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Periodic properties of functions and coloured sets

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Dedicated to Professor K. Győry on his 60th birthday

Abstract. Let $f : R^n \to R^1$ be a non-negative bounded function vanishing outside of a bounded set. If f has got a finite range $\{0, c_1, \dots, c_m\}$, such that $c_i > 0$ and the sets $A_i := \{x \in R^n : f(x) = c_i\}$ are non-empty and bounded, $i = 1, 2, \dots, m$, then we call f a *coloured set* and c_i its colours. Let $L \subset R^n$ be a point lattice of full dimension. In the paper the following question is studied: under what conditions can the coloured set f be extended periodically $\pmod L$ to the whole space R^n? For general f some sufficient and [or] necessary conditions for its periodic extendability are proved. Five conditions that are equivalent to the periodic extendability of a coloured set are found. The results of the present paper both extend some of those proved in [20] for one colour sets and refine some of those proved in [18], [19] for general real valued bounded functions. The results of [18], [19], [20] seem to be interesting for at least three reasons. Firstly, they yield substantial improvements of some basic results in the geometry of numbers, e.g., the theorem of Minkowski–Blichfeldt–V. d. Corput, or the theorem of Siegel–Bombieri. Secondly, in [20] a new connection of the results with partitions of a bounded set in R^n has been discovered, which connection might contribute to new developments in both fields (the theory of partitions and the geometry of numbers, respectively). Thirdly, for the proofs, new results in the field of multi-dimensional Fourier analysis also had to be proved.

1. Introduction

In what follows V means the volume (Lebesgue measure, shortly measure) in R^n, $\int \cdot\, dx$ stands for the integral and "a.e." stands for almost everywhere, respectively, with respect to the V.

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$A + B := \{a + b : a \in A,\ b \in B\}$ is the algebraic (Minkowski) sum of $A, B \subseteq R^n$, in particular $A - B := A + (-B)$. $\theta \in R^n$ is the zero vector. If for all $x \in A + B$ there are unique $a \in A$ and $b \in B$ such that $x = a + b$, then we shall write $A + B$ as $A \oplus B$ (the direct algebraic sum of A and B).

$|\cdot|$ is either the cardinality of a set or the absolute value of a real or complex number (the meaning will be clear from the context).

Put $\langle y, z\rangle := \sum_{i=1}^n y_i z_i$ for the usual scalar product in R^n.

Given n linearly independent vectors $b_1, \ldots, b_n \in R^n$, the set

$$L := \left\{ \sum_{i=1}^n u_i b_i : u_i \text{ integers},\ i = 1, \ldots, n \right\}$$

is the point-lattice generated by the basis (b_i). The set $P := \{\sum_{i=1}^n \lambda_i b_i : 0 \le \lambda_i < 1,\ i = 1, \ldots, n\}$ is the basic cell of L defined by the basis (b_i) (as one can easily see, P is in a one-to-one correspondence with the quotient space R^n/L). The number $d(L) := V(P)$ is the determinant of L. P and L also give a direct decomposition of R^n, $R^n = P \oplus L$, i.e., any $x \in R^n$ can be written uniquely as

$$x = \varphi(x) + [x], \qquad \varphi(x) \in P,\ [x] \in L.$$

This defines two "canonical" projections $\varphi : R^n \to P$, $[\ ] : R^n \to L$. The definition of φ shows that for any $y, z \in R^n$ we have

$$\varphi(y) = \varphi(z) \ \Leftrightarrow\ y - z \in L. \tag{1.0}$$

For any set $A \subseteq R^n$ write

$$\varphi(A) := \bigcup_{x \in A} \{\varphi(x)\}, \qquad [A] := \bigcup_{x \in A} \{[x]\}.$$

The set

$$L^* := \left\{ \sum_{i=1}^n u_i b_i^* : u_i \text{ integers},\ i = 1, \ldots, n \right\}$$

is the point lattice polar to L, where (b_i^*) is the system of vectors orthonormal to the system (b_i). As one can easily see, $L^* = \{x \in R^n : \langle x, u\rangle \in Z\ \forall u \in L\}$. On point-lattices consult the classical books [2], [3], [5], [9].

One of the basic theorems on point lattices is the theorem of Minkowski–Blichfeldt–V. d. Corput (see, e.g., [5]): Let $A \subset R^n$ be a bounded measurable set of positive measure $V(A)$. Then the condition

$$(A - A) \cap L = \{\theta\} \tag{1.1}$$

implies

$$V(P) - V(A) \geq 0. \tag{1.2}$$

A refinement of this theorem is the so called Siegel–Bombieri formula, Siegel [8], (for convex symmetric A) and Bombieri, [1], (for general A): (1.1) implies that

$$V(P) - V(A) \ = \ \frac{1}{V(A)} \sum_{\theta \neq v \in L^*} \left| \int_A e^{-2\pi i \langle v,y \rangle} dy \right|^2, \tag{1.3}$$

where i in the latter integral stands for the imaginary unit.

An improvement of another type of (1.2) is a consequence of the following inequality, proved in [11] (see also [12]): For any bounded measurable A with positive measure we have

$$|(A - A) \cap L| \ \geq \ 2\frac{V(A)}{V(\varphi(A))} \ - \ 1 \ \geq 1. \tag{1.4}$$

Now, (1.1) and (1.4) imply

$$V(P) - V(A) \ = \ V(P) - V(\varphi(A)). \tag{1.5}$$

As the quatities $V(A)$, $\int_A \cdot \, dy$ and $V(\varphi(A))$ are invariant upon replacing the set A by a set of measure zero, the results (1.2)$\div$ (1.5) still hold, if we take any set $A' \subset A$ s.t. $V(A') = V(A)$ and substitute $(A' - A') \cap L$ instead of $(A - A) \cap L$. The open kernel A^o of A, i.e., an open subset of A such that $V(A^o) = V(A)$, is (when it exists) an example of A'. [In fact, Siegel [8] proved (1.3) for $A := \frac{1}{2}K$ under this weaker assumption $K^o \cap L = \{\theta\}$, where K is a convex body such that $K = -K$.] This invariance is not true for the condition (1.1), i.e., the conditions (1.2), (1.3) and (1.5) are far from being sufficient for (1.1). This (as well as Siegel's original result) supports the feeling that the above results might be further improved.

And indeed, substantial improvements of the above results have been proved recently in [20]. As it is shown there, (1.3) is a simple consequence of (1.5), hence the property (1.5), i.e., $V(A) = V(\varphi(A))$, is the clue for (1.3), on one hand. On the other hand, a deeper analysis of the set $(A - A) \cap L$ shows that the condition (1.5) is equivalent to the following one:

$$\hat{\mathcal{L}}(A) := \{u \in L : V(A \cap (A + u)) > 0\} = \{\theta\}, \tag{1.6}$$

moreover, that the condition (1.5) is in fact also equivalent to (1.3), so the three conditions (1.3), (1.5) and (1.6) are equivalent (see, [20], Theorem 2.10). The above set $\hat{\mathcal{L}}(A)$ is a natural subset of the set $(A - A) \cap L$, because, as it can easily be seen, for any set $A \subset R^n$ we have

$$\mathcal{L}(A) := (A - A) \cap L = \{u \in L : A \cap (A + u) \neq \emptyset\}. \tag{1.7}$$

It is clear that the set $\hat{\mathcal{L}}(A)$ is invariant upon changing A by a set of measure zero.

An even more interesting feature of [20] is that for all the results (e.g., [20], the Theorem 2.10 mentioned above) a new notion introduced in [20] is responsible: the so called *inner aperodicity (w.r.t. L)* of A. Namely, [20], Theorem 2.10 says that a fourth condition, the almost everywhere inner aperiodicity of A, is equivalent to (1.3), (1.5) and (1.6). The second crucial discovery of [20] is that this inner aperiodicity is only a special case of a more general concept of *partitions of bounded subsets of* R^n (see [20], Theorem 3.2).

It is clear that all the above results proved for a set $A \subset R^n$ (e.g., all results of [20]) can be formulated in terms of the characteristic function χ_A of A. Now, a natural question arises: what can we say, when instead of χ_A we take an arbitrary bounded real-valued function f defined on R^n? The papers [18], [19] show that many results are true for this more general function. For this, the concept of inner aperiodicity had to be refined taking into account the values of the function, and also a new notion had to be introduced: the so called *periodic extendability (w.r.t. L)* of f, (this property is automatically true for any characteristic function).

We have to note that the idea of extending a result for sets to functions is not new in the geometry of numbers. For example, the identity (1.3) is in fact a consequence of a more general identity (1.12) due to

Bombieri (see below). Neither are the investigations on periodic properties of a function w.r.t. a point-lattice new: multi-dimensional Fourier analysis is dealing with such questions. However, we feel that our approach *principially differs* from the usual ones. Here is a short explanation, why we think so.

First, two known notions. For any real valued function defined and bounded on R^n the set $\mathrm{supp}(f) := \{x \in R^n : f(x) \neq 0\}$ is usually called the *support* of f.

In the literature f is called *periodic* (mod L), shortly *L-periodic*, if

$$f(x) = f(x+u), \qquad x \in R^n,\ u \in L. \tag{1.8}$$

It is clear that if $\mathrm{supp}(f)$ is bounded, then f cannot be L-periodic, on one hand. On the other hand, any such function defines an L-periodic function $\bar{f}$ as follows:

$$\bar{f}(x) := \sum_{u \in L} f(x+u), \qquad x \in R^n. \tag{1.9}$$

The function $\bar{f}$ plays an interesting role both in multi-dimensional Fourier analysis and in the geometry of numbers.

Namely, if f is Lebesgue-measurable (shortly measurable) and has bounded support, then the Fourier expansion of $\bar{f}$ is equal to

$$\sum_{v \in L^*} \hat{f}(v)\ e^{2\pi i \langle v,x\rangle}, \qquad x \in R^n, \tag{1.10}$$

where

$$\hat{f}(z) := \frac{1}{d(L)} \int_{R^n} f(y) e^{-2\pi i \langle y,z\rangle} dy, \qquad z \in R^n. \tag{1.11}$$

This statement is in fact essentially equivalent to the Poisson summation formula (see, e.g., [10], p. 251, Theorem 2.4).

As to the role of $\bar{f}$ in the geometry of numbers, let us recall an identity involving $\bar{f}$ which is due to Bombieri [1]: if f is bounded, measurable, non-negative and has bounded support, then

$$\int_{R^n} f(x)\bar{f}(x) dx \ = \ d(L) \cdot \sum_{v \in L^*} |\hat{f}(v)|^2, \tag{1.12}$$

where $\hat{f}$ is the function (1.11).

The latter identity is an important tool for some improvements of basic theorems in the geometry of numbers (see, [5], pp. 128–131). Among other things, applying (1.12) to the charecteristic function of a bounded measurable set A and using (1.7) we get the formula (1.3) at once. Surprisingly enough, it seems that the identity (1.7) (i.e., the simple fact that the set $(A - A) \cap L$ is equal to the right hand side set of (1.7)) has not been recognized for a long time. This simple identity was used first in [13], implying many new results and in fact this identity led us to the definition (1.6) of $\hat{\mathcal{L}}(A)$.

Let us note that (1.12) is a simple consequence of the classical Parseval formula and of another identity

$$\int_{R^n} g(x)dx = \int_P \Big(\sum_{u \in L} g(y+u)\Big) dy, \tag{1.13}$$

where g is any complex-valued function g defined on R^n and integrable there.

The latter identity holds in a much more general setting for functions defined on any topological group G and its discrete subgroup Γ s.t. the quotient group G/Γ is compact, see e.g. [22].

Two more L-periodic functions generated by f have been introduced in [12]; those functions, together with $\bar{f}$, also are quite useful in the geometry of numbers, see [12] or [5], pp. 128–131.

In the results just mentioned (as well as in all other results concerning the periodicity of a function) different L-periodic functions *generated* by f, rather than the periodic structure of f were important. The idea to understand and describe the *inner periodic structure of f itself* first appeared in the papers [18], [19], [20], where many results on the latter periodic structure have been proved already. The present paper may be considered as a further development of the results of the above three papers.

The paper is divided into three more sections. In Section 2, after defining precisely two periodic properties of a function (its periodic extendability and periodic part, respectively), the connections between these and with the basic problem in the geometry of numbers are investigated. In Section 3 some results proved in [20] for one colour sets (especially those which are related to partitions of sets) are extended to general coloured sets. The results of Sections 2 and 3 refine those proved for general bounded functions f in [18], [19]. The results and methods of Sections 2 and 3 are elementary and combinatorial in nature. In the papers [18], [19], [20] many analytic type results also are proved, where all concepts used are taken "almost everywhere". Section 4 contains a short account of those analytic results, showing some basic ideas and hinting some posibilities for further improvements.

2. Two periodic properties of a function

Remember all notations and definitions from the Section 1. In this section, if not stated otherwise, $L \subset R^n$ is any point-lattice of dimension n and f is any function mentioned in the Abstract. The periodic extendability of a function can be precised as follows:

Definiton 2.1. We call f [almost] extendable to an L-periodic function (shortly [almost] L-extendable), if there is a function $g : R^n \to R^1$ which is L-periodic (in the classical sense, see (1.8)) and such that

$$f(x) = g(x), \quad \text{[a.e.]} \quad x \in \mathrm{supp}(f). \tag{2.1}$$

Our first theorem is a local (or inner) characterization of periodic extendability.

Theorem 2.2 ([18], [19]). *The coloured set f is L-extendable if and only if*

$$\begin{gathered} f(x) = f(x+u) \\ \forall\ u \in \mathcal{L}(\mathrm{supp}(f)) \quad \text{and} \quad \forall\ x \in \mathrm{supp}(f) \cap (\mathrm{supp}(f) - u). \end{gathered} \tag{2.2}$$

Proof ([18], [19]). Denote for short $A := \mathrm{supp}(f)$.

Assume that (2.2) is not true, i.e., there are $\theta \neq v \in \mathcal{L}(A)$ and $y \in A \cap (A - v)$ such that $f(y) \neq f(y+v)$. Now, let h be an L-extension of f, i.e., an L-periodic function such that $f(x) = h(x)$, $x \in A$. It is clear that $y + v \in A$, hence $h(y+v) = f(y+v)$, consequently $f(y) = h(y) \neq h(y+v)$ that contradicts the L-periodicity of h. This proves one direction of the theorem.

To prove the converse direction, assume that (2.2) is true and define the function $g : R^n \to R^1$ as follows:

$$g(x) := \begin{cases} f(x), & x \in A, \\ f(y), & x \notin A,\ \exists\, y \in A \text{ s.t. } \varphi(x) = \varphi(y), \qquad x \in R^n. \\ 0, & x \notin A,\ \varphi(x) \notin \varphi(A). \end{cases} \tag{2.3}$$

This function is well defined, because

$$\{y, z \in A \text{ and } \varphi(y) = \varphi(z)\} \;\Rightarrow\; f(y) = f(z). \tag{$*$}$$

Indeed, (1.0) implies that $y = z + v$ for some $v \in L$. The definition (1.7) of $\mathcal{L}(A)$ and the fact that $y, z \in A$ in turn imply that $v \in \mathcal{L}(A)$ and $z \in A \cap (A - v)$. Finally, property (2.2) implies $f(z) = f(z + v) = f(y)$, so $(*)$ is true.

To prove that g is L-periodic, first let $x \in A$. For any $u \in L$ we have $\varphi(x) = \varphi(x+u)$, hence, if $x+u \notin A$ then we have $g(x+u) = f(x) = g(x)$, and if $x + u \in A$ then property (2.2) implies $g(x) = f(x) = f(x + u) = g(x + u)$.

Secondly, let $x \notin A$. Then, for any $u \in L$ we have either $x + u \in A$ or $x + u \notin A$.

In the first case $g(x) = f(x + u) = g(x + u)$. In the second case $g(x+u) = f(y)$ for some $y \in A$ with $\varphi(x+u) = \varphi(y)$ and $g(x) = f(z)$ for some $z \in A$ with $\varphi(x) = \varphi(z)$. But $\varphi(x+u) = \varphi(x) = \varphi(y) = \varphi(z)$, hence by $(*)$ $f(y) = f(z)$, yielding $g(x) = f(z) = f(y) = g(x + u)$.

By this the theorem is proved. □

Theorem 2.2 and its proof show two interesting things:

– We see that one direction of the theorem is nearly a triviality, but the other direction is not so simple, moreover, (2.3) *gives a construction for extending f to the whole space* when condition (2.2) is true.

– Another interesting thing is that *condition (2.2) is automatically satisfied if f is such that*

$$\mathcal{L}(\mathrm{supp}(f)) = \{\theta\}, \tag{2.4}$$

i.e. we have

Corollary 2.3. *If the coloured set f satisfies* (2.4) *then it is L-extendable.*

This corollary suggests the following question: is there some other periodicity property of the function f, which, together with its L-extendability, would imply the condition (2.4)?

The answer is yes, as it is shown in [18], [19], [20]. A special notion has been introduced in the latter papers (called in [19] an L-periodic part of f, in [20] an L-periodic part of a set or in [18] the set of pL-points of $\mathrm{supp}(f)$), and it has been proved that the needed above property is the lack of L-periodic parts of f (of the set), or equivalently the lack of pL-points in $\mathrm{supp}(f)$.

Definition 2.4 (see [20]). Let $A \subset R^n$ be any bounded set. We call the set $D \subseteq P$ an L-periodic part of A if

$$\begin{cases} |(A - D) \cap L| \geq 2 \quad \text{and} \\ D + u \subseteq A, \quad \forall\, u \in (A - D) \cap L. \end{cases} \tag{2.5}$$

Definition 2.5 (see [18]). Let f be any coloured set. We call $D \subseteq P$ an L-periodic part of f if

$$\begin{cases} |(\mathrm{supp}(f) - D) \cap L| \geq 2 \quad \text{and} \\ f(x+u) = f(x+v) \neq 0, \quad x \in D,\ u, v \in (\mathrm{supp}(f) - D) \cap L. \end{cases} \tag{2.6}$$

One can easily see that for one-colour sets the Definitions 2.4 and 2.5 are equivalent (i.e., give the same sets D).

An L-periodic part of a function has the following quite interesting and useful property.

Assertion 2.6. Let f be a function and D its L-periodic part. Then

$$(\mathrm{supp}(f) - D) \cap L = (\mathrm{supp}(f) - x) \cap L, \quad \forall x \in D. \tag{2.7}$$

Proof. The definition of D shows that

$$D + u \subseteq \mathrm{supp}(f), \quad \forall\, u \in (\mathrm{supp}(f) - D) \cap L, \tag{2.8}$$

and this implies (2.7). □

Condition (2.5) has the following equivalent form:

Assertion 2.7. Let A be any bounded set. Then (2.5) is equivalent to

$$\begin{cases} |(A-D)\cap L| \geq 2 \quad \text{and} \\ (A-D)\cap L = (A-x)\cap L \quad \forall x \in D. \end{cases} \tag{2.9}$$

PROOF. Applying Assertion 2.6 to the one-colour set $f := \chi_A$ we see that (2.5) implies (2.9). To prove the converse implication, assume that (2.5) is not true. If $|(A-D)\cap L| \leq 1$, then (2.9) cannot be true either. So let $|(A-D)\cap L| \geq 2$, and assume that there is $u \in (A-D)\cap L$ s.t. $D+u \not\subseteq A$. This implies that there is $y \in D$ s.t. $y+u \notin A$ hence $u \notin A-y$, implying that $(A-D)\cap L \setminus (A-y)\cap L \neq \emptyset$, i.e., (2.9) is not true. $\square$

Property (2.9) suggests the following notation: For any bounded set $A \subset R^n$ denote

$$\begin{aligned} \hat{\mathcal{P}}(A) := \{D \subseteq P : |(A-D)\cap L| \geq 2 \text{ and} \\ (A-D)\cap L = (A-x)\cap L \ \forall x \in D\} \end{aligned} \tag{2.10}$$

The family $\hat{\mathcal{P}}(A)$ has got the following interesting property.

Assertion 2.8. Let $D, E \in \hat{\mathcal{P}}(A)$, $D \neq E$. Then either $D \cap E = \emptyset$ or there is $F \in \hat{\mathcal{P}}(A)$ such that $D, E \subseteq F$ and $(A-F)\cap L = (A-D)\cap L = (A-E)\cap L$.

PROOF. Denote by G and H the maximal (w.r.t. set inclusion) members of $\hat{\mathcal{P}}(A)$ containing D and E, respectively. We claim that either $G \cap H = \emptyset$ or $G = H$. Indeed, assume $G \cap H \neq \emptyset$ and $G \setminus H \neq \emptyset$. Take $x \in G \cap H$, $y \in G \setminus H$. Then, by the definition of sets in $\hat{\mathcal{P}}(A)$, we have $(A-G)\cap L = (A-H)\cap L = (A-x)\cap L$ and $(A-G)\cap L = (A-y)\cap L$. These imply that $(A-H)\cap L = (A-z)\cap L$ holds for all $z \in H \cup \{y\}$, which contradicts the maximality of H. Similarly we come to a contradiction if we assume that $G \cap H \neq \emptyset$ and $H \setminus G \neq \emptyset$. We see that either $D \cap E = \emptyset$ or $D, E \subseteq F := G = H$. In the second case by the definition of F we have $(A-F)\cap L = (A-y)\cap L = (A-x)\cap L$, $\forall\, x \in D$, $\forall\, y \in E$, hence $(A-F)\cap L = (A-D)\cap L = (A-E)\cap L$. $\square$

It is clear that in general properties (2.7) and $|(\mathrm{supp}(f)-D)\cap L| \geq 2$ together do not imply that D is a periodic part of f, i.e., the family $\hat{\mathcal{P}}(\mathrm{supp}(f))$ does not coincide with the family of all periodic parts of f. One more condition for a $D \in \hat{\mathcal{P}}(\mathrm{supp}(f))$ to be a periodic part of f should be satisfied, namely, we have

Assertion 2.9. Let f be any coloured set. The set $D \subseteq P$ is an L-periodic part of f if and only if

$$\begin{cases} D \in \hat{\mathcal{P}}(\operatorname{supp}(f)) \quad \text{and} \\ f(x+u) = f(x+v), \quad x \in D,\ u,v \in (\operatorname{supp}(f) - D) \cap L. \end{cases} \tag{2.11}$$

Proof. If the set $D \subset P$ is an L-periodic part of f, then the second part of (2.11) is clear and by Assertion 2.6 condition (2.7) holds, hence $D \in \hat{\mathcal{P}}(\operatorname{supp}(f))$. Conversely, let $D \in \hat{\mathcal{P}}(\operatorname{supp}(f))$. Then using Assertion 2.7 with $A := \operatorname{supp}(f)$ we get that (2.5) is true with $A := \operatorname{supp}(f)$. But this implies that

$$f(x+u) \neq 0, \quad x \in D,\ u \in (\operatorname{supp}(f) - D) \cap L,$$

hence condition (2.11) implies (2.6). □

It can happen that f has no periodic parts but $\hat{\mathcal{P}}(\operatorname{supp}(f)) \neq \emptyset$. As we shall see, this will have an interesting consequence: in this case f is not periodically L-extendable. The latter statement is a simple corollary of the following

Theorem 2.10. *f is extendable to an L-periodic function and has no L-periodic parts if and only if condition* (2.4) *is satisfied.*

Proof. Assume first that f has got an L-periodic part $D \subseteq P$. This implies that there are $x \in D$ and $u,v \in L$ such that $u \neq v$ and $x + u, x + v \in \operatorname{supp}(f)$. This means that $\theta \neq u - v \in \operatorname{supp}(f) - \operatorname{supp}(f)$, contradicting (2.4).

Secondly, assume that f cannot be extended to any periodic function. This implies that $\bar{f}$, being a periodic function, cannot be an extension of f, hence there is $x \in \operatorname{supp}(f)$ with $f(x) \neq \bar{f}(x)$. This yields

$$\sum_{\theta \neq u \in L} f(x+u) \neq 0,$$

consequently there is $u \in L, u \neq \theta$, so that $x+u \in \operatorname{supp}(f)$. This, together with $x \in \operatorname{supp}(f)$, implies $\theta \neq u \in \operatorname{supp}(f) - \operatorname{supp}(f)$, cotradicting (2.4).

These prove one direction of the theorem.

To prove the converse direction, observe first that

$$\mathcal{L}(P) = \{\theta\}. \tag{2.12}$$

Assume (2.4) is not true, i.e., there is $\theta \neq u = x - z \in L$ with $x, z \in \text{supp}(f)$. Using the projections φ and $[\cdot]$ defined in Section 1, this implies $u - [x] + [z] = \varphi(x) - \varphi(z) \in L$ which is by (2.12) possible only if $\varphi(x) = \varphi(z)$. The last two equalities together with $u \neq \theta$ also imply that $[x] \neq [z]$. Denote $y = \varphi(x) = \varphi(z)$. We have $x = y + [x], z = y + [z] \in \text{supp}(f)$. Now, if $f(x) = f(z)$, then either $D := \{y\}$ is a periodic part of f or there are $u, v \in (\text{supp}(f) - y) \cap L$ such that $f(u+y) \neq f(v+y)$. If $f(x) \neq f(z)$, then $f(x) = f(z+u) \neq f(z)$, where $z \in \text{supp}(f)$. In both cases we see that f cannot be extended to any periodic function.

By this the converse direction and the whole theorem are proved. $\square$

Corollary 2.11. *Assume $\hat{\mathcal{P}}(\text{supp}(f)) \neq \emptyset$ and that f has no L-periodic parts. Then f cannot be extended to an L-periodic function.*

Proof. If f could be extended to an L-periodic function, then by Theorem 2.10 condition (2.4) is true. But $\hat{\mathcal{P}}(\text{supp}(f)) \neq \emptyset$ implies that there is $x \in P$ s.t. $|(\text{supp}(f) - x) \cap L| \geq 2$, consequently there are $u, v \in L$, $u \neq v$, and $a, b \in \text{supp}(f)$ such that $u = a - x$ and $v = b - x$. This implies $\theta \neq a - b \in L$, contradicting (2.4). $\square$

It is interesting to "check" both theorems for one-colour sets f, the constant functions, i.e., $f(x) = c$, $x \in A \subset R^n$, $f(x) = 0$, $x \in R^n \setminus A$. For these functions Theorem 2.2 is a triviality, because $g(x) = c$, $x \in R^n$, is trivially an L-periodic extension of f and f trivially satisfies the condition (2.2). This is in a sharp contrast with Theorem 2.10 , where now the first part of the condition ("f is extendable to an L-periodic function") is trivially satisfied, consequently it is the second part ("f has no L-periodic parts") that is equivalent to (2.4) and this is not a triviality, on one hand. On the other hand, as one indeed expects, the more special f is, the more sophisticated results can be proved. These deeper results for sets ($\sim$ characteristic functions of sets) have been proved in [20] and, as it is shown in the next section, similar results are true for coloured sets.

3. L-extendability of coloured sets and set partitions

In this section L is as in Section 2 and reminds the definition of a coloured set in Abstract.

We call the family $\mathcal{B} := \{B_k\}_{k=1}^p$ of bounded sets $B_k \subset R^n$ a partition of the set $B \subset R^n$ if $B = \bigcup_{k=1}^p B_k$ and $B_j \cap B_k = \emptyset$ for all $1 \leq j < k \leq p$.

In [20] two special families $\{B_k\}_{k=1}^p$, and $\{B_k+u_k\}_{k=1}^p$, have been introduced for any bounded $A\subset R^n$ and L, where the B_k and u_k are

$$B_k := (A-u_k)\cap P, \qquad u_k\in [A],\ k=1,\dots,p,\ p:=\big|[A]\big|. \tag{3.1}$$

[By the way, the definition of B_k in [20], (see, [20], (3.11)) contains a misprint, $(A-u_k)\cap L$ is correctly $(A-u_k)\cap P$.]

One can easily see that for the first family we have $\varphi(A)=\bigcup_{k=1}^p B_k$. One of the results proved in [20] is that the set $\bigcup_{k=1}^p(B_k+u_k)$ has no L-periodic parts (as defined by (2.5)) if and only if the family $\{B_k\}_{k=1}^p$ is a partition of $\varphi(A)$.

The following two theorems show that a similar connection between periodic properties and partitions is true also in our more complex coloured case.

Definition 3.1. Let f be a coloured set and $\{A_i\}_{i=1}^m$ its level sets. Let $\{u_1,\dots,u_m\}$ be an arbitrary set of m different points of L. Then we call the set

$$A(f) := \bigcup_{i=1}^m(\varphi(A_i)+u_i) \tag{3.1}$$

an L-spread of f.

The family $\{A_i\}_{i=1}^m$ is a partition of $\mathrm{supp}(f)$. It is clear that

$$\varphi(A(f))=\varphi(\mathrm{supp}(f))=\bigcup_{i=1}^m\varphi(A_i),$$

so the family of sets $\varphi(A_i)\subseteq\varphi(\mathrm{supp}(f))$ might be a partition of $\varphi(\mathrm{supp}(f))$. The question is what conditions assure that it is a partition. The answer is given by

Theorem 3.2. *Let f be a coloured set and $\{A_i\}_{i=1}^m$ its level sets. Then condition* (2.2) *of Theorem 2.2 holds if and only if the family $\{\varphi(A_i)\}_{i=1}^m$ is a partition of $\varphi(\mathrm{supp}(f))$.*

Proof. Assume $\varphi(A_i)\cap\varphi(A_j)\neq\emptyset$ for some $i\neq j$. This implies that there is $z\in\varphi(A_i)\cap\varphi(A_j)$, consequently there are $x\in A_i$, $y\in A_j$, such that $z=\varphi(x)=\varphi(y)$. Clearly $x\neq y$ and condition (1.0) shows that

$\theta \neq u := y - x \in L$, hence $u \in \mathcal{L}(\text{supp}(f))$, $x \in \text{supp}(f) \cap (\text{supp}(f) - u)$, implying that $c_i = f(x) \neq c_j = f(y) = f(u+x)$, i.e., condition (2.2) is not true.

Assume now that condition (2.2) is not true, i.e., that there are $u \in \mathcal{L}(\text{supp}(f)$ and $x \in \text{supp}(f) \cap (\text{supp}(f) - u)$ such that $f(x) \neq f(x+u)$. These imply that $f(x) = c_i \neq f(y) = c_j$ for some $i \neq j$, $\varphi(x) = \varphi(y)$ (because $y := x + u$) and $x \in A_i$, $y \in A_j$. Hence, denoting $z := \varphi(x) = \varphi(y)$, we have $z \in \varphi(A_i) \cap \varphi(A_j)$, so the family $\{\varphi(A_i)\}_{i=1}^m$ cannot be a partition of $\varphi(\text{supp}(f))$. $\square$

Theorem 3.3. *Let f be a coloured set and $A(f)$ an L-spread of f. Then the one-colour set $A(f)$ has no L-periodic parts if and only if the family $\{\varphi(A_i)\}_{i=1}^m$ is a partition of $\varphi(\text{supp}(f))$.*

Proof. Applying Theorem 2.10 to the one-colour set $A(f)$, we see that $A(f)$ has no periodic parts if and only if

$$\mathcal{L}(A(f)) = \{\theta\}. \tag{3.2}$$

Now it is enough to prove that the family $\{\varphi(A_i)\}_{i=1}^m$ is a partition of $\varphi(\text{supp}(f))$ if and only if (3.2) is true. Denote for short $B_i := \varphi(A_i)$, $i = 1, \ldots, m$, and let $u_1 \ldots, u_m \in L$ be the family defining $A(f)$.

Assume $B_k \cap B_l \neq \emptyset$ for some $1 \leq k < l \leq m$, and let $y \in B_k \cap B_l$. It is clear that $B_k = (A(f) - u_k) \cap P$, $1 \leq k \leq m$, hence $y \in (A(f) - u_k) \cap (A(B) - u_l) \cap P$, yielding $y = a - u_k = b - u_l$, $a, b \in A(f)$, in contradiction to (3.2), because $u_k \neq u_l$.

Conversely, assume (3.2) is not true, i.e., there are $a \in B_k + u_k$, $b \in B_l + u_l$, $\theta \neq u \in L$, s.t. $a - b = u$.

As $B_k, B_l \subseteq P$, we see that there are $x_k, x_l \in P$ such that $a = x_k + u_k$, $b = x_l + u_l$, consequently $x_k - x_l = u + u_l - u_k$. By (2.12) this implies that $u + u_l - u_k = \theta$, hence $x_k = x_l$ and $u = u_k - u_l \neq \theta$, yielding that $k \neq l$ and $a - u_k = b - u_l \in B_k \cap B_l$, i.e., $\{B_i\}_{i=1}^m$ is not a partition of $\varphi(\text{supp}(f))$. $\square$

The next theorem collects the results proved up to now and proves two more relations.

Theorem 3.4. *Let f be a coloured set, A_i, $i = 1, \dots, m$, its level sets and $A(f)$ an L-spread of f. Then the following six conditions are equivalent:*

(a) *f is L-extendable;*

(b) *$A(f)$ has no L-periodic parts;*

(c) *$\{\varphi(A_i)\}_{i=1}^m$ is a partition of $\varphi(\operatorname{supp}(f))$;*

(d) *condition* (2.2) *is true;*

(e) *$\mathcal{L}(A(f)) = \{\theta\}$;*

(f) *$(A_i - A_j) \cap L = \emptyset$, $\forall\ 1 \le i < j \le m$.*

If f is L-extendable then

(g) *$\mathcal{L}(\operatorname{supp}(f)) = \bigcup_{i=1}^m (A_i - A_i) \cap L$.*

Proof. Theorem 2.2 shows that (a) is equivalent to (d). By Theorem 3.2, (c) is equivalent to (d). The equivalence of (b) and (e) is assured by Theorem 2.10 applied to the one-colour set $A(f)$. Theorem 3.3 gives the equivalence of (b) and (c).

We prove now the equivalence of (d) and (f). Denote for short $A := \operatorname{supp}(f) = \bigcup_{i=1}^m A_i$. Assume first that (f) is not true, i.e., for some $i < j$ there is $u \in (A_i - A_j) \cap L$. It is clear that $u \ne \theta$ and that there are $x \in A_i,\ y \in A_j$, s.t. $y = x + u$. This implies that $c_j = f(y) = f(x+u) \ne c_i = f(x)$, which contradicts the condition (2.2), i.e. (d), is not true.

To prove the contrary, assume (2.2), i.e. (d), is not true. So there are $\theta \ne u \in (A - A) \cap L$ and $x \in A \cap (A - u)$ such that $f(x) \ne f(x+u)$. These imply that $x \in A_i,\ x + u \in A_j$ for some $i \ne j$, hence $x \in A_i \cap (A_j - u)$, contradicting (f).

As to the proof of (g), it is a simple consequence of condition (f), i.e., (f) implies (g). (Let us note that in general condition (g) does not imply (f).) $\square$

4. Some analytic results and remarks

Let us give a brief account of some analytic results proved in [18], [19], [20]. As we have already noted in Section 1, some results are independent of changing the sets by sets of measure zero. The two theorems below reflect this independence.

In this section L is as in Section 2 and f is a non-negative bounded measurable function with bounded support.

Theorem 4.1 ([18], [19]). *Let f be such that $V(\mathrm{supp}(f)) > 0$. Then*

$$\int_{R^n} f^2(x)dx \leq^a \frac{1}{d(L)} \sum_{v\in L^*} \int_{R^n}\int_{R^n} \cos(2\pi\langle v, x-t\rangle) \min\{f^2(x), f^2(t)\}dxdt$$

$$\leq^b \frac{1}{d(L)} \sum_{v\in L^*} \int_{R^n}\int_{R^n} \cos(2\pi\langle v, x-t\rangle) f(x)f(t)\, dxdt. \tag{4.1}$$

$\leq^a$ *is equality if and only if f is almost L-extendable and f has no L-periodic parts of positive measure.*

$\leq^b$ *is equality if and only if f is almost L-extendable.*

The proof of Theorem 3.9 is based on the following.

Lemma 4.2 ([18], [19]). *The following two identities hold:*

$$\sum_{u\in L}\int_{R^n} f(x)f(x+u)dx = \frac{1}{d(L)} \sum_{v\in L^*} \int_{R^n}\int_{R^n} \cos(2\pi\langle v, x-t\rangle) f(x)f(t)dxdt \tag{4.2}$$

and

$$\sum_{u\in L}\int_{R^n} \min\{f(x), f(x+u)\}dx = \frac{1}{d(L)} \sum_{v\in L^*} \int_{R^n}\int_{R^n} \cos(2\pi\langle v, x-t\rangle) \min\{f(x), f(t)\}dxdt. \tag{4.3}$$

The identity (4.2) is in fact a consequence of Bombieri's identity (1.12), because the right hand side of (1.12) can, after some calculations, be transformed to that of (4.2). Nevertheless, in [18], [19] an independent proof is given (which is not so complicated, it is a relatively simple consequence of the generalized Parseval formula and the identity (1.13)). This is in a sharp contrast with the identity (4.3), the proof of which is very complicated, see [18], [19]. The proof of (4.3) uses also the generalized Parseval formula and the identity (1.13), but in the course of the proof a series of (sometimes tricky) precise transformations and calculations were needed. One of the basic tricks in the proof of (4.3) was to transform the question into a similar one for hypographs of f in the space R^{n+1}. More exactly, the hypograph of a non-negative function f is

$$\mathrm{hyp}(f) := \{ (x,\xi) \in R^n \times R^1 : 0 \le \xi \le f(x),\ x \in \mathrm{supp}(f)\}. \tag{4.4}$$

After this the following quite interesting identity was needed:

Lemma 4.3 ([18], [19]). *For any $(x,0) \in R^{n+1} = R^n \times R^1$ we have*

$$\begin{aligned} &\bar{V}(\mathrm{hyp}(f) \cap (\mathrm{hyp}(f) - (x,0))) \\ &\quad = \int_{\mathrm{supp}(f)\cap(\mathrm{supp}(f)-x)} \min\{f(y), f(y+x)\}dy, \end{aligned} \tag{4.5}$$

where $\bar{V}$ is the Lebesgue measure in R^{n+1}.

The second main trick was to define a new $(n+1)$-dimensional lattice $W \subset R^{n+1}$ (an extension of L) and to work with $\mathrm{hyp}(f)$ and W. After a series of transformations one arrives at the very core of the proof, the following well-known identity (see, e.g., [4]): for $-1 \le \eta \le +1$ we have

$$\sum_{k\ge 1} \frac{\cos(2\pi k\eta)}{k^2} = \frac{\pi^2}{6} - \pi^2|\eta| + \pi^2\eta^2. \tag{4.6}$$

Let us close the paper with three remarks.

Remark 1. An unexpected, totally new consequence of the two nested inequalities (4.1) has been discovered recently. Roughly speaking, after fixing the left hand side of $(4.1,\le^a)$ ($\sim L_2$-norm of f), the expression on the right hand side of $(4.1,\le^b)$ ($\sim l_2$-norm of $\hat{f}$ restricted to L^*, see the right hand side of (1.12)) is to be minimized over all f having a fixed L_2-norm. Theorem 4.1 solves this minimization problem in the sense the it gives an exact characterization of the solution (in terms of L-periodic properties of f). This minimization problem is studied and solved in [21].

Remark 2. In fact, the left hand sides of (4.2) and (4.3) are special cases of a "continuous" family of expressions. Namely, let a, b, α, λ be real numbers such that $a, b \ge 0$, $0 < \lambda < 1$ and $-\infty < \alpha < \infty$, $\alpha \ne 0$. Then we define $M_\alpha^{(\lambda)}(a,b)$ to be equal to zero if one of a, b is zero and to be the number $(\lambda a^\alpha + (1-\lambda)b^\alpha)^{1/\alpha}$ otherwise. Taking limits in α one arrives at three more numbers: $M_{-\infty}^{(\lambda)}(a,b) = \min\{a,b\}$; $M_{+\infty}^{(\lambda)}(a,b) = \max\{a,b\}$ if

$a, b > 0$ and $= 0$ if $a \cdot b = 0$; $M_0^{(\lambda)}(a, b) = a^{\lambda} \cdot b^{(1-\lambda)}$. (These numbers might be called "extended means" of a, b. On "usual" means $(\lambda a^{\alpha} + (1-\lambda) b^{\alpha})^{1/\alpha}$, $-\infty \leq \alpha \leq +\infty$, see, e.g., [6].)

The properties of usual means (see [6]) show that $M_\alpha^{(\lambda)}(a, b)$ is for positive a, b, $a \neq b$, a strictly increasing continuous function of α on $[-\infty, +\infty]$. Now, we have

Theorem 4.4 ([18], [19]). *Let $-\infty \leq \alpha \leq \beta \leq +\infty$ and assume $V(\operatorname{supp}(f)) > 0$. Then*

$$\begin{aligned} \int_{R^n} f(x)dx &\leq^1 \sum_{u \in L} \int_{R^n} M_\alpha^{(\lambda)}(f(x), f(x+u))dx \\ &\leq^2 \sum_{u \in L} \int_{R^n} M_\beta^{(\lambda)}(f(x), f(x+u))dx. \end{aligned} \tag{4.7}$$

$\leq^1$ is equality if and only if f is almost extendable to an L-periodic function and f has no periodic parts of positive measure.

For $\alpha < \beta$, $\leq^2$ is equality if and only if f is almost extendable to an L-periodic function.

It turned out that the identities (4.2), (4.3) also extend to this more general situation, i.e., a general identity is true, obtained by taking the means $M_\alpha^{(\lambda)}(f(x), f(x+u))$ and $M_\alpha^{(\lambda)}(f(x), f(t))$ instead of $f(x)f(x+u)$ and $f(x)f(t)$ in (4.2). The proofs are not quite simple, a paper on these more general Fourier analytic identities is in preparation.

Remark 3. All results proved in this paper can be extended (e.g. using an extension of Fourier techniques proposed in [14]) to the case when L is any discrete subgroup of R^n ($\sim$ point lattice of any dimension). In fact, in the papers [18], [19] L is a point lattice of arbitrary dimension. In this case some more tricks are needed for the proofs, see [14], [18], [19]. This is true for the basic result of [11] mentioned in Section 1 (say, (1.4)) as well, see, e.g. [15]. Moreover, the basic result (1.4) holds not only for any L, but also in any Abelian locally compact topological group G for any so called "sufficiently large" discrete subgroup Γ of G (see [17] for more details). This suggests a very interesting question: to what extent can the results proved for (R^n, L), e.g. those proved in the present paper, be extended to (G, Γ)? As our techniques work in many situations also for general G, Γ, one has a feeling that some of the results might be extended to these more general structures.

An interesting "extension of the geometry of numbers" has been introduced and studied in [16]: one takes instead of L an arbitary closed subgroup of R^n. It is sure that some basic theorems of the geometry of numbers can, to some extent, be extended to this more general situation, [16]. What about the "periodic w.r.t. the closed subgroup" structure of a set or function?

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