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On a representation of the projective connections of Finsler manifolds

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Dedicated to Professor Lajos Tamássy on his 70th birthday

The idea of introducing a single affine connection on the associated manifold involving one additional dimension, corresponding to the family of the projectively related affine connections on an original manifold, is due to T.Y. THOMAS [1]. Since then, many authors, for instance J.H.C. WHITEHEAD [2], and K. YANO [3], discussed and developed this theme. Later, S. KOBAYASHI and T. NAGANO [4] studied it from a modern viewpoint by means of the fiber bundle. In the present paper I am going to establish the foundations of the projective theory of Finsler manifolds on the basis of our theory [4], [5] of the non-affine connection by use of fiber bundles. That is, I shall try to represent the Berwald connection of an n-dimensional Finsler manifold M^n as a non-affine connection of the relative line bundle LM^{n+1}, which is called a "projective connection". To this purpose, it is necessary to base my theory on various kinds of connections of some bundles on M.

Since the connection on LM constructed from a Berwald connection of M is invariant by the projective change of the Berwald connection, all its torsion and curvature tensors are also invariant by this projective change. We see that the famous generalized Weyl and Douglas tensors are included as part of the components of these tensors. Moreover, we obtain that the constructed connection is linear (affine) and affinely flat if and only if the Berwald connection is projectively flat. Finally, I am going to show that the paths of a constructed connection with affine parameter correspond to the geodesics of the original Finsler manifold with projective parameter. These global results are explicitly expressed by means of the canonical coordinates.

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DIAGRAM OF BUNDLES

Fig. 1

0. Introduction

We introduce some notations (see *Fig. 1*):

M : an n-dimensional differentiable manifold.

$(F(M), M, p_F)$: the frame bundle of M.

$(T(M), M, p_T)$: the tangent bundle of M.

(LM, M, p_L) : the relative line bundle of M, or the associated bundle of $F(M)$ with standard fiber $\mathbb{R}$: i.e. $LM = (F(M) \times \mathbb{R})/\sim$, where the equivalence $\sim$ means $(z, r) \sim (zg, -w \log |g| + r)$ for $(z, r) \in F(M) \times \mathbb{R}$, $g \in GL(n, \mathbb{R})$, $|g|$ = the absolute value of the determinant g, w = a constant real number, (in §5 we put $w = -1/(n+1)$).

$(F(LM), LM, p_{FL})$: the frame bundle of LM.

$(T(LM), LM, p_{TL})$: the tangent bundle of LM.

$(T(LM) \times F(M), T(LM), p_{TLF})$: the induced bundle $(p_L p_{TL})^{-1} F(M)$ of bundle $F(M)$ by $p_L p_{TL}$.

$(T(LM) \times F(LM),\ T(LM),\ p_{TLFL})$: the induced bundle ${p_{TL}}^{-1} F(LM)$ of bundle $F(LM)$ by p_{TL}.

Let $\{U, (x^i)\}$ be a local coordinate neighborhood system on M, and the following be its canonical coordinate systems on the corresponding bundles:

$\{{p_F}^{-1} U,\ (x^i, {z^i}_j)\}$ on $F(M)$,
$\{{p_T}^{-1} U,\ (x^i, y^i)\}$ on $T(M)$,
$\{{p_L}^{-1} U,\ (x^i, x^0)\}$ on $L(M)$,
$\{(p_L p_{TL})^{-1} U,\ (x^i, x^0, y^i, y^0)\}$ on $T(LM)$,
$\{(p_L p_{FL})^{-1} U,\ (x^a, {w^a}_b)\}$ on $F(LM)$,
$\{(p_L p_{TL} p_{TLF})^{-1} U,\ (x^i, x^0, y^i, y^0, {z^i}_j)\}$ on $T(LM) \times F(M)$,
$\{(p_L p_{TL} p_{TLFL})^{-1} U,\ (x^a, y^a, {w^a}_b)\}$ on $T(LM) \times F(LM)$,

where $i, j = 1, 2, \ldots, n$; $a, b = 1, 2, \ldots, n, 0$.

For instance, a coordinate transformation on $(p_L p_{TL} p_{TLF})^{-1} U \cap (p_L p_{TL} p_{TLF})^{-1} \underline{\mathrm{U}} \neq \phi$ in $T(LM) \times F(M)$ is expressed as

$$\underline{\mathrm{x}}^i = \underline{\mathrm{x}}^i(x^j), \qquad \underline{\mathrm{x}}^0 = w\ \log |\partial \underline{\mathrm{x}} / \partial x| + x^0,$$
$$\underline{\mathrm{y}}^i = \frac{\partial \underline{\mathrm{X}}^i}{\partial x^j} y^j, \qquad \underline{\mathrm{y}}^0 = w\ \frac{\partial^2 \underline{\mathrm{X}}^k}{\partial x^j \partial x^m}\ \frac{\partial x^m}{\partial \underline{\mathrm{X}}^k}\ y^j + y^0,$$
$${\underline{\mathrm{z}}^i}_j = \frac{\partial \underline{\mathrm{X}}^i}{\partial x^k}\ {z^k}_j,$$

where $|\partial \underline{\mathrm{x}} / \partial x|$ = the absolute value of the determinant $\left[\frac{\partial \underline{\mathrm{X}}^i}{\partial x^j}\right]$, $i, j, k, m = 1, 2, \ldots, n$.

The "projective connection" of the Finsler manifold M is represented as a regular "pair-connection [5]" without $(h)hv$-torsion of the manifold LM, which is defined by a pair of horizontal distributions on $T(LM) \times$

$F(LM)$ (See 3). On the way to do it, we shall use a "tetra-connection" in $T(LM) \times F(M)$ (See 1).

1. Tetra-connection in the bundle $T(LM) \times F(M)$

The total space of the induced bundle $(p_L p_{TL})^{-1} F(M)$ is a set $\{(Y, Z);\ Y \in T(LM),\ Z \in F(M),\ p_L p_{TL}(Y) = p_F(Z)\}$, so we write it as $T(LM) \times F(M)$, and its right translations R_g, $g \in GL(n, \mathbb{R})$, mean $R_g(Y, Z) = (Y, R_g Z)$. On this bundle we shall consider four invariant distributions by the right translations.

Definition. A *tetra-connection* in the bundle $T(LM) \times F(M)$ is a collection $\{\Gamma^h, \Gamma^{h0}, \Gamma^v, \Gamma^{v0}\}$ of four distributions such that

(a) the tangent space $(T(LM) \times F(M))_p$ at $p \in T(LM) \times F(M)$ is the direct sum of $\Gamma^h, \Gamma^{h0}, \Gamma^v, \Gamma^{v0}$ and of the tangent space F_p of the fiber through p: i.e. $(T(LM) \times F(M))_p = {\Gamma^h}_p + {\Gamma^{h0}}_p + {\Gamma^v}_p + {\Gamma^{v0}}_p + F_p$,

(b) $p_{TLF}(\Gamma^v + \Gamma^{v0})$ is the tangent space of the fiber through $p_{TLF}(p)$ in the tangent bundle $T(LM)$, and $T(p_L) p_{TLF}(\Gamma^{v0}) = 0$, where $T(p_L)$ is the tangential map of p_L,

(c) $T(p_L) p_{TLF}(\Gamma^{h0}) = 0$,

(d) each of the four distributions is right invariant: i.e.

$$R_g {\Gamma^h}_p = {\Gamma^h}_{pg}, \quad R_g {\Gamma^{h0}}_p = {\Gamma^{h0}}_{pg},$$
$$R_g {\Gamma^v}_p = {\Gamma^v}_{pg}, \quad R_g {\Gamma^{v0}}_p = {\Gamma^{v0}}_{pg}.$$

The h-basic, $h0$-basic, v-basic and $v0$-basic vector fields of this connection and the fundamental vector fields, which span respectively $\Gamma^h, \Gamma^{h0}, \Gamma^v$, Γ^{v0} and F_p, are defined as follows:

Let $l_L : M \to LM$, $l_{TL} : LM \to T(LM)$, $l_{TLF} : T(LM) \to T(LM) \times F(M)$ be the horizontal lifts with respect to this connection, then

$H_J = l_{TLF} l_{TL} l_L (Z e_J)$ for $(Y, Z) \in T(LM) \times F(M)$, where $\{e_J\}$ are the basis of the standard fiber $\mathbb{R}^n$ of the tangent bundle $T(M)$ and $Z \in F(M)$ means the principal map $Z : \mathbb{R}^n \to T(M)$.

$H = l_{TLF} l_{TL}(Ze)$, where $\{e\}$ is the base of the standard fiber $\mathbb{R}$ of the associated bundle (line bundle) LM with respect to the frame bundle $F(M)$ and $Z, Z \in F(M)$, means the tangential map of the principal map $Z : \mathbb{R} \to LM$, $r \to w \log |Z| + r = x^0$.

$V_J = l_{TLF} l^v l_L (Z e_J)$, where $l^v : LM \to T(LM)$ is the vertical lift of the tangent bundle $T(LM)$.

$V = l_{TLF} l^v (Ze)$.

$L_I{}^J = L_p(g_I{}^J)$, where $\{g_I{}^J\}$ are the basis of the Lie algebra of $GL(n,\mathbb{R})$ and $L_p, P = (Y,Z) \in T(LM) \times F(M)$, means the tangential map of the left translation $L_p : GL(n,\mathbb{R}) \to T(LM) \times F(M)$, $g \to R_g P = (Y, R_g Z)$.

In terms of canonical coordinates, they are expressed by

$$H_J = z^j{}_J \left(\frac{\partial}{\partial x^j} - H_j \frac{\partial}{\partial x^0} - N^i{}_j \frac{\partial}{\partial y^i} - N_j \frac{\partial}{\partial y^0} - F_k{}^l{}_j \; z^k{}_m \frac{\partial}{\partial z^l{}_m} \right),$$

$$H = \frac{\partial}{\partial x^0} - E^l{}_k \; z^k{}_m \frac{\partial}{\partial z^l{}_m},$$

$$V_J = z^j{}_J \left(\frac{\partial}{\partial y^j} - H_j \frac{\partial}{\partial y^0} - C_k{}^l{}_j \; z^k{}_m \frac{\partial}{\partial z^l{}_m} \right),$$

$$V = \frac{\partial}{\partial y^0} - C^l{}_k \; z^k{}_m \frac{\partial}{\partial z^l{}_m},$$

$$L^K{}_J = z^j{}_J \; \delta^K{}_k \frac{\partial}{\partial z^j_k},$$

where $i,j,k,l,m,I,J,K = 1,2,\ldots,n$.

Let i_Z for fixed $Z \in F(M)$ be the tangential map of $i_Z : T(LM) \to T(LM) \times F(M)$, $Y \to (Y,Z)$, then

$$* H = i_Z l_{TL}(Ze) = \partial/\partial x^0,$$

$$* V_J = i_Z l^v l_L(Ze_J) = z^j{}_J \; (\partial/\partial y^j - H_j \; \partial/\partial y^0),$$

$$* V = i_Z l^v (Ze) = \partial/\partial y^0$$

are also special $h0$-basic, v-basic and $v0$-basic vector fields without torsions E and C respectively.

The structure equations are written as

$$[L^J{}_I, \; H_K] = \delta^J{}_K H_I, \quad [L^J{}_I, \; *H] = 0, \quad [L^J{}_I, \; *V_K] = \delta^J{}_K * V_I,$$

$$[L^J{}_I, \; *V] = 0, \quad [L^J{}_I, \; L^L{}_K] = \delta^J{}_K L^L{}_I - \delta^L{}_I L^J{}_K,$$

and the torsion and curvature tensors are given by

$$\begin{aligned}
[H_I, H_J] &= T_I{}^K{}_J H_K + R_{IJ} * H + R_I{}^M{}_J * V_M + R_I \circ_J * V \\
&\quad + R_K{}^M{}_{IJ} L^K{}_M, \\
[H_I, *H] &= \odot H_I * H + \odot N^M{}_I * V_M + \odot N_I * V + \odot F_K{}^M{}_I L^K{}_M, \\
[H_I, *V_J] &= H_{IJ} * H + P_I{}^M{}_J * V_M + P_{IJ} * V + P_K{}^M{}_{IJ} L^K{}_M, \\
[H_I, *V] &= \ominus H_I * H + \ominus N^M{}_I * V_M + \ominus N_I * V + \ominus F_K{}^M{}_I L^K{}_M, \\
[*V_I, *H] &= \odot H_I * V,
\end{aligned}$$

$$[*V, *H] = 0,$$
$$[*V_I, *V_J] = (H_{IJ} - H_{JI}) * V,$$
$$[*V, *V_I] = \ominus H_I * V.$$

There we have

$$T_I{}^K{}_J = z^i{}_I z^j{}_J z^{-1K}{}_k T_i{}^k{}_j, \qquad R_{IJ} = z^i{}_I z^j{}_J T_{ij},$$
$$R_I{}^M{}_J = z^i{}_I z^j{}_J z^{-1M}{}_m R_i{}^m{}_j, \quad \cdots \quad , \ominus H_I = z^i{}_I \ominus H_i,$$

where

$$T_i{}^k{}_j = F_i{}^k{}_j - F_j{}^k{}_i,$$
$$R_{ij} = \delta_j H_i - \delta_i H_j,$$
$$R_i{}^m{}_j = \delta_j N^m{}_i - \delta_i N^m{}_j,$$
$$R_i{}^0{}_j = \delta_j N_i - \delta_i N_j + H_m R_i{}^m{}_j,$$
$$R_k{}^m{}_{ij} = \delta_j F_k{}^m{}_i - \delta_i F_k{}^m{}_j - F_l{}^m{}_i F_k{}^l{}_j + F_l{}^m{}_j F_k{}^l{}_i,$$
$$\odot H_i = \partial_0 H_i,\ \odot N^m{}_i = \partial_0 N^m{}_i,\ \odot N_i = \partial_0 N_i + H_m \partial_0 N^m{}_i,$$
$$\odot F_k{}^m{}_i = \partial_0 F_k{}^m{}_i,$$
$$H_{ij} = \dot{\delta}_j H_i,$$
$$P_i{}^m{}_j = \dot{\delta}_j N^m{}_i - F_j{}^m{}_i,$$
$$P_{ij} = -\delta_i H_j + \dot{\delta}_j N_i + H_m \dot{\delta}_j N^m{}_i,$$
$$P_k{}^m{}_{ij} = \dot{\delta}_j F_k{}^m{}_i,$$
$$\ominus H_i = \dot{\partial}_0 H_i,\ \ominus N^m{}_i = \dot{\partial}_0 N^m{}_i,\ \ominus N_i = \dot{\partial}_0 N_i + H_m \dot{\partial}_0 N^m{}_i,$$
$$\ominus F_k{}^m{}_i = \dot{\partial}_0 F_k{}^m{}_i;$$

using some abbreviated notations

$$\delta_j = \frac{\partial}{\partial x^j} - H_j \frac{\partial}{\partial x^0} - N^m{}_j \frac{\partial}{\partial y^m} - N_j \frac{\partial}{\partial y^0},$$
$$\dot{\delta}_j = \frac{\partial}{\partial y^j} - H_j \frac{\partial}{\partial y^0},$$
$$\partial_0 = \partial/\partial x^0, \qquad \dot{\partial}_0 = \partial/\partial y^0.$$

The invariance of the tetra-connection with respect to x^0 and y^0 is defined by $[*H, H_I] = 0$ and $[*V, H_I] = 0$ respectively, or $\odot H_I = \odot N^M{}_I = \odot N_I = \odot F_K{}^M{}_I = 0$ and $\ominus H_I = \ominus N^M{}_I = \ominus N_I = \ominus F_K{}^M{}_I = 0$.

On the bundle $T(LM) \times F(M)$ there exist globally a relative scalar $\gamma(P) = x^0 - w \log |z^j{}_i|$, a contravariant vector $\gamma^l(P) = z^{-1I}{}_i y^i$ and a scalar

$\gamma^0(P) = y^0 + H_m y^m$ for $P = (x^i, x^0, y^i, y^0, z^i{}_j) \in T(LM) \times F(M)$. Their covariant derivatives are

$$
\begin{aligned}
&\gamma;_I = H_I \cdot \gamma = z^i{}_I(-H_i + wF_m{}^m{}_i),\\
&\gamma; = *H \cdot \gamma = 1,\\
&\gamma_{:I} = *V_I \cdot \gamma = 0,\\
&\gamma_{:} = *V \cdot \gamma = 0,\\
&\gamma^I;_J = H_J \cdot \gamma^I = z^{-1I}{}_i z^j{}_J(-N^i{}_j + F_m{}^i{}_j y^m),\\
&\gamma^I; = *H \cdot \gamma^I = 0,\\
&\gamma^I{}_{:J} = *V_J \cdot \gamma^I = \delta^I{}_J,\\
&\gamma^I{}_{:} = *V \cdot \gamma^I = 0,\\
&\gamma^0;_J = H_J \cdot \gamma^0 = z^j{}_J(-N_j + (\delta_j H_k)y^k - H_k N^k{}_j),\\
&\gamma^0; = *H \cdot \gamma^0 = \odot H_K \gamma^K,\\
&\gamma^0{}_{:J} = *V_J \cdot \gamma^0 = H_{KJ}\gamma^K,\\
&\gamma^0{}_{:} = *V \cdot \gamma^0 = 1 + \ominus H_M \gamma^M.
\end{aligned}
$$

The tetra-connection is said to be *regular* when $\gamma;_I = 0$, $\gamma^I;_J = 0$ and $\gamma^0;_J = 0$, or $H_i = wF_m{}^m{}_i$, $N^i{}_j = F_m{}^i{}_j y^m$, $N_j = (\delta_j H_k)y^k - H_k N^k{}_j$. Therefore, let $\{F_k{}^i{}_j\}$ be a given Berwald connection, then it may uniquely be regarded as an x^0-, y^0-invariant and regular tetra-connection without torsion E and C. (Details come later.)

2. Berwald tetra-connection

A manifold M with a metric $ds = L(x^i, dx^i)$ is said to be a Finsler manifold. Its geodesics are given by

$$\frac{dy^i}{dt} + 2G^i(x^j, y^j) = ky^i, \qquad y^i = \frac{dx^i}{dt},$$

where $k = (d^2s/dt^2)/(ds/dt)$ and $G^i(x^i, y^i)$ are positively homogeneous of degree 2 with respect to (y^i). Then $G_j{}^i{}_k = (\partial^2 G^i)/(\partial y^j \partial y^k)$ are said to be the coefficients of the Berwald connection.

We shall consider the Berwald connection as a special tetra-connection determined uniquely by the six axioms:

(a) It is *h-metrical*, or the h-covariant derivatives of the fundamental function $F = \frac{1}{2}L(x^i, y^i)^2$ all vanish: $F;_I = 0$.
Here the function F is regarded as an x^0-, y^0-invariant scalar positively

homogeneous of degree 2 with respect to (y^i) on the bundle $T(LM) \times F(M)$.

(b) It is *regular*: i.e.

$$\text{(b1)}\quad \gamma;_I = 0, \quad \text{(b2)}\quad \gamma^I;_J = 0, \quad \text{(b3)}\quad \gamma^0;_I = 0.$$

(c) The $(h)h$-torsion tensor $T_I{}^K{}_J = 0$.

(d) The $(v)hv$-torsion tensor $P_I{}^K{}_J = 0$.

In terms of canonical coordinates these axioms are written as

$$\begin{aligned}
&\text{(a)} && F;_i = \frac{\partial F}{\partial x^i} - \frac{\partial F}{\partial y^k} N^k{}_i = 0,\\
&\text{(b1)} && \gamma;_i = -H_i + wF_m{}^m{}_i = 0,\\
&\text{(b2)} && \gamma^i;_j = -N^i{}_j + F_m{}^i{}_j y^m = 0,\\
&\text{(b3)} && \gamma^0;_j = -N_j + (\delta_j H_k) y^k - H_k N^k{}_j = 0,\\
&\text{(c)} && T_i{}^k{}_j = F_i{}^k{}_j - F_j{}^k{}_i = 0,\\
&\text{(d)} && P_i{}^k{}_j = \partial N^k{}_i / \partial y^j - F_j{}^k{}_i = 0,
\end{aligned}$$

Differentiating (a) by y^j and contracting by y^i, we see by use of (a), (b2), (c) and (d) that

$$g_{jk} N^k{}_i y^i = \frac{\partial^2 F}{\partial x^i \partial y^j} y^i - \frac{\partial F}{\partial x^j}, \qquad \text{where } g_{jk} = \frac{\partial^2 F}{\partial y^j \partial y^k}.$$

Therefore we have

$$N^k{}_i y^i = g^{kj} \left(\frac{\partial^2 F}{\partial x^i \partial y^j} y^i - \frac{\partial F}{\partial x^j} \right) \equiv 2G^k,$$

and differentiating the last equations by y^m and using (b2), (c) and (d) we obtain

$$N^k{}_m = F_i{}^k{}_m y^i = \partial G^k / \partial y^m \equiv G^k{}_m.$$

Then (d) implies

$$F_l{}^k{}_m = \partial N^k{}_m / \partial y^l = \partial^2 G^k / \partial y^l \partial y^m \equiv G_l{}^k{}_m.$$

Moreover from (b1) and (b3) we see

$$H_i = wF_m{}^m{}_i = wG_m{}^m{}_i \qquad \text{and}$$

$$N_j = \left(\frac{\partial H_k}{\partial x^j} - N^m{}_j \frac{\partial H_k}{\partial y^m} \right) y^k - H_m N^m{}_j = \frac{\partial H}{\partial x^j} - G^m{}_j \frac{\partial H}{\partial y^m},$$

where $H = H_k y^k = wN^m{}_m = wG^m{}_m$.

Let ${}^G\Gamma^h$ be spaned by the $\{H_I\}$ and ${}^G*\Gamma^v$ by the $\{*V_I\}$ determined above, then the tetra-connection $\{{}^G\Gamma^h, *\Gamma^{h0}, {}^G*\Gamma^v, *\Gamma^{v0}\}$ is called *Berwald tetra-connection*, which is an $x0$-, $y0$-invariant regular tetra-connection without torsion E and C and positively homogeneous of degree o with respect to (y^i).

In this case, since it is $x0$-, $y0$-invariant and $T_i{}^k{}_j = 0$, $P_i{}^k{}_j = 0$, $P_{ij} = 0$ and $R_i{}^o{}_j = 0$, the bracket equations are written as

$$[H_I, H_J] = R_{IJ} * H + R_I{}^K{}_J * V_K + R_K{}^M{}_{IJ}\, L^K{}_M,$$
$$[H_I, *V_J] = H_{IJ} * H + P_K{}^M{}_{IJ}\, L^K{}_M;$$

the others all vanish, where

$$R_{ij} = \frac{\partial H_{[i}}{\partial x^{j]}} - N^m{}_{[j}\frac{\partial H_{i]}}{\partial y^m},$$
$$R_i{}^k{}_j = \frac{\partial N^k{}_{[i}}{\partial x^{j]}} - N^m{}_{[j}\frac{\partial N^k{}_{i]}}{\partial y^m},$$
$$R_k{}^m{}_{ij} = \frac{\partial F_k{}^m{}_{[i}}{\partial x^{j]}} - N^I{}_{[j}\frac{\partial F_k{}^m{}_{i]}}{\partial y^l} - F_l{}^m{}_{[i}F_k{}^l{}_{j]},$$
$$H_{ij} = \frac{\partial H_i}{\partial y^j} = w\frac{\partial^2 G^k{}_k}{\partial y^i y^j},$$
$$P_k{}^m{}_{ij} = \frac{\partial F_k{}^m{}_i}{\partial y^j} = \frac{\partial^3 G^m}{\partial y^k \partial y^i \partial y^j}.$$

The relations $R_{IJ} = wR_K{}^K{}_{IJ}$, $H_{IJ} = wP_K{}^K{}_{IJ}$ and $R_I{}^K{}_J = R_M{}^K{}_{IJ}\gamma^M$ hold. Their essential Bianchi's equations are given by

$$S_{(ijk)}\{R_k{}^m{}_{ij}\} = 0,$$
$$S_{(ijk)}\{R_l{}^m{}_{ij};_k + R_i{}^p{}_j P_l{}^m{}_{kp}\} = 0,$$
$$R_k{}^m{}_{ij} = R_i{}^m{}_{j:k},$$
$$R_p{}^q{}_{ij:k} = P_p{}^q{}_{ik};_j - P_p{}^q{}_{jk};_i,$$

where $S_{(ijk)}$ denotes the cyclic permutation of i, j, k and summation, and

$$R_l{}^m{}_{ij};_k = \frac{\partial R_l{}^m{}_{ij}}{\partial x^k} - N^p{}_k\frac{\partial R_l{}^m{}_{ij}}{\partial y^p} - R_p{}^m{}_{ij}F_l{}^p{}_k - R_l{}^m{}_{pj}F_i{}^p{}_k - R_l{}^m{}_{ip}F_j{}^p{}_k + R_l{}^p{}_{ij}F_p{}^m{}_k,$$
$$R_l{}^m{}_{ij:k} = \partial R_l{}^m{}_{ij}/\partial y^k.$$

3. Pair-connection in the bundle $T(LM) \times F(LM)$

Definition. A *pair-connection* in the bundle $T(LM) \times F(LM) \to T(LM)$ is a collection $\{\bar{\Gamma}^h, \bar{\Gamma}^v\}$ of two distributions such that

(a) the tangent space $(T(LM) \times F(LM))_Q$ at $Q \in T(LM) \times F(LM)$ is the direct sum of $\bar{\Gamma}^h, \bar{\Gamma}^v$ and the tangent space G_Q of the fiber through Q :

$$(T(LM) \times F(LM))_Q = \bar{\Gamma}^h{}_Q + \bar{\Gamma}^v{}_Q + G_Q,$$

(b) $p_{TLFL}(\bar{\Gamma}^v)$ is tangential to the fiber through $p_{TLFL}(Q)$ of the tangent bundle $T(LM)$,

(c) each of the two distributions is right invariant: i.e. if R_a, $a \in GL(n+1, \mathbb{R})$, is the right translation of the induced bundle $T(LM) \times F(LM)$, then $R_a \bar{\Gamma}^h{}_Q = \bar{\Gamma}^h{}_{Qa}$, $R_a \bar{\Gamma}^v{}_Q = \bar{\Gamma}^v{}_{Qa}$.

The h-basic, v-basic and fundamental vector fields are expressed in terms of canonical coordinates $(x^a, y^a, w^a{}_b)$ as follows:

$$\begin{aligned}
\bar{H}_A &= w^a{}_A \left(\frac{\partial}{\partial x^a} - \bar{N}^b{}_a \frac{\partial}{\partial y^b} - \bar{F}_d{}^b{}_a w^d{}_c \frac{\partial}{\partial w^b{}_c} \right), \\
\bar{V}_A &= w^a{}_A \left(\frac{\partial}{\partial y^a} - \bar{C}_d{}^b{}_a w^d{}_c \frac{\partial}{\partial w^b{}_c} \right), \\
\bar{L}^B{}_A &= w^a{}_A \delta^B{}_b \frac{\partial}{\partial w^a{}_b},
\end{aligned}$$

where $A, B, a, b, c, d = 1, 2, \ldots, n, 0$.

When $\gamma^B{}_{|A} = w^a{}_A(-\bar{N}^b{}_a + \bar{F}_c{}^b{}_a y^c) = 0$ and $\gamma^B{}_{\|A} = \delta^B{}_A + \bar{C}_C{}^B{}_A \gamma^C = \delta^B{}_A$ hold for $\gamma^B = w^{-1B}{}_b y^b$, it is said that the pair-connection is regular.

We see that $\#\bar{V}_A = w^a{}_A \frac{\partial}{\partial y^a}$ are also special v-basic vector fields without torsion $\bar{C}$, and we have $\bar{V}_A = \#\bar{V}_A - \bar{C}_D{}^B{}_A \bar{L}^D{}_B$.

The structure equations are written as

$$\begin{aligned}
\left[\bar{L}^B{}_A, \bar{H}_C\right\} &= \delta^B{}_C \bar{H}_A, \\
\left[\bar{L}^B{}_A, \#\bar{V}_C\right] &= \delta^B{}_C \#\bar{V}_A, \\
\left[\bar{L}^B{}_A, \bar{L}^D{}_C\right] &= \delta^B{}_C \bar{L}^D{}_A - \delta^D{}_A \bar{L}^B{}_C,
\end{aligned}$$

and the torsion and curvature tensors are given by

$$\begin{aligned}
\left[\bar{H}_A, \bar{H}_B\right] &= \bar{T}_A{}^C{}_B \bar{H}_C + \bar{R}_A{}^D{}_B \#\bar{V}_D + \bar{R}_C{}^D{}_{AB} \bar{L}^C{}_D, \\
\left[\bar{H}_A, \#\bar{V}_B\right] &= \bar{P}_A{}^C{}_B \#\bar{V}_C + \bar{P}_C{}^D{}_{AB} \bar{L}^C{}_D, \\
\left[\#\bar{V}_A, \#\bar{V}_B\right] &= 0,
\end{aligned}$$

where these tensors are written as

$$\bar{T}_A{}^C{}_B = w^a{}_A w^b{}_B w^{-1C}{}_c \bar{T}_a{}^c{}_b, \quad \bar{T}_a{}^c{}_b = \bar{F}_a{}^c{}_b - \bar{F}_b{}^c{}_a,$$
$$\bar{R}_A{}^C{}_B = w^a{}_A w^b{}_B w^{-1C}{}_c \bar{R}_a{}^c{}_b, \quad \bar{R}_a{}^c{}_b = \bar{\delta}_b \bar{N}^c{}_a - \bar{\delta}_a \bar{N}^c{}_b,$$
$$\bar{R}_C{}^D{}_{AB} = w^c{}_C w^a{}_A w^b{}_B w^{-1D}{}_d \bar{R}_c{}^d{}_{ab},$$
$$\bar{R}_c{}^d{}_{ab} = \bar{\delta}_b \bar{F}_c{}^d{}_a - \bar{\delta}_a \bar{F}_c{}^d{}_b - \bar{F}_e{}^d{}_a \bar{F}_c{}^e{}_b + \bar{F}_e{}^d{}_b \bar{F}_c{}^e{}_a,$$
$$\bar{P}_A{}^C{}_B = w^a{}_A w^b{}_B w^{-1C}{}_c \bar{P}_a{}^c{}_b, \quad \bar{P}_a{}^c{}_b = \dot{\partial}_b \bar{N}^c{}_a - \bar{F}_b{}^c{}_a,$$
$$\bar{P}_C{}^D{}_{AB} = w^a{}_A w^b{}_B w^c{}_C w^{-1D}{}_d \bar{P}_c{}^d{}_{ab}, \quad \bar{P}_c{}^d{}_{ab} = \dot{\partial}_b \bar{F}_c{}^d{}_a,$$

using some abbreviated notations

$$\bar{\delta}_a = \frac{\partial}{\partial x^a} - \bar{N}^b{}_a \frac{\partial}{\partial y^b}, \qquad \dot{\partial}_a = \partial/\partial y^a.$$

4. The induced pair-connection in $T(LM) \times F(LM)$.

A map $\phi_H : T(LM) \times F(M) \to T(LM) \times F(LM)$ is defined as $(Y, Z) \to (Y, (l_H(Z), T(L_Z)(\partial/\partial r)))$, where l_H is the lift $T(M) \to T(LM)$ with respect to the distribution $p_{TL} p_{TLF}(\Gamma^h + \Gamma^{h0})$, or $l_H : z^i{}_I(\partial/\partial x^i) \to z^i{}_I(\partial/\partial x^i - H_i \partial/\partial x^0)$, and $T(L_Z)$ is the tangent map of the principal map $L_Z : \mathbb{R} \to LM$; $r \to w \log|z| + r = x^0$, or $T(L_Z) : T(\mathbb{R}) \to T(LM)$; $\partial/\partial r \to \partial/\partial x^0$.

By the expression in canonical coordinates we have

$$\phi_H : (x^i, x^0, y^i, y^0, z^i{}_j) \to (x^i, x^0, y^i, y^0, w^a{}_b),$$

where

$$(w^a{}_b) = \begin{pmatrix} z^i{}_j & 0 \\ -H_i z^i{}_j & 1 \end{pmatrix},$$

or

$$w^i{}_j = z^i{}_j,\ w^i{}_0 = 0,\ w^0{}_j = -H_i z^i{}_j,\ w^0{}_0 = 1,$$

$i, j = 1, 2, \ldots, n$; $a, b = 1, 2, \ldots, n, 0$.

The map ϕ_H is a bundle map such that

$$\text{(a)} \quad p_{TLFL}\phi_H = p_{TLF}, \qquad \text{(b)} \quad \phi_H R_g = R_{f(g)}\phi_H,$$

where $f : GL(n, \mathbb{R}) \to GL(n+1, \mathbb{R}); (g^i{}_j) \to \begin{pmatrix} g^i{}_j & 0 \\ 0 & 1 \end{pmatrix}$.

Now, we shall induce by ϕ_H from a tetra-connection $\{\Gamma^h, \Gamma^{h0}, \Gamma^v, \Gamma^{v0}\}$ a pair-connection $\{\bar{\Gamma}^h, \bar{\Gamma}^v\}$ on the bundle $T(LM) \times F(LM)$. It is to be noticed that ϕ_H is dependent of Γ^h.

The basic vector fields $(\bar{H}_A, \bar{V}_A)$ of the induced pair-connection are given from the basic vector fields (H_I, H, V_I, V) of a tetra-connection by ϕ_H as follows:

$$\bar{H}_I = \phi_H(H_I),\ \bar{H}_0 = \phi_H(H),\ \bar{V}_I = \phi_H(V_I),\ \bar{V}_0 = \phi_H(V).$$

Therefore, in canonical coordinates at $\phi_H(Y, Z)$, the coefficients of the induced pair-connection are obtained as

$$\begin{aligned}
&\bar{N}^i{}_j = N^i{}_j, \qquad \bar{N}^i{}_0 = 0, \qquad \bar{N}^0{}_j = N_j, \qquad \bar{N}^0{}_0 = 0,\\
&\bar{F}_j{}^i{}_k = F_j{}^i{}_k + E^i{}_j H_k,\ \bar{F}_0{}^i{}_k = 0,\ \bar{F}_j{}^i{}_0 = E^i{}_j,\ \bar{F}_0{}^i{}_0 = 0,\\
&\bar{F}_j{}^0{}_k = \frac{\partial H_j}{\partial x^k} - N^m{}_k \frac{\partial H_j}{\partial y^m} - N_k \frac{\partial H_j}{\partial y^0} - (F_j{}^m{}_k + E^m{}_j H_k) H_m,\ \bar{F}_0{}^0{}_k = 0,\\
&\bar{F}_j{}^0{}_0 = \frac{\partial H_j}{\partial x^0} - E^m{}_j H_m, \quad \bar{F}_0{}^0{}_0 = 0,\\
&\bar{C}_j{}^i{}_k = C_j{}^i{}_k + C^i{}_j H_k, \quad \bar{C}_0{}^i{}_k = 0, \quad \bar{C}_j{}^i{}_0 = C^i{}_j, \quad \bar{C}_0{}^i{}_0 = 0,\\
&\bar{C}_j{}^0{}_k = \frac{\partial H_j}{\partial y^k} - (C_j{}^m{}_k + C^m{}_j H_k) H_m, \qquad \bar{C}_0{}^0{}_k = 0,\\
&\bar{C}_j{}^0{}_0 = \frac{\partial H_j}{\partial y^0} - C^m{}_j H_m, \qquad \bar{C}_0{}^0{}_0 = 0.
\end{aligned}$$

In particular, when it is induced from a Berwald tetra-connection, we have $({}^G\bar{H}_A, {}^G\bar{V}_A)$ whose coefficients are as follows:

$$\begin{aligned}
&\bar{N}^i{}_j = N^i{}_j, \qquad \bar{N}^i{}_0 = 0, \qquad \bar{N}^0{}_j = N_j, \qquad \bar{N}^0{}_0 = 0,\\
&\bar{F}_j{}^i{}_k = F_j{}^i{}_k, \qquad \bar{F}_0{}^i{}_k = 0, \qquad \bar{F}_j{}^i{}_0 = 0, \qquad \bar{F}_0{}^i{}_0 = 0,\\
&\bar{F}_j{}^0{}_k = \frac{\partial H_j}{\partial x^k} - N^m{}_k \frac{\partial H_j}{\partial y^m} - F_j{}^m{}_k H_m = \frac{\partial N_k}{\partial y^j} \equiv N_{kj},\\
&\bar{F}_0{}^0{}_k = 0, \qquad \bar{F}_j{}^0{}_0 = 0, \qquad \bar{F}_0{}^0{}_0 = 0,\\
&\bar{C}_j{}^i{}_k = 0, \qquad \bar{C}_0{}^i{}_k = 0, \qquad \bar{C}_j{}^i{}_0 = 0, \qquad \bar{C}_0{}^i{}_0 = 0,\\
&\bar{C}_j{}^0{}_k = H_{jk}, \qquad \bar{C}_0{}^0{}_k = 0, \qquad \bar{C}_j{}^0{}_0 = 0, \qquad \bar{C}_0{}^0{}_0 = 0.
\end{aligned}$$

Notice that this pair-connection is regular.

5. Projective connection of a Finsler manifold

In this section, by use of the bundles we get a new interpretation on the representation of the projective connection of a Finsler manifold. From the Berwald tetra-connection, which is essentially an ordinary Berwald connection G, we shall construct a pair-connection $({}^p\bar{\Gamma}^h, \#\bar{\Gamma}^v)$, which is invariant by any projective change of the Berwald connection. Throughout this section, we put $w=-1/(n+1)$.

Now we take a $(1,2)$ type tensor ${}^GA=({}^GA_B{}^A{}_C)$ on $T(LM)\times F(LM)$ depending on the Berwald connection G as follows:

$$\begin{aligned}{}^GA=\ & y^iH_{jk}(\partial/\partial x^i-H_i\partial/\partial x^0)\otimes dx^j\otimes dx^k\\ &+\delta^i{}_k(\partial/\partial x^i-H_i\partial/\partial x^0)\otimes(dx^0+H_mdx^m)\otimes dx^k\\ &+\delta^i{}_j(\partial/\partial x^i-N_i\partial/\partial x^0)\otimes dx^j\otimes(dx^0+H_ldx^l)\\ &+K_{jk}(\partial/\partial x^0)\otimes dx^j\otimes dx^k+(\partial/\partial x^0)\otimes(dx^0+H_mdx^m)\otimes(dx^0+H_idx^l),\end{aligned}$$

where $H_{jk}=\frac{\partial^2H}{\partial y^j\partial y^k}$, $K_{jk}=\frac{1}{n^2-1}\left(nR_j{}^m{}_{mk}+R_k{}^m{}_{mj}+\frac{\partial R_m{}^m{}_{kp}}{\partial y^j}y^p\right)$.
That is to say

$$\begin{aligned}&{}^GA_J{}^I{}_K(\phi_H(P))=\gamma^I(P)H_{JK}(P),\ {}^GA_0{}^I{}_K=\delta^I{}_K,\ {}^GA_J{}^I{}_0=\delta^I{}_J,\ {}^GA_0{}^I{}_0=0,\\ &{}^GA_J{}^0{}_K(\phi_H(P))=K_{JK}(P),\qquad {}^GA_0{}^0{}_K=0,\quad {}^GA_J{}^0{}_0=0,\quad {}^GA_0{}^0{}_0=1.\end{aligned}$$

Then we put

$${}^p\bar{H}_A={}^G\bar{H}_A-{}^GA_B{}^C{}_A\left(\gamma^B\#\bar{V}_C+\bar{L}^B{}_C\right).$$

The pair-connection $({}^p\bar{\Gamma}^h,\#\bar{\Gamma}^v)$, where ${}^p\bar{\Gamma}^h$ is the distribution spaned by $\{{}^p\bar{H}_A\}$, is called the *projective connection* of the Finsler manifold M. We see that ${}^p\bar{H}_A=w^a{}_A\left(\frac{\partial}{\partial x^a}-{}^p\bar{N}^b{}_a\frac{\partial}{\partial y^b}-{}^p\bar{F}_d{}^b{}_aw^d{}_c\frac{\partial}{\partial w^b{}_c}\right)$, where

$$\begin{aligned}&{}^p\bar{N}^j{}_i=N^j{}_i+\delta^j{}_i(y^0+H)+y^jH_i, && {}^p\bar{N}^j{}_0=y^j,\\ &{}^p\bar{N}^0{}_i=N_i-HH_i+K_{mi}y^m, && {}^p\bar{N}^0{}_0=y^0,\\ &{}^p\bar{F}_k{}^j{}_i=F_k{}^j{}_i+H_k\delta^j{}_i+H_i\delta^j{}_k+y^jH_{ki}, && {}^p\bar{F}_0{}^j{}_i=\delta^j{}_i,\\ &{}^p\bar{F}_k{}^j{}_0=\delta^j{}_k, && {}^p\bar{F}_0{}^j{}_0=0,\\ &{}^p\bar{F}_k{}^0{}_i=N_{ik}+K_{ki}-HH_{ki}-H_kH_i, && {}^p\bar{F}_0{}^0{}_i=0,\\ &{}^p\bar{F}_k{}^0{}_0=0, && {}^p\bar{F}_0{}^0{}_0=1.\end{aligned}$$

Proposition 1. *In the projective connection, the relations*

(1) $\gamma^B{}_{|A}={}^p\bar{H}_A\cdot\gamma^B=0$ *(i.e. regular),*

(2) *the* $(h)h-$ *torsion tensor* ${}^p\bar{T}_{AB}=0$,

(3) *the*$(v)hv-$ *torsion tensor* ${}^p\bar{P}_C{}^B{}_A=0$,

(4) $v^B{}_{|A}=\delta^B{}_A$ for $v=\partial/\partial x^0$,

hold, where the notation $_{|A}$ *denotes the covariant derivative by* ${}^p\bar{H}_A$.

Proof. It is shown immediately that (1) ${}^p\bar{N}^b{}_a = {}^p\bar{F}_c{}^b{}_a y^c$, (3) ${}^p\bar{F}_c{}^b{}_a = \partial^p\bar{N}^b{}_a/\partial y^c$. (2) follows from the equations $K_{jk} - K_{kj} = -wR_m{}^m{}_{jk}$, $N_{kj} - N_{jk} = wR_m{}^m{}_{jk}$. (4) as $v^i = 0$, $v^0 = 1$, therefore $v^b{}_{;a} = {}^p\bar{F}_0{}^b{}_a = \delta^b{}_a$.

Theorem 1. *The projective connection* ${}^p\bar{\Gamma}^h(G)$ *is invariant with respect to the projective change of the Berwald connection G, that is to say, let* ${}^p\bar{\Gamma}^h(\bar{G})$ *correspond to* $\bar{G}^i = G^i + y^i b(x^k, y^k)$, *where* $b(x^k, y^k)$ *is positively homogeneous of degree 1 with respect to* (y^k), *then* ${}^p\bar{\Gamma}^h(\bar{G}) = {}^p\bar{\Gamma}^h(G)$.

Proof. It is obvious that $f^i \equiv G^i + y^i H$, where $H = wG^m{}_m$, is invariant by the projective change of G^i, therefore $f^i{}_j \equiv \partial f^i/\partial y^j = G^i{}_j + \delta^i{}_j H + y^i H_j$ and $f_k{}^i{}_j \equiv \partial f^i{}_j/\partial y^k = G_k{}^i{}_j + \delta^i{}_k H_j + \delta^i{}_j H_k + y^i H_{jk} = {}^p\bar{F}_k{}^i{}_j$ are also invariant. To show that the coefficients ${}^p\bar{F}_k{}^0{}_j$ are projectively invariant, we calculate the invariants

$$\begin{aligned}
{}^f R_j{}^i{}_{kl} &\equiv A_{[kl]}\{\partial f_j{}^i{}_k/\partial x^l - f^m{}_l \partial f_j{}^i{}_k/\partial y^m - f_j{}^m{}_l f_m{}^i{}_k\} \\
&= R_j{}^i{}_{kl} + \delta^i{}_j(N_{lk} - N_{kl}) + \delta^i{}_k(N_{lj} - H_j H_l - HH_{jl}) \\
&\quad - \delta^i{}_l(N_{kj} - H_j H_k - HH_{jk}) + y^i \partial(N_{lk} - N_{kl})/\partial y^j, \\
{}^f K_{jk} &\equiv 1/(n^2-1)\ (n\,{}^f R_j{}^m{}_{mk} + {}^f R_k{}^m{}_{mj}) \\
&= K_{jk} + N_{kj} - H_j H_k - HH_{jk} = {}^p\bar{F}_j{}^0{}_k,
\end{aligned}$$

where $A_{[jk]}$ denotes the interchange of i, j and subtraction, and

$$N_{lk} = \partial H_k/\partial x^l - G^m{}_l(\partial H_k/\partial y^m) - H_m G_k{}^m{}_l, \quad H_k = wG_m{}^m{}_k.$$

The next theorem is a immediate result of Theorem 1.

Theorem 2. *The curvature tensors* ${}^p R_A{}^B{}_{CD}$ *and* ${}^p P_A{}^B{}_{CD}$ *of the projective pair-connection* $({}^p\bar{\Gamma}^h, \#\bar{\Gamma}^v)$ *in* $T(LM) \times F(LM)$ *are also invariant by the projective change of the Berwald connection G.*

By some calculations we have the curvature tensors:

$${}^p R_j{}^i{}_{kl} = W_j{}^i{}_{kl} \text{ (Weyl's generalized projective curvature tensor)},$$

$${}^p R_j{}^0{}_{kl} = K_{jk;l} - K_{jl;k} - H_m W_j{}^m{}_{kl} - H_{jm} W_p{}^m{}_{kl} y^p,$$

the other components all vanish, where

$$\begin{aligned}
W_j{}^i{}_{kl} &= R_j{}^i{}_{kl} - \delta^i{}_j(K_{kl} - K_{lk}) - \delta^i{}_k K_{jl} + \delta^i{}_l K_{jk} - y^i(K_{kl} - K_{lk})_{:j} \\
&= R_j{}^i{}_{kl} - w\delta^i{}_j R_m{}^m{}_{kl} + wy^i R_m{}^m{}_{kl:j} \\
&\quad + w/(n-1)\{\delta^i{}_k(nR_j{}^m{}_{ml} + R_l{}^m{}_{mj} + y^p R_m{}^m{}_{lp:j}) \\
&\quad - \delta^i{}_l(nR_j{}^m{}_{mk} + R_k{}^m{}_{mj} + y^p R_m{}^m{}_{kp:j})\},
\end{aligned}$$

and

$^pP_j{}^i{}_{kl} = D_j{}^i{}_{kl}$ (Douglas' tensor),
$^pP_j{}^0{}_{kl} = -1/(n-1)D_j{}^m{}_{kl};_m - H_m D_j{}^m{}_{kl}$,
the other components all vanish, where

$$D_j{}^i{}_{kl} = G_j{}^i{}_{kl} + \delta^i{}_j H_{kl} + \delta^i{}_k H_{jl} + \delta^i{}_l H_{jk} + y^i H_{jkl} = \partial^3(G^i + y^i H)/\partial y^j \partial y^k \partial y^l.$$

Lemma. *The following identities hold:*

(1) $W_j{}^i{}_{kl} = -W_j{}^i{}_{lk}$, $W_j{}^i{}_{kl} + W_l{}^i{}_{jk} + W_k{}^i{}_{lj} = 0$, *hence, all* $W_j{}^i{}_{kl} = 0$ *when* $n = 2$.

(2) $W_j{}^m{}_{ml} = 0$, *so* $W_j{}^m{}_{km} = 0$, $W_m{}^m{}_{kl} = 0$.

(3) $W_j{}^m{}_{kl};_m = -(n-2)(K_{jk};_l - K_{jl};_k) + R_k{}^t{}_m D_j{}^m{}_{lt} - R_l{}^t{}_m D_j{}^m{}_{kt}$
$+\{(n-1)H_{jt}W_p{}^t{}_{kl} - H_{kt}W_j{}^t{}_{lp} + H_{lt}W_j{}^t{}_{kp}\}y^p$.

(3) is obtained by a long calculation.

Proposition 2. *Some properties are satisfied as follows:*

(a) *the projective connection* $(^p\bar{\Gamma}^h, \#\bar{\Gamma}^v)$ *determined here is normal, i.e.* $^pR_A{}^C{}_{CB} = 0$, *so* $^pR_A{}^C{}_{BC} = 0$, $^pR_C{}^C{}_{AB} = 0$.

(b) *the vector* $v = \partial/\partial x^0$ *is affine in this projective connection, or* $v^A{}_{|B|C}$ $= {}^pR_B{}^A{}_{DC}v^D - {}^pP_B{}^A{}_{CD}v^D{}_{|E}\gamma^E$, $v^A{}_{|B\|C} = {}^pP_B{}^A{}_{DC}v^D$.

Proof. (a) follows immediately by Lemma (2). (b) is implied by $^pR_B{}^A{}_{0C} = 0$, $^pP_B{}^A{}_{CD}\gamma^D = 0$ and $^pP_B{}^A{}_{0C} = 0$.

The results of calculating $^pR_B{}^A{}_{CD}$ and $^pP_B{}^A{}_{CD}$ and the Lemma lead us to

Proposition 3. (a) *The projective pair-connection* $(^p\bar{\Gamma}^h, \#\bar{\Gamma}^v)$ *is linear, i.e.* $^pP_B{}^A{}_{CD} = 0$, *if and only if the Douglas tensor* $D_j{}^i{}_{kl} = 0$.

(b) *The projective pair-connection is linear and affinely flat, i.e.* $^pR_B{}^A{}_{CD} = 0$ *and* $^pP_B{}^A{}_{CD} = 0$, *if and only if*

(1) *the Weyl tensor* $W_j{}^i{}_{kl} = 0$ *and the Douglas tensor* $D_j{}^i{}_{kl} = 0$ *when* $n \geq 3$.

(2) $K_{jk};_l - K_{jl};_k = 0$ *and the Douglas tensor* $D_j{}^i{}_{kl} = 0$ *when* $n = 2$.

Proof. In showing (b), let us notice that if $n \neq 2$ the equations $K_{jk};_l - K_{jl};_k = 0$ follow from $W_j{}^i{}_{kl} = 0$ and $D_j{}^i{}_{kl} = 0$ by Lemma (3), if $n = 2$ then the Weyl tensor always satisfies $W_j{}^i{}_{kl} = 0$.

Considering together Proposition 3 with the classical results in [7] we see

Theorem 3. *A Berwald connection G on M is projectively flat if and only if the associated projective pair-connection $({}^p\bar{\Gamma}^h, \#\bar{\Gamma}^v)$ is linear (affine) and affinely flat.*

Finally we obtain

Theorem 4. *The geodesics of a Finsler manifold (M, L) are the projections of the paths of LM with respect to the projective connection associated with the Berwald connection.*

That is to say, the path equations

$$\frac{d^2x^a}{dt^2} + {}^p\bar{F}_b{}^a{}_c \frac{dx^b}{dt}\frac{dx^c}{dt} = 0, \qquad a, b, c = 1, 2, \dots, n, 0,$$

with regard to the projective connection ${}^p\bar{\Gamma}^h$ and with the affine parameter t, are rewritten as the geodesic equations on a Finsler manifold (M, L) with the projective parameter s

$$\frac{d^2x^i}{ds^2} + 2G^i\left(x^j, \frac{dx^j}{ds}\right) = 0, \quad \{s, t\} = \frac{2}{n-1} R_m{}^m{}_k \left(x^j, \frac{dx^j}{dt}\right)\frac{dx^k}{dt},$$

where $\{s, t\} \equiv \left(\frac{d^3s}{dt^3}/\frac{ds}{dt}\right) - \frac{3}{2}\left(\frac{d^2s}{dt^2}/\frac{ds}{dt}\right)^2$ (Schwarz's derivative), and i, j, k, $m = 1, 2, \dots, n$.

Proof. The path equations

$$\begin{aligned}
&\frac{d^2x^i}{dt^2} + {}^p\bar{F}_j{}^i{}_k \frac{dx^j}{dt}\frac{dx^k}{dt} + 2\frac{dx^0}{dt}\frac{dx^i}{dt} = 0,\\
&\frac{d^2x^0}{dt^2} + {}^p\bar{F}_j{}^0{}_k \frac{dx^j}{dt}\frac{dx^k}{dt} + \left(\frac{dx^0}{dt}\right)^2 = 0
\end{aligned}$$

become

$$\begin{aligned}
&\frac{d^2x^i}{dt^2} + 2G^i + 2\left(\frac{dx^0}{dt} + H\right)\frac{dx^i}{dt} = 0,\\
&\frac{d}{dt}\left(\frac{dx^0}{dt} + H\right) + \left(\frac{dx^0}{dt} + H\right)^2 + K_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.
\end{aligned}$$

Here, putting $s = \int \exp\left[-2(x^0 + \int H dt)\right] dt$, we get the result of Theorem 4.

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