcap_{P\in\mathcal{P}} R_P$ is a GCD domain.*

Proof. According to [7]; 5.4, the group of divisibility $G = G(R)$ of R admits a quasi-divisor theory of a finite character. According to [19]; 3.2, G admits also a strong theory of quasi-divisors of a finite character. Then for some o-ideals H_P, $P \in \mathcal{P}$, the group of divisibility $G(R_P)$ is o-isomorphic to the quotient po-group G/H_P, $G(R_P) \cong_o G/H_P$. Let Γ be the l-group from the strong theory of quasi-divisors of G. According to [20]; 3.5, there are prime l-ideals Δ_P; $P \in \mathcal{P}$, of Γ such that $\Gamma/\Delta_P \cong_o \Lambda_{t_{H_P}}(G/H_P)$, where $\Lambda_r(T)$ is the Lorenzen r-group of a po-group T with ideal system r and t_{H_P} is the ideal system on G/H_P derived from a t-system (see [20], for more details about Lorenzen r-groups and [18] for more details about r-systems defined on quotient po-groups). It is clear that since G/H_P is an o-group, $\Lambda_t(G/H_P) \cong_o \Lambda_{t_{H_P}}(G/H_P) \cong_o G/H_P$. In fact, according to [20]; 3.2, the t_{H_P}-system is regularly closed and it follows that the elements of $\Lambda_{t_{H_P}}(G/H_P)_+$ are of the form A_t/H_P for some finite set A in G. Hence, $A_t/H_P = (aH_P)$ for some $a \in A$. Moreover, according to 2.1, $H = h^{-1}(\bigcap_{P\in\mathcal{P}} \Delta_P)$ is an o-ideal of G and G/H is an l-group. Then $G(R_S) \cong_o G/H$ and R_S is a GCD domain. □

We now deal with the following problem : If Δ is an l-ideal of Γ and $h : G \longrightarrow \Gamma$ is a theory of quasi-divisors, is it possible to say that Γ/Δ is also a value group of some quasi-divisor theory associated with G? The following proposition solves partially this question.

Proposition 2.4. *Let $h : G \longrightarrow \Gamma$ be a theory of quasi-divisors and let $\Delta \subset \Gamma$ be a G-dense l-ideal of Γ. Then there exists an l-ideal $\Delta' \subset \Gamma$ such that $\Delta \subseteq \Delta'$, the following diagram commutes and h' from this diagram is a theory of quasi-divisors:*

$$\begin{array}{ccc} G & \xrightarrow{h} & \Gamma \\ \text{nat}\downarrow & & \downarrow\text{nat} \\ G/h^{-1}(\Delta) & \xrightarrow{h'} & \Gamma/\Delta' \end{array}$$

Proof. According to [20]; 3.3, $H = h^{-1}(\Delta)$ is an o-ideal of G and the inclusion map $h'' : G/H \longrightarrow \Lambda_t(G/H)$ is a theory of quasi-divisors.

Moreover, according to the same proposition, $\Gamma/\Delta \cong_o \Lambda_{t_H}(G/H)$ and the corresponding map $h' : G/H \to \Lambda_{t_H}(G/H)$ is a (t_H, t)-morphism. Since h'' is a (t,t)-morphism and $t_H \le t$, it follows that h'' is a (t_H, t)-morphism as well and from the universality of the Lorenzen r-group (see [2]) it follows that there exists a (t,t)-morphism ψ such that the following diagram commutes:

$$\begin{array}{ccc} G/H & \xrightarrow{h'} & \Lambda_{t_H}(G/H) \cong_o \Gamma/\Delta \\ \| & & \downarrow \psi \\ G/H & \xrightarrow{h''} & \Lambda_t(G/H). \end{array}$$

Now, according to [20]; 3.2, the r-system t_H is regularly closed in G/H and it follows that the elements of $\Lambda_{t_H}(G/H)$ are of the form $\mathcal{A}_{t_H}/\mathcal{B}_{t_H}$, where $\mathcal{A}$, $\mathcal{B}$ are finite subsets in G/H. Analogously, the elements of $\Lambda_t(G/H)$ are of a form $\mathcal{A}_t/\mathcal{B}_t$ and ψ is then defined such that $\psi(\mathcal{A}_{t_H}/\mathcal{B}_{t_H}) = \mathcal{A}_t/\mathcal{B}_t$. Therefore, ψ is an l-epimorphism and there exists an l-ideal Δ' of Γ such that $\Delta \subseteq \Delta'$ and $\Lambda_t(G/H) \cong_o \Gamma/\Delta'$. □

In the previous proposition we tried to investigate whether Γ/Δ is a value group of some quasi-divisor theory associated with the original quasi-divisor thaory $G \xrightarrow{h} \Gamma$. An analogous question can be stated for the divisor class group $\mathcal{C}_h$ of h, i.e. if R is a subgroup of $\mathcal{C}_h$, does there exist another quasi-divisor theory associated with the original one and such that $\mathcal{C}_h/R$ is its divisor class group? In the following proposition we partly solve this question.

Proposition 2.5. *Let $G \xrightarrow{h} \Gamma \xrightarrow{\varphi_h} \mathcal{C}_h$ be a strong theory of quasi-divisors of a finite character and let R be a subgroup of the divisor class group $\mathcal{C}_h$ of h. Then the following statements are equivalent:*

(1) *There exists an l-ideal Δ of Γ such that $R = \varphi_h(\Delta)$.*

(2) *There exist morphisms φ, ψ, σ such that φ is surjective, ψ is an l-epimorphism and σ is an epimorphism such that the following diagram commutes:*

$$\begin{array}{ccccc} G & \xrightarrow{h} & \Gamma & \xrightarrow{\varphi_h} & \mathcal{C}_h \\ \varphi \downarrow & & \downarrow \psi & & \downarrow \sigma \\ G' & \xrightarrow{h'} & \Gamma' & \xrightarrow{\varphi_{h'}} & \mathcal{C}_h/R. \end{array}$$

Moreover, the lower row in this diagram is a strong theory of quasi-divisors of a finite character with corresponding divisor class group.

Proof. (1) $\Longrightarrow$ (2). Let $\Delta \subset \Gamma$ be an l-ideal and let $R = \varphi_h(\Delta)$. Let a homomorphism $\varphi_\Delta : \Gamma/\Delta \to \mathcal{C}_h/R$ be defined so that

$$(\forall \alpha\Delta \in \Gamma/\Delta) \quad \varphi_\Delta(\alpha\Delta) = \sigma.\varphi_h(\alpha),$$

where $\mathcal{C}_h \xrightarrow{\sigma} \mathcal{C}_h/R$ is the canonical morphism. Then the following diagram commutes:

$$\begin{array}{ccccc} G & \xrightarrow{h} & \Gamma & \xrightarrow{\varphi_h} & \mathcal{C}_h \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\sigma} \\ G' = \ker\varphi_\Delta & \xrightarrow{h'} & \Gamma/\Delta & \xrightarrow{\varphi_\Delta} & \mathcal{C}_h/R, \end{array}$$

where ψ and σ are canonical morphisms and φ is defined so that for any $g \in G$, we set $\varphi(g) = \psi.h(g)$. It is then clear that φ_Δ is defined correctly and that $\varphi(g) \in \ker\varphi_\Delta$, since $\varphi_\Delta.\psi.h(g) = \sigma.\varphi_h.h(g) = 0$. Moreover, φ is surjective. In fact, if $\alpha\Delta \in \ker\varphi_\Delta$, then $0 = \sigma.\varphi_h(\alpha)$ and $\varphi_h(\alpha) \in \varphi_h(\Delta)$. Now there exists $\beta \in \Delta$ such that $\alpha = \beta.h(g)$ for some $g \in G$ and $\varphi(g) = \psi.h(g) = h(g)\Delta = \alpha\Delta$.

We now show that the inclusion map h' is a strong theory of quasi-divisors of a finite character. Let $T_1, \dots, T_n$ be prime l-ideals of Γ/Δ. Then there exist prime l-ideals $\Delta_1, \dots, \Delta_n$ of Γ such that $T_i \cong_o \Delta_i/\Delta$ and $\Delta \subset \Delta_i$. According to 2.1, $\varphi_h(\bigcap_i \Delta_i^+) = \mathcal{C}_h$. Let $\boldsymbol{a} + R \in \mathcal{C}_h/R$, then for some $\alpha \in \bigcap_i \Delta_i^+$ we have $\varphi_h(\alpha) = \boldsymbol{a}$ and it follows that $\alpha\Delta \in \bigcap_i T_i^+$. Hence, $\boldsymbol{a}+R = \sigma.\varphi_h(\alpha) = \varphi_\Delta(\alpha\Delta)$ and it follows that $\mathcal{C}_h/R = \varphi_\Delta(\bigcap_i T_i^+)$. Therefore, h' is a strong theory of quasi-divisors of a finite character and $\mathcal{C}_h/R$ is its divisor class group.

(2) $\Longrightarrow$ (1). Let $\Delta = \ker\psi$. Then $R = \varphi_h(\Delta)$. In fact, let $\boldsymbol{a} \in R$. Then there exists $\alpha \in \Gamma$ such that $\varphi_h(\alpha) = \boldsymbol{a}$. Now $\varphi_{h'}\psi(\alpha) = \sigma\varphi_h(\alpha) = 0$ and $\psi(\alpha) \in \ker\varphi_{h'} = \operatorname{Im} h'$. Therefore, for some $g' \in G'$ we have $h'(g') = \psi(\alpha)$ and since φ is surjective, there exists $g \in G$ such that $\varphi(g) = g'$. Then $\psi(\alpha) = h'(g') = h'\varphi(g) = \psi h(g)$ and there exists $\beta \in \Delta$ such that $\beta = \alpha.h(g)$. Hence, $\varphi_h(\beta) = \boldsymbol{a}$ and $R \subseteq \varphi_h(\Delta)$. The converse inclusion is trivial. $\square$

F. Lucius [25] proved in 1998 that there exists a proper generalization of quasi-divisor theory. Namely he introduced the notion of greatest common divisor theory which is an o-isomorphism h of a po-group G into an l-group Γ such that the following condition is satisfied:

$$(\forall \alpha, \beta \in \Gamma) \quad \alpha = \beta \Leftrightarrow \{g \in G : h(g) \geq \alpha\} = \{g \in G : h(g) \geq \beta\}.$$

In [26] we proved that this condition is equivalent to the condition

$$\forall \alpha \in \Gamma, \quad \alpha = \inf_{\Gamma}(h(G) \cap (\alpha)_t).$$

In the following proposition we show that some "density" property can also be used for investigating of this GCD theory.

Proposition 2.6. *Let G be a po-group, h be an o-isomorphism of G into an l-group Γ and let $\varphi_h : \Gamma \to \mathcal{C}_h$ be the corresponding divisor class group. Then the following statements are equivalent:*

(1) *For any defining family W of t-valuations of Γ and any $w \in W$, $\varphi_h(\Gamma_+ \setminus M_w) = \mathcal{C}_h$ holds, where $M_w = \{g \in \Gamma : w(g) > 1\}$.*

(2) *For any prime l-ideal Δ of Γ, $\varphi_h(\Delta_+) = \mathcal{C}_h$ holds.*

(3) *For any minimal prime l-ideal Δ of Γ, $\varphi_h(\Delta_+) = \mathcal{C}_h$ holds.*

Moreover, from any of these statements it follows that h is a GCD theory.

Proof. (1) $\Longrightarrow$ (2). Let Δ be a prime l-ideal of Γ. Then $W = \{w_{\Delta'} : \Gamma \xrightarrow{w_{\Delta'}} \Gamma/\Delta'$ is the canonical morphism, Δ' a prime $l-$ ideal of $\Gamma\}$ is a defining family of t-valuations of Γ. Hence, $w_\Delta \in W$ and (2) follows from $\Gamma_+ \setminus M_{w_\Delta} = \Delta_+$.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). Let W be a defining family of t-valuations of Γ with the required property and let $w \in W$. Then M_w is a prime t-ideal of Γ and according to [5] there exists a prime l-ideal Δ' such that $\Gamma_+ \setminus M_w = \Delta'_+$. Let Δ be a minimal prime l-ideal such that $\Delta \subseteq \Delta'$. Then we have

$$\mathcal{C}_h = \varphi_h(\Delta_+) \subseteq \varphi_h(\Delta'_+) = \varphi_h(\Gamma_+ \setminus M_w) \subseteq \mathcal{C}_h.$$

Now, let (1) hold. We show that for any $\alpha \in \Gamma_+, \alpha = \inf_\Gamma(h(G) \cap (\alpha)_t)$ holds. In fact, it is clear that $\alpha \leq h(G) \cap (\alpha)_t$. Let $\beta \leq h(G) \cap (\alpha)_t$ and let us suppose that $\alpha \ngeq \beta$. Since W is a defining family of Γ, there exists $w \in W$ such that $1 \leq w(\alpha) < w(\beta)$. Since $-\varphi_h(\alpha) \in \mathcal{C}_h$, there exists $\gamma \in \Gamma_+$ such that $w(\gamma) = 1$ and $\varphi_h(\gamma) + \varphi_h(\alpha) = 0$. Hence, there exists $g \in G$ such that $\alpha \leq \gamma\alpha = h(g)$ and $h(g) \in h(G) \cap (\alpha)_t$. On the other hand, we have $w(h(G)) = w(\gamma\alpha) = w(\alpha) < w(\beta)$ and it follows that $\beta \nleq h(g)$, a contradiction. Therefore, $\alpha = \inf_\Gamma(h(G) \cap (\alpha)_t)$ and according to [26]; 2.2, h is a GCD theory. □

Proposition 2.7. *Let G be a po-group and let $G \xrightarrow{h} \Gamma$ be an o-isomorphism of G into an l-group Γ. Let us consider the following property:*

(∗) *For arbitrary finitely generated t-ideals $J, J_1, \dots, J_n$ of G such that $J \subseteq \bigcup_i J_i$, there exists i such that $J \subseteq J_i$.*

Then

(1) *If h is a theory of quasi-divisors, Γ is finitely atomic and G satisfies the property (∗), then h is a strong theory of quasi-divisors.*

(2) *If h is a strong theory of quasi-divisors of a finite character, then G satisfies the property (∗).*

Proof. (1) Let $\varphi_h : \Gamma \longrightarrow \mathcal{C}_h$ be the canonical map of Γ onto the divisor class group of h. Let us suppose that h is not a strong theory of quasi-divisors. Then according to [17]; 2.9, there exist atoms $\alpha_1, \dots, \alpha_n \in \Gamma_+$ such that

$$\varphi_h\bigg(\bigcap_i (\Gamma_+ \setminus (\alpha_i)_t)\bigg) \subset \mathcal{C}_h.$$

Hence, there exists $\boldsymbol{a} \in \mathcal{C}_h$ such that for any $\beta \in \Gamma_+$ with the property $\varphi_h(\beta) = \boldsymbol{a}$ we have $\beta \geq \alpha_i$ for some i. Let $\alpha \in \Gamma_+$ be such that $\varphi_h(\alpha) = -\boldsymbol{a}$ and let $g \in G$ be such that $h(g) \geq \alpha$. Then $h(g) = \alpha.\beta$ for some $\beta \in \Gamma_+$ and it follows that $\varphi_h(\beta) = \boldsymbol{a}$. Hence, $h(g).\alpha^{-1} \in (\alpha_i)_t$ for some i and we have $h(g) \in (\alpha_i\alpha)_t$ for some i. Therefore, $h(G) \cap (\alpha)_t \subseteq \bigcup_i ((\alpha\alpha_i) \cap h(G))$. Since h is a (t,t)-morphism, $J = h^{-1}((\alpha)_t)$ and $J_i = h^{-1}((\alpha\alpha_i)_t)$ are t-ideals in G and $J \subseteq \bigcup_i J_i$. Moreover, J, J_i are finitely generated. In fact, since h is a theory of quasi-divisors, for $\alpha > 1$ there exist $g_1, \dots, g_n \in G_+$ such that $\alpha = h(g_1) \wedge \dots \wedge h(g_n)$. Then $h^{-1}((\alpha)_t) = (g_1, \dots, g_n)_t$ in G and analogously for the t-ideal $h^{-1}((\alpha\alpha_i)_t)$. Therefore, according to the statement (∗), there exists i such that $h(G) \cap (\alpha)_t \subseteq (\alpha\alpha_i) \cap h(G)$. Now, according to [17]; 2.7, we have

$$\alpha = \inf_\Gamma (h(G) \cap (\alpha)_t) \geq \inf_\Gamma (h(G) \cap (\alpha\alpha_i)) = \alpha\alpha_i,$$

a contradiction with $\alpha > 1$.

(2) Let $J, J_1, \dots, J_n$ be finitely generated t-ideals of G and let $\alpha_i = \inf_\Gamma h(J_i)$, $\alpha = \inf_\Gamma h(J)$. Then we have

$$J = \{g \in G : h(g) \geq \alpha\}, \quad J_i = \{g \in G : h(g) \geq \alpha_i\}.$$

Let $J \subseteq \bigcup_i J_i$ and let us assume that $\alpha \not\geq \alpha_i$ for all i. Let W be a defining family of t-valuations of G of a finite character and let $\widehat{W}$ be the set of extensions of elements of W onto the t-valuations of Γ. Then for any i there exists $w_i \in W$ such that $\hat{w}_i(\alpha_i) > \hat{w}_i(\alpha) \geq 1$. Let $W_1 = \{w_1, \ldots, w_m\} \cup \{w \in W : \hat{w}(\alpha) \neq 1\}$. Then W_1 is finite and $(\hat{w}(\alpha))_w$ is compatible and a W_1-complete family. According to [16]; 3.5, there exists $g \in G$ such that

$$w(g) = \hat{w}(\alpha); \quad w \in W_1,$$

$$w(g) \geq 1; \qquad w \in W \setminus W_1.$$

Hence, $g \geq 1$ and $w(g) \geq \hat{w}(\alpha)$ for all $w \in W$. Since $\widehat{W}$ is a defining family of Γ, we have $h(g) \geq \alpha$ and since $g \in J \subseteq \bigcup_i J_i$, there exists i_0 such that $g \in J_{i_0}$. Thus, $h(g) \geq \alpha_{i_0}$ and we have

$$\hat{w}_{i_0}(\alpha_{i_0}) \leq w_{i_0}(g) = w_{i_0}(\alpha) < \hat{w}_{i_0}(\alpha_{i_0}),$$

a contradiction. Therefore, $\alpha \geq \alpha_i$ for some i and it follows that $J \subseteq J_i$. □

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