Publ. Math. Debrecen
56 / 3-4 (2000), 643–655

Point Finsler spaces with metrical linear connections

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Dedicated to Professor Kálmán Győry
my colleague and an old friend of mine on his 60th birthday

Abstract. We present a geometric construction of and constructive investigations in Finsler spaces $\tilde{F}^n = (M, \tilde{\mathcal{L}})$ whose indicatrices $\tilde{I}(x)$ are affine images of a single indicatrix: $\tilde{I}(x) = \mathfrak{a}(x) I_0$. These are Finsler spaces modeled on a Minkowski space [4] or Finsler spaces with 1-form metric [8]. Vectors are considered not at line elements: $\xi(x, y)$, but at points: $\xi(x)$ (i.e. we work in a point Finsler space [3]).

We characterize these $\widetilde{F}^n$ in a geometric way as those Finsler spaces which admit linear and at the same time metrical connections in the tangent bundle τ_M. Also the linear automorphisms (the affine rigidity) of the indicatrices are investigated.

Introduction

Linear (called also affine) and at the same time metrical connections in the tangent bundle τ_M (i.e. among the tangent vectors $\xi(x) \in T_x M$ of the manifold M) play a basic role in the theory of Riemannian spaces $V^n = (M, g)$. In a Finsler space $F^n = (M, \mathcal{L})$ with fundamental (metric) function $\mathcal{L}(x, y)$, $x \in U \subset M$, $y \in T_x M$ metrical and linear connections among the tangent vectors $\xi(x)$ do not exist in general. This can easily be seen from the fact that the indicatrices $I(x_0) = \{y \mid \mathcal{L}(x_0, y) = 1\}$ which play the role of the unit spheres cannot, in general, be taken into each other by linear mappings (e.g. if $I(x_0)$ is an ellipsoid and $I(x_1)$ is not).

Mathematics Subject Classification: 53C60.
Key words and phrases: Finsler spaces, metrical linear connection.
Supported by OTKA 32058 and FKFP 0312/1997.

This necessitated the introduction of the n-dimensional Finsler vectors $\xi(x,y) \in VTM = TM \times_M TM$ defined at line elements (x,y). Among the $\xi(x,y)$ there exist metrical and linear connections (in fact with a variety of choices, for a recent survey see [11]). Introduction of vectors defined at line elements makes the theory of Finsler spaces a little more complicated. – This problem does not emerge until we are concerned with the metric only (arc length, geodesic, etc.) and we do not use parallelism. – We can obtain in an F^n also metrical connections among the more simple tangent vectors $\xi(x)$, of course not linear, but only homogeneous ones [12], [14], [7]. Finsler spaces with vectors from the tangent bundle are called point Finsler spaces [3], [12].

In this work we present a geometric construction of and constructive investigations in the Finsler spaces with metrical and linear connections $\overset{am}{\Gamma}(x)$. These spaces must be special Finsler spaces, for, as we have seen, metrical and at the same time linear connections $\overset{am}{\Gamma}(x)$ do not exist in general in an F^n. We want to contribute to their constructive geometric theory. These spaces are closely related to and concerning the metric coincident with Finsler spaces of 1-form metric [8] or with Finsler spaces modeled on Minkowski spaces [4]–[6], however the space of our investigation is a point Finsler space, and not a line element space. So our connections relate to τ_M and not to $\mathcal{V}TM$.

In Section 1 we give a geometric construction of the spaces to be investigated. Then in Section 2 we show that these spaces are exactly those Finsler spaces which admit linear and at the same time metrical connections in τ_M. The proof of this characterization will be performed by constructive geometric means. Finally in Section 3 we investigate the consequences of the fact that the holonomy group of our connection consists of the identity alone or it has more elements (the affine rigidity of the indicatrix).

1. Geometric construction of $\tilde{F}^n$

Let $F^n = (M, \mathcal{L})$ be a locally Minkowskian Finsler space denoted by $\ell\mathcal{M}^n$ over the connected and paracompact manifold M. Then there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}$, $\alpha \in \mathcal{A}$ of M, where the φ_α are diffeomorphisms from E^n or from an open simply-connected part of it onto U_α. We concentrate on one U_α denoted by U. The situation on the other U_α is similar. On U there exists an adapted coordinate system (x), such that $\mathcal{L}(x,y)$, and thus also the indicatrix I_0 of $F^n = \ell\mathcal{M}^n$ are independent of x. We consider (x) as a Euclidean coordinate system. By $d\varphi$ we can indentify $T_xU = T_xM$ with $T_xE^n \approx E^n$. Then these indicatrices I_0 in the different T_xU are congruent and parallel displaced in the Euclidean sense.

We attach to each $x \in U$ a nondegenerate affine transformation $\mathfrak{a}(x) : T_xM \to T_xM$ (affine always means centro-affine). Then

$$\mathfrak{a}(x)I_0 =: \tilde{I}(x) \tag{1}$$

is a new indicatrix and we get on U a Finsler space $(U,\tilde{I}(x)) \equiv (U,\tilde{\mathcal{L}})$ determined by $\mathcal{L}$ or I_0 and $\mathfrak{a}$, and denoted by $\tilde{F}^n = (M,\mathcal{L},\mathfrak{a})$ or $(M,I_0,\mathfrak{a})$. If the fundamental function of the locally Minkowski space $F^n = \ell\mathcal{M}^n$ in the adapted coordinate system (x) is $\mathcal{L}(y)$, then $\tilde{\mathcal{L}}(x,\mathfrak{a}(x)y) = 1 = \mathcal{L}(y)$, and $\tilde{\mathcal{L}}(x,y) = \mathcal{L}(\mathfrak{b}(x)y)$, where $\mathfrak{b}$ is the inverse of $\mathfrak{a}$. Thus the $\tilde{F}^n$ on U are the Finsler spaces with 1-form metric introduced and investigated by Y. Ichijyo [4]–[6], M. Matsumoto and H. Shimada [8], [9] etc., see also P. Antonelli–R.S. Ingarden–M. Matsumoto [1].

$\mathfrak{a}(x)$ takes the unit sphere $S : \delta_{ik}y^iy^k = 1$ of T_xU into an ellipsoid

$$Q(x) := \mathfrak{a}(x)S \tag{2}$$

with the equation $g_{ik}(x)y^iy^k = 1$, $y \in T_xU$ which is equivalent to a symmetric nondegenerate 2-form $g(x)$ or a tensor of type $(0,2)$. Thus we obtain a Riemannian space $V^n = (U,g)$ which is called conjugate to $\tilde{F}^n$. However, to a $Q(x) \subset T_xM$ there is a number of $\mathfrak{a}(x)$ satisfying (2), namely if $\mathfrak{a}_0S = Q$, then also $\mathfrak{a}_0\mathfrak{f}S = Q$, where $\mathfrak{f}$ is a rotation. – In the case of $I_0 = S$ we obtain $\tilde{F}^n = V^n$, however such an $\tilde{F}^n = (M,S,\mathfrak{a})$ can be generated by many different fields $\mathfrak{a}$ of affine transformations.

An interesting special case occurs when all $\mathfrak{a}(x)$ are rotations. Then all $\tilde{I}(x)$ are congruent, but not parallel translated of each other. In this case $\tilde{\mathcal{L}}(x,y)$ does depend of x. – Another special case is when $\mathfrak{a}(x)$ are equiaffine, i.e. $\|S\|_E = \|Q(x)\|_E \Rightarrow \|\tilde{I}(x)\|_E = \text{const}$.

In order to construct globally – not only on a chart U – the affine deformated of an $F^n = (M,\mathcal{L}) = \ell\mathcal{M}^n$, let us consider a field $\mathfrak{A}$ of affine, transformations on M. Such fields exist. Since M is paracompact, it can be equipped with Riemannian metrics. Let g and h be two such metrics having local components $g_{ij}(x)$ and $h_{ij}(x)$ on a chart $U(x)$. Then $a_i^k(x) = g_{ij}(x)h^{jk}(x)$ are regular matrices, for both (g_{ij}) and (h^{jk}) are so, and thus they determine affine transformations $\mathfrak{a}(x) : T_xM \to T_xM$, $\forall\, x \in M$. So (1) is defined at each point of M, and this yields a Finsler space denoted by $\tilde{F}^n = (M, \tilde{\mathcal{L}})$ or by $(M, \tilde{I})$, $(M; \mathcal{L}, \mathfrak{A})$, $(M; I, \mathfrak{A})$ and called the affine deformated of $\ell\mathcal{M}^n$. Of course affine deformation can be performed on every Finsler space F^n, however $\tilde{F}^n$ denote in this paper affine deformated of locally Minkowski spaces $\ell\mathcal{M}^n$.

Let $V^n = (M, g)$ and $\hat{V}^n = (M, \hat{g})$ be two Riemannian manifolds on M. $g(p)$, $p \in M$ is equivalent to an ellipsoid $Q(p) \subset T_pM$, and $\hat{g}(p)$ to $\hat{Q}(p) \subset T_pM$. Then there exist always affine transformations $\mathfrak{a}(p)$ such that $\hat{Q}(p) = \mathfrak{a}(p)Q(p)$. So the arbitrary $\hat{V}^n = (M, \hat{g}) = (M, \hat{Q})$ is the affine deformated $\tilde{V}^n = (M; Q, \mathfrak{A})$ of the arbitrary $V^n = (M, Q)$. A locally Euclidean space $\ell E^n = (M, \overset{*}{g})$ is a special Riemannian space, where there exist to each point $p \in M$ a neighbourhood U and a coordinate system (x) on U such that $\overset{*}{g}_{ij}(x) = \delta_{ij}$ which is equivalent to a sphere $S(p) \subset T_pM$: $\ell E^n = (M, S)$. Thus every Riemannian space is an affine deformated by an appropriate $\mathfrak{A}$ of a locally Euclidean space, where (2) $\hat{Q}(p) = \mathfrak{a}(p)S(p)$:

$$\widetilde{\ell E^n} = (M; S, \mathfrak{A}) = (M, \hat{Q}) = (M, \hat{g}) = \hat{V}^n.$$

$\mathfrak{a}(p)$ is not unique, for $\mathfrak{a}\mathfrak{f}$ in place of $\mathfrak{a}$ also satisfies (2). However the affine deformated of the locally Minkowskian spaces $\ell\mathcal{M}^n$ yield the $\tilde{F}^n$ only, and not all F^n can be generated in this way:

$$\begin{array}{lll} \mathfrak{A} : \ell E^n \longrightarrow V^n & \mathfrak{A} : \ell\mathcal{M}^n \longrightarrow \tilde{F}^n & \\ V^n, \ell E^n \Longrightarrow \mathfrak{A} & \tilde{F}^n, \ell\mathcal{M}^n \Longrightarrow \mathfrak{A} & \{\tilde{F}^n\} \subset \{F^n\} \\ \exists\mathfrak{A} : V^n \longrightarrow \hat{V}^n & & \end{array}$$

We remark that as a Euclidean space E^n is a special Minkowski space $\mathcal{M}^n$, so an $\ell E^n = (M, \overset{*}{g}) = (M, S)$ is also a special locally Minkowskian space $\ell\mathcal{M}^n = (M, \overset{*}{\mathcal{L}}) = (M, \overset{*}{I})$, where $\overset{*}{I} = S$ in certain adapted coordinate systems.

Finally we remark that a field $\mathfrak{a}(p)$, $p \in U$ can smoothly be extended to a field $\mathfrak{A}$ on a paracompact M. Let namely $\{U_i\}$ $i \in A$ be a locally finite open cover of M with $U = U_1$, and $\underset{1}{g}$ and $\underset{1}{h}$ two Riemannian metrics on U satisfying $\underset{1}{g}_{jk}(x) = \underset{1}{h}_{kr}(x)a^r_j(x)$ in a local coordinate system (x) on U. We denote by U_β $\beta \in B \subset A$ those elements of $\{U_i\}$ (including U_1) which intersect U_1. We define $\underset{\beta}{g} \in C^\infty$ and $\underset{\beta}{h} \in C^\infty$ on U_β such that (a) $\underset{\beta}{g} \restriction (U_\beta \cap U_1) = \underset{1}{g}$ and (b) $\underset{\beta}{h} \restriction (U_\beta \cap U_1) = \underset{1}{h}$. On the other U_i the $\underset{i}{g}$ and $\underset{i}{h}$ may be arbitrary. Let now $f_i \in C^\infty(M)$ be a partition of unity on M subordinate to $\{U_i\}$. Then $\sum_i f_i \underset{i}{g} = g$ and $\sum_i f_i \underset{i}{h} = h$ are two Riemannian metrics on M, and ${a_j}^k(x) = g_{jr}(x)h^{rk}(x)$ gives a smooth field $\mathfrak{A}$ on M. Then for any $x_0 \in U$ $(\mathfrak{A}(x_0))^k_j = \Big(\sum_i f_i(x_0)g_{jr}(x_0)\Big)\Big(\sum_i f_i(x_0)\underset{i}{h}^{rk}(x_0)\Big)$. Here $f_i(x_0) = 0$ if $i \neq \beta \in B$. Then from (a) and (b)

$$\big(\mathfrak{A}(x_0)\big)^k_j = \sum_{\beta \in B} f_\beta(x_0)\Big(\underset{1}{g}_{jr}(x_0)\underset{1}{h}^{rk}(x_0)\Big) = \underset{1}{g}_{jr}(x_0)\underset{1}{h}^{rk}(x_0) = {a_j}^k(x_0),$$

i.e. $\mathfrak{A} \restriction U = \mathfrak{a}$.

2. Characterization of $\tilde{F}^n$ by geometric means

We want to characterize the just constructed $\tilde{F}^n = (M; \overset{*}{I}, \mathfrak{A}) = (M, \tilde{I})$ spaces among the Finsler spaces $F^n = (M, \mathcal{L})$. $\tilde{F}^n$ is an affine deformated by $\mathfrak{A}$ of an $\ell\mathcal{M}^n = (M, \overset{*}{I})$ space. So M must admit a locally Minkowskian structure. This means that every $p \in M$ has a neighbourhood U and coordinate systems (x) on U such that $\mathcal{L}(x, y)$ is independent of x. These coordinate systems are called adapted. Adapted coordinate systems are related to each other by linear transformations, and every coordinate system obtained from an adapted one by a linear transformation is also adapted ([10] p. 158). Hence if $U_1(x)$ and $U_2(z)$ are adapted, then on $U_1 \cap U_2 \neq \emptyset$ $(x) \leftrightarrow (z)$ is linear. It is clear that in case of a locally Minkowskian space $\ell\mathcal{M}^n = (M, \overset{*}{I})$ M has a cover $\{U_\alpha(x_\alpha)\}$ with adapted coordinate systems (x_α) on U_α. Then (a) $(x_{\alpha_1}) \to (x_{\alpha_2})$ on $U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset$ must be linear. A differentiable structure $\{(U_\alpha, \varphi_\alpha)\}$, $\alpha \in A$, $\varphi_\alpha : U_\alpha \to E^n(x_\alpha)$ on M will be called affine if the transformations (a) are linear. If $(M, \overset{*}{I})$ is a locally Minkowskian space, then M clearly admits an affine differentiable structure. This is clearly necessary for an F^n to be an $\tilde{F}^n$.

Theorem 1. *The $\tilde{F}^n$ spaces are those Finsler spaces $F^n = (M, \mathcal{L})$ which admit a metrical linear connection $\overset{am}{\Gamma}(x)$ and an affine differentiable structure on M.*

Proof. A) Every $\tilde{F}^n$ has a metrical linear connection.

Since $\tilde{F}^n = (M, \tilde{I})$ is the affine deformated of an $\ell\mathcal{M}^n = (M, \overset{*}{I})$, we have affine transformations $\mathfrak{a}(p)$, $p \in M$ and charts (U, φ) $p_0 \in U$ for any $p_0 \in M$ with adapted coordinates (x) on U. Then $\overset{*}{I}(p)$ is a parallel translate of $\overset{*}{I}(p_0)$ in any adapted coordinate system on U, and also (1) holds in the form: $\mathfrak{a}\overset{*}{I} = \tilde{I}$. Let $\mathfrak{t}$ be the Euclidean parallel translation in an adapted coordinate system from $p_0 \in U$ to $p \in U$. Then

$$\mathfrak{g}(p_0, p) := \mathfrak{a}(p) \circ \mathfrak{t} \circ \mathfrak{a}^{-1}(p_0) \tag{3}$$

is an affine transformation, and $\mathfrak{g}(p_0, p)\underset{0}{\vec{\xi}}(p_0) = \vec{\xi}(p)$, $\underset{0}{\vec{\xi}} \in T_{p_0}U = T_{p_0}M$ is a vector field on U. Let $\underset{0}{\vec{\varepsilon}}_i \in \tilde{I}_0$ be a basis of $T_{p_0}M$, $\vec{\varepsilon}_i(p) = \mathfrak{g}(p_0, p)\underset{0}{\vec{\varepsilon}}_i(p_0)$ and $\underset{0}{\vec{\xi}} = \underset{0}{\xi^i}\underset{0}{\vec{\varepsilon}}_i$. Then $\vec{\xi}(p) = \underset{0}{\xi^i}\vec{\varepsilon}_i(p)$, for $\mathfrak{g}$ is affine, and the differential $d\vec{\xi}$ has the form

$$\left(d\vec{\xi}\right)(p_0, dp) = \underset{0}{\xi^i} d\vec{\varepsilon}_i(p_0, dp), \quad \forall\, p_0, \underset{0}{\vec{\xi}}. \tag{4}$$

Since $\mathfrak{g}$ is independent of the adapted coordinate system used, so are $\vec{\xi}$ and $d\vec{\xi}$. So (3) determines by $\underset{0}{\vec{\xi}} \to \underset{0}{\vec{\xi}} + d\vec{\xi} = \mathfrak{g}(p_0, dp)\underset{0}{\vec{\xi}}$ a mapping $T_{p_0}M \to T_{p_0+dp}M$, $\forall\, p_0, dp$ linear in $\underset{0}{\vec{\xi}}$, and thus a linear connection $\overset{a}{\Gamma}$. The coefficients of this connection in an adapted coordinate system (x) can be obtained from (4) by writing its components in (x):

$$d\xi^j(x_0, dx) = \underset{0}{\xi^i} d\varepsilon_i^j(x_0, dx) = \underset{0}{\xi^i}\overset{a}{\Gamma}{}_i{}^j{}_k(x_0)dx^k, \qquad \forall\, x_0, \underset{0}{\xi^i},$$

where $\overset{a}{\Gamma}{}_i{}^j{}_k(x_0) = \dfrac{\partial \varepsilon_i^j}{\partial x^k}(x_0)$.

Moreover

$$\mathfrak{g}(p_0, p)\tilde{I}(p_0) = \mathfrak{a}(p) \circ \mathfrak{t} \circ \mathfrak{a}^{-1}(p_0)\tilde{I}(p_0) = \mathfrak{a}(p) \circ \mathfrak{t}\overset{*}{I}(p_0) = \mathfrak{a}(p)\overset{*}{I}(p) = \tilde{I}(p),$$

i.e. $\mathfrak{g} : T_{p_0}M \to T_pM$ is metrical, and thus so is the constructed $\overset{a}{\Gamma}$. □

B) Every $F^n = (M, \mathcal{L}) = (M, I)$ admitting an affine differentiable structure and a metrical affine connection $\overset{am}{\Gamma}(x)$ on M is a Finsler manifold of type $\tilde{F}^n$.

$F^n = \tilde{F}^n$ means that it is an affine deformated by $\mathfrak{A}$ of a locally Minkowskian space $\ell\mathcal{M}^n = (M; \overset{*}{I}, \mathfrak{A})$. So we have to give $\overset{*}{I}$ and $\mathfrak{A}$ on M.

First we consider the local case (when M is covered by a single chart (U, φ) with local coordinates x).

Let (x) be considered as a Descartes coordinate system (i.e. the $\frac{\partial}{\partial x^i}(x)$ are considered as an orthonormal system in T_xM). This makes M into an E^n. $\mathfrak{t}$ denotes a translation in this E^n sending x_0 to x. We define $\mathfrak{t}I_0 = \overset{*}{I}(x) \equiv \overset{*}{I}$, where $I_0 = I(x_0)$ is the indicatrix of F^n at x_0, and thus $\overset{*}{I}(x_0) = I_0$. This process makes M into a Minkowski space $\mathcal{M}^n$. Let γ be a curve in E^n from x_0 to x, and denote by $P(\gamma)$ the parallel transport along γ according to the metrical connection $\overset{am}{\Gamma}(x)$. Then $P(\gamma)I(x_0) = I(x)$ and $P(\gamma) \circ \mathfrak{t}^{-1} : T_xM \to T_xM$ is an affine transformation:

$$\mathfrak{a}(x) := P(\gamma) \circ \mathfrak{t}^{-1} \quad \forall\, x \in U = M, \tag{5}$$

and

$$\mathfrak{a}(x)\overset{*}{I}(x) = P(\gamma) \circ \mathfrak{t}^{-1}\overset{*}{I}(x) = P(\gamma)I(x_0) = I(x). \tag{6}$$

This means that $\tilde{F}^n = (M, \overset{*}{I}, \mathfrak{A}) = (M, I) = F^n$.

We start with the global case. Let $\{(U_\alpha, \varphi_\alpha(x_\alpha))\}$, $\alpha \in A$ be an affine differentiable structure on M such that the U_α are connected. Let p_0 be a fixed and p an arbitrary point of M. If we have translations $\mathfrak{t} : T_{p_0}M \to T_pM$, then we can construct affine transformations $\mathfrak{a}(p)$ as we did it in (5) in the local case, and then we can finish our proof similarly as in (6). Translations $\mathfrak{t}$ (i.e. linear transformations $\mathfrak{t} : T_{x_0}M \to T_xM$ defined by $\mathfrak{t}\frac{\partial}{\partial x^i}(x_0) = \frac{\partial}{\partial x^i}(x)$) depend on the coordinate systems, except they are related by linear transformation. So we have to use such coordinate systems. Fortunately, coordinate systems belonging to an affine defferentiable structure (used by us) have this property. Moreover p_0 and p may be in different U_α. However this difficulty can be surmounted relatively easily.

Let $C(t)$, $t \in [t_0, \hat{t}]$, $t =$ arc length, be a curve of F^n joining p_0 and p. Then C is bounded and closed in F^n, so it is compact. Hence it is covered by finitely many U_i, $i = 1, 2, \ldots, N$. We renumber a subclass of them.

Let U_1 be such a U_i which contains $p_0 = C(t_0)$ and cuts out the longest segment $[t_0, t_1]$ from C such thtat the endpoint $C(t_1) = \widetilde{p}(t_1)$ still belongs to ∂U_1. Suppose that $\tilde{p}_1 \neq C(\hat{t}\,)$. Then among the U_i containing $\tilde{p}_1$ there is a U_2 which cuts out again the longest segment from $C_1(t)$, $t \in [t_1, \hat{t}\,]$. We denote the endpoint of this segment by $\tilde{p}_2$. It is $C(t_2) \in C_1 \cap \partial U_2$. Continuing this, the process arrives to $p = C(\hat{t}\,)$, and breaks down in $\bar{N} \leq N$ steps. The U_i used in the above process are U_k, $k = 1, 2, \dots, \bar{N}$. Then $U_j \cap U_{j+1} \cap C \neq \emptyset$, $1 \leq j < \bar{N}$ is a segment of C. Let $p_j \in U_j \cap U_{j+1}$ be a point of this segment. We denote by $\mathfrak{t}_j$ the translation from p_{j-1} to p_j in the coordinate system (x_j) over U_j. $\mathfrak{t}_j$ is independent of the coordinate system (x_j) used, since we allow coordinate systems of the affine differentiable structure only, which are related to each other by linear transformations. Then we define $\mathfrak{t} := \mathfrak{t}_{\bar{N}} \circ \cdots \circ \mathfrak{t}_2 \circ \mathfrak{t}_1$. With the aid of $\mathfrak{t}$ we can construct an $\ell\mathcal{M}^n = (M, \overset{*}{I})$ on M. Let $\overset{*}{I}_0 = I(p_0)$ be the indicatrix of F^n at p_0, and $\overset{*}{I}(p) = \mathfrak{t}^{-1}\overset{*}{I}_0$. Then we have an $\overset{*}{I}(p)$ on M. They are parallel translate on U_α, and the coordinate systems (x_α) relate to each other by linear transformations, so they are adapted. Thus we obtain an $\ell\mathcal{M}^n$. Now let γ be a curve (coincident with or different from C) joining p_0 with p, and let $P(\gamma)$ denote the parallel transport along γ according to the metrical connection $\overset{am}{\Gamma}(x)$ of F^n. Then $P(\gamma) \circ \mathfrak{t}^{-1} : T_pM \to T_pM$ is an affine transformation $\mathfrak{a}(p)$. This can be done for every $p \in M$, and we obtain a field $\mathfrak{A}$. These $\mathfrak{a}(p)$ take indicatrices $\overset{*}{I}(p)$ of the locally Minkowskian $\ell\mathcal{M}^n$ into the indicatrices $I(p)$ of F^n:

$$\mathfrak{a}(p)\overset{*}{I}(p) = P(\gamma) \circ \mathfrak{t}^{-1}\overset{*}{I}(p) = P(\gamma)\overset{*}{I}(p_0) = P(\gamma)I(p_0) = I(p).$$

Thus $F^n = (M, I) = (M; \overset{*}{I}, \mathfrak{A}) = \tilde{F}^n$. $\square$

We remark that the constructed $\mathfrak{A}$ depends on the curves C connecting p_0 and p. This is related to the affine automorphisms of the indicatrices of $\tilde{F}^n$ discussed in Section 3.

Y. Ichijyo ([4]–[6]) obtained by a completely different approach, results quite similar to Theorem 1.

We have constructed $\mathfrak{A}$, but we have not proved the smoothness of $\mathfrak{A}$. This is assured if the family of curves $\{C(p_0, p)\}$ joining the fixed p_0 with the arbitrary p is such that $P(C)\underset{0}{\vec{\xi}} = \vec{\xi}(p)$ depends smoothly on p.

In an adapted coordinate system (x) the coefficients of the metrical linear connection $\overset{a}{\Gamma}(x)$ of part A) can also be expressed by the components $a^j_m(x)$ of the affine transformations $\mathfrak{a}(x)$ as follows: Let us denote the matrix of $\mathfrak{a}(x)$ by $a^i_h(x)$ and that of the inverse $\mathfrak{b}(x)$ by $b^k_j(x)$. By (3) the parallel displaced of $y_0 \in \tilde{I}(x_0)$ from x_0 to x along a curve $\gamma : x^i = x^i(t)$ is

$$P^{\overset{a}{\Gamma}}_{x_0,x;\gamma} y_0 = y(x(t)) = \mathfrak{a}(x(t)) \circ \mathfrak{b}(x_0) y_0,$$

or in components

$$y^j(t) = a^j_m(x(t)) {b_0}^m_s {y_0}^s. \tag{7}$$

Then $\overset{a}{\Gamma}{}_k{}^j{}_i(x)$ must satisfy

$$\frac{dy^j}{dt} = \overset{a}{\Gamma}{}_k{}^j{}_i y^k \frac{dx^i}{dt}, \tag{8}$$

i.e.

$$\frac{\partial a^j_m}{\partial x^i} \frac{dx^i}{dt} {b_0}^m_s {y_0}^s = \overset{a}{\Gamma}{}_k{}^j{}_i a^k_m {b_0}^m_s {y_0}^s \frac{dx^i}{dt}.$$

From this we have ([8] p. 162)

$$\overset{a}{\Gamma}{}_k{}^j{}_i(x) = b^m_k(x) \frac{\partial a^j_m}{\partial x^i}(x).$$

Then (8) is satisfied by (7).

3. Automorphisms (rigidity) of the indicatrix

1. Suppose that $\overset{am}{\Gamma} \equiv \Gamma$ is a metrical linear connection on M for the Finsler manifold $F^n = (M, \mathcal{L})$. We know that the parallel transport $P(\gamma) : T_{p_0}M \to T_{p_1}M$, according to the given connection Γ along a curve γ connecting p_0 with p_1, takes $I(p_0)$ into $I(p_1)$:

$$P(\gamma) I(p_0) = I(p_1), \tag{9}$$

for Γ is supposed to be metrical. This also means that $P(\gamma)I(p_0)$ is independent of the curve connecting p_0 and p_1. Thus $P(\gamma)y = \overline{y} \in I(p_1)$ for any $y \in I(p_0)$ and for any γ joining p_0 with p_1. However this does not assure that

$$P(\gamma_1)y = P(\gamma_2)y, \quad \forall y \in I(p_0) \tag{10}$$

for two different curves γ_1 and γ_2 joining p_0 and p_1, i.e. (9) does not imply the independence of $\overline{y}$ of γ.

It is easy to see that if $P(\gamma)y = \overline{y}$ $\forall y \in I(p_0)$ is independent of γ, then so is $P(\gamma)z$ for any $z \in T_{p_0}M$. Indeed, for a given $z \in T_{p_0}M$ let r be the ray in $T_{p_0}M$ through the origin p_0 and z, and let $y = r \cap I(p_0)$. Then $z = \lambda y$, $\lambda \in R$, and $P(\gamma)z = \overline{z} = \lambda\overline{y}$ on the ray $P(\gamma)r = \overline{r}$, independently of γ.

In this case $P(\gamma)$ is independent of γ, hence $\mathfrak{a}(p) = P(\gamma) \circ \mathfrak{t}^{-1}$, and thus $\overset{a}{\Gamma}(p)$, and also $\mathfrak{A}$ are unique, and $\mathfrak{A}$ determines a single $V^n = (M, g)$ conjugate to $\tilde{F}^n$.

If (10) holds for any p_0, p_1 and γ_1, γ_2, then the holonomy group of Γ consists of the indentity alone. In this case we also can say that $I(p)$ is rigid with respect to Γ. In this case Γ has a vanishing curvature, and conversely. If, moreover, Γ is torsion-free, then its coefficients vanish in an appropriate coordinate system, and the indicatrices $I(p)$ are congruent and parallel displaced in this coordinate system. Hence F^n is a Minkowski space.

If in a Finsler manifold of type $\tilde{F}^n$

$$\overline{y}_1 = P(\gamma_1)y \neq P(\gamma_2)y = \overline{y}_2, \quad y \in \tilde{I}(x_0)$$

occurs, then $\tilde{I}(p_1)$ admits a proper (nontrivial) affine (linear) automorphism $\mathfrak{c}$

$$\mathfrak{c}\tilde{I}(p_1) = \tilde{I}(p_1), \quad \mathfrak{c} \neq \text{id}.$$

Namely $P(\gamma_1) \circ (P(\gamma_2))^{-1} : T_{p_1}M \to T_{p_1}M$ is an affine transformation, and it takes $\tilde{I}(p_1)$ into $\tilde{I}(p_1)$. Thus $\mathfrak{c}(p_1) = P(\gamma_1) \circ P(\gamma_2^{-1})$ is an affine automorphism of $\tilde{I}(p_1)$ (γ_2^{-1} means γ_2 running in the opposite direction). Moreover, it takes $\overline{y}_2$ into $\overline{y}_1$, and thus it is not the identity. This $\mathfrak{c}(p_1)$ can be transferred to any other point $p \in M$. Namely let γ_3 be a curve joining p_1 with p. Then $P(\gamma_3) \circ P(\gamma_1) \circ P(\gamma_2^{-1}) \circ P(\gamma_3^{-1}) = P(\gamma_3) \circ \mathfrak{c}(p_1) \circ P(\gamma_3)^{-1} = \mathfrak{c}(p)$ is a proper affine automorphism of $\tilde{I}(p)$, similar to $\mathfrak{c}(p_1)$ and generated by a paralllel translation along a closed curve. Since in $\tilde{F}^n$ every $\tilde{I}(p)$ is affine to $\tilde{I}(p_1)$, every $\tilde{I}(p)$ admits the same proper affine automorphisms. – We denote by $\mathfrak{C}(p)$ the set $\{\mathfrak{c}(p)\}$ of the affine automorphisms of the indicatrices of an $\tilde{F}^n = (M, \tilde{I})$ generated by parallel translations along closed curves.

So if an indicatrix $\tilde{I}(p)$ of the $\tilde{F}^n = (M, \tilde{I})$ admits no proper affine automorphism generated by $\overset{am}{\Gamma}$ (i.e. $\mathfrak{C} = \text{id.}$), then the curvature of its linear metrical connection $\overset{am}{\Gamma}$ vanishes, and conversely. If moreover $\overset{am}{\Gamma}$ is torsion-free, then $\tilde{F}^n$ is a Minkowski space.

2. Let us suppose that there exists a $\mathfrak{c}_0(p_0) \in \mathfrak{C}$, $\mathfrak{c}_0 \neq \text{id.}$ generated by parallel translation along a closed curve γ_0 which, through a family of curves γ_t in M, can continuously be shrunk to p_0. (This is certainly the case if M is simply connected.) Then the γ_t yield a continuous set $\mathfrak{c}_t(p_0)$ of proper automorphisms of $\tilde{I}(p_0)$ consisting of orientation preserving affine transformations. The same is true for all $\tilde{I}(p)$. Thus we obtain

Theorem 2. *If in a simply connected $\tilde{F}^n$ there exists an automorphism $\mathfrak{c}(p_0) \neq$ id. of the indicatrix $\tilde{I}(p_0)$ generated by parallel translation, then there exists a continuous set $\mathfrak{c}_t(p)$ of proper automorphisms of every $\tilde{I}(p)$.*

3. We want to show that the case of $n = 2$ is exceptional.

Theorem 3. *If in a simply connected $\tilde{F}^2$ an indicatrix admits a proper affine automorphism $\mathfrak{C} \neq$ id. (i.e. it is not affinely rigid), then $\tilde{F}^2$ is a Riemannian space.*

Proof. $\tilde{F}^n$ makes $T_{p_0}M$ into a Minkowski space with indicatrix $\tilde{I}(p_0) : \tilde{F}^n(T_{p_0}M) = \mathcal{M}^n(T_{p_0}M, \tilde{I}(p_0))$. The Minkowski norm (length) of a vector $\overrightarrow{AB}$, $A, B \in T_{p_0}M$ is the Minkowski (or Finsler) length $\|O\hat{B}\|_M$, where $\overrightarrow{O\hat{B}}$ is the parallel translate of $\overrightarrow{AB}$ to the origin O of $T_{p_0}M$. $\|O\hat{B}\|_M$ equals the value of the ratio $(O, \hat{B}, P) = \frac{\|O\hat{B}\|_E}{\|OP\|_E}$, where P is the intersection point of the ray r through O and $\hat{B}$ with the indicatrix $\tilde{I}(p_0) : P = r \cap \tilde{I}(p_0)$, and $\| \, \|_E$ is the Euclidean length with respect to a Euclidean metric in $T_{p_0}M$. Then any centroaffine transformation $\mathfrak{a}$ of $T_{p_0}M$ which leaves $\tilde{I}(p_0)$ (as a whole) invariant (e.g. $\mathfrak{c} \in \mathfrak{C}$) is an isometry of $\mathcal{M}_0^n = \mathcal{M}^n(T_{p_0}M, \tilde{I}(p_0))$. Namely, $\mathfrak{a}$ takes P into $\mathfrak{a}(P) =: P' \in \tilde{I}(p_0)$, and leaves the parallelism, and also the ratios invariant: $(O, P, \hat{B}) = (O, P', \mathfrak{a}(\hat{B}))$. Thus, denoting the images by $\mathfrak{a}$ by a dash, we have

$$\|\overrightarrow{AB}\|_M = (O, P, \hat{B}) = (O, P', \hat{B}') = \|O\hat{B}'\|_M = \|A'B'\|_M.$$

This means that any $\mathfrak{c}_t(p_0) \in \mathfrak{C}$ is an isometry of $\mathcal{M}^n(T_{p_0}M, \tilde{I}(p_0))$.

We recall that the linear isometry group of a not Euclidean two-dimensional Minkowski space is finite ([15] p. 83 or [2]). However, if in a simply connected $\tilde{F}^n$ $\mathfrak{C}$ consists not of the identity alone (i.e. the indicatrices are not affinely rigid with respect to $\overset{am}{\Gamma}$), then, according to our Theorem 2, in $T_{p_0}M$ there exist infinitely many different centroaffine transformations $\mathfrak{c}_t(p_0)$. Each of them generates a linear isometry of $\mathcal{M}^n$. Then according to the recalled result, in the case of $n = 2$ the geometry in $T_{p_0}M$ must be Euclidean, since the cardinality of $\mathfrak{c}_t(p_0)$ is not finite. This is true also for all T_pM. Thus this $\tilde{F}^2$ ($\mathfrak{C} \neq$ id.) is a Riemannian space. $\square$

4. Let us consider an $\tilde{F}^n = (M, I_0, \mathfrak{A}) = (M, \tilde{I})$, where $\mathfrak{A}$ consists of rotations. We denote such a manifold by $\hat{\mathcal{M}}^n$.

Theorem 4. *$\tilde{F}^n$ is manifold of type $\hat{\mathcal{M}}^n$ iff the Riemannian manifold $V^n = (M, g)$ conjugate to the $\tilde{F}^n$ is a Euclidean space E^n: $\tilde{F}^n = \hat{\mathcal{M}}^n \Leftrightarrow V^n = E^n$.*

A) If $\tilde{F}^n = \hat{\mathcal{M}}^n$, then $\mathfrak{f}(p_0)I_0 = \tilde{I}(p)$, where $\mathfrak{f}$ is a rotation. Hence the indicatrix $Q(p)$ of the V^n conjugate to $\tilde{F}^n$ is $\mathfrak{f}(p)S = S$, and thus $V^n = E^n$.

B) If $V^n = E^n$, then in an appropriate coordinate system (x)
$Q(x) \equiv S$, and from $\mathfrak{a}(x)S = Q(x)$ we get $\mathfrak{a}(x)S = S$. Thus $\mathfrak{a}(x) = \mathfrak{f}(x)$, and $\tilde{F}^n = \hat{\mathcal{M}}^n$. $\square$

Similarly, one can also see that $\tilde{F}^n = \mathcal{M}^n \Leftrightarrow \exists$ a coordinate system $(x) : \mathfrak{a}(x) = \text{id}$.

a) If $\tilde{F} = \mathcal{M}^n$ then in an adapted coordinate system (x) $\tilde{I}(x) \equiv I_0$. Thus (1) is satisfied by $\mathfrak{a}(x) \equiv \text{id}$.

b) If $\mathfrak{a}(x) \equiv$ id. then $\tilde{I}(x) \equiv I_0$, and $\tilde{F}^n = \mathcal{M}^n(M, I_0)$.

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(Received November 26, 1999; revised March 13, 2000)