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The quasiasymptotic expansion at zero and generalized Watson lemma for Colombeau generalized functions

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Abstract. Quasiasymptotic expansion at zero in the Colombeau algebra of generalized functions and its coherence with this notion for Schwartz distributions is given. A version of the Watson lemma related to the expansion of the Laplace transformation of an appropriate generalized Colombeau function is proved. In particular, the asymptotic expansion of δ^2 and the expansion of its Laplace transformation is given.

1. Introduction

Asymptotic analysis is an old subject (cf. [1]) which has a lot of applications in applied mathematics, physics and engineering. It approximates integral expressions or solutions of differential equations.

Since a generalized function g is represented by an ε-net of smooth functions g_ε with the power order growth with respect to ε, $(\varepsilon \to 0)$, the order growth of ε reflects in some sense its singularity. We found that the singularity at zero is characterized through the analysis of the behaviour of $g_\varepsilon(\varepsilon x)$ as $\varepsilon \to 0$. This was already done for Schwartz distributions, using the quasiasymptotics and quasiasymptotic expansion at zero [9], but since the Colombeau space $\mathcal{G}$ contains elements which are no distributions (δ^2, for example) we reconsider the concept of quasiasymptotic expansion

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in $\mathcal{G}$. Note that the quasiasymptotic behaviour in $\mathcal{G}$ is used in [8] for the analysis of a non-linear Cauchy problem.

Generalized asymptotic expansion in the space of Schwartz distributions is studied in [1–3], [5], [9], (see also references in [9]), and the asymptotic expansion related to geometric optics in $\mathcal{G}$ is analyzed in [6].

The results of this paper are the following.

First, by using the definition of quasiasymptotic behaviour at zero of Colombeau generalized functions [8], we derive the definition and the properties of quasiasymptotic expansions at zero for Colombeau generalized functions. We compare this notion with the corresponding one for Schwartz distributions. An $f \in \mathcal{D}'$ has a quasiasymptotic expansion if and only if the embedded Colombeau generalized function has a quasiasymptotic expansion in $\mathcal{G}$.

Second, we give an Abelian-type result for the Laplace transformation of an $f \in \mathcal{G}$ which has appropriate quasiasymptotic expansion. This is a generalized version of the classical Watson lemma [1].

Third, we find the quasisasymptotic expansion of the generalized function $\delta^2 \in \mathcal{G} \setminus \mathcal{D}'$ and the asymptotic expansion of its Laplace transformation.

2. Preliminaries

Notation

Schwartz spaces of test functions and distributions on the real line $\mathbb{R}$ are denoted by $\mathcal{D}$ and $\mathcal{D}'$, respectively; $\mathcal{S}$ is the space of rapidly decreasing functions and its dual $\mathcal{S}'$ is the space of tempered distributions. Also, we use the notions $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$.

Let $\alpha \in \mathbb{R}$. Denote

$$f_{\alpha+1}(x) = \begin{cases} \dfrac{x^\alpha H(x)}{\Gamma(\alpha+1)}, & \alpha > -1 \\ f^{(n)}_{\alpha+n+1}(x), & \alpha \leq -1, \end{cases}$$

where n is the smallest integer for which $\alpha+n > -1$, and H is the Heaviside function.

Recall that $C^\infty(\Omega)$ is a topological vector space whose topology is given by a countable set of seminorms

$$\mu_k(\varphi) = \sup\left\{ \left|\varphi^{(i)}(x)\right|;\ |i| \leq k,\ x \in \Omega_k,\ k \in \mathbb{N} \right\}.$$

Colombeau algebra of generalized functions

We define $\mathcal{E}_M$ as the space of locally bounded functions $R(\varepsilon) = R_\varepsilon : (0,1) \to C^\infty(\Omega)$ such that for every $k \in \mathbb{N}$ there exists $a \in \mathbb{R}$ such that

$$\sup\left\{\left|R_\varepsilon^{(i)}(x)\right|;\ |i| \le k,\ x \in \Omega_k\right\} = O(\varepsilon^a), \quad \left(\Omega_k \subset\subset \Omega_{k+1},\ \bigcup_{k=1}^{\infty} \Omega_k = \Omega\right).$$

By $\mathcal{N}$ is denoted the space of all elements $H_\epsilon \subset \mathcal{E}_M$ with the property that for every $k \in \mathcal{N}$ and $a \in \mathbb{R}$

$$\sup\left\{\left|R_\varepsilon^{(i)}(x)\right|;\ |i| \le k,\ x \in \Omega_k\right\} = O(\varepsilon^a).$$

The quotient space $\mathcal{G} = \mathcal{E}_M/\mathcal{N}$ is a Colombeau space.

In an appropriate way (R_ε and H_ε above do not depend on x) are defined spaces of moderate complex numbers $\mathcal{E}_0$, null spaces $\mathcal{N}_0$, and the space of generalized complex numbers $\mathcal{C} = \mathcal{E}_0/\mathcal{N}_0$.

It is easy to verify that $\mathcal{G}(\Omega)$ is a differential algebra, where derivations are defined by $R^{(\alpha)} = [R_\varepsilon^{(\alpha)}]$ ($[\cdot]$ denotes equivalence class).

The support of a generalized function $H, \operatorname{supp} H$, is defined as the complement of the largest open subset Ω' such that $H_{|\Omega'} = 0$.

It is said that f belongs to $\mathcal{E}_{t,M}(\mathbb{R}^n)$ if for any $k \in \mathbb{N}$ there exist $a \in \mathbb{R}$ and $m \in \mathbb{N}_0$ such that

$$\sup_{|\alpha| \le k} \sup_{\mathbb{R}^n} (\langle x\rangle^{-m} |\partial^\alpha f_\epsilon(x)|) = O(\epsilon^a). \tag{1}$$

(note $\langle x\rangle \stackrel{\text{def}}{=} (1 + |x|^2)^{1/2}$).

The space of elements g of $\mathcal{E}_{t,M}(\mathbb{R}^n)$ with the property that for every k there exists $m \in \mathbb{N}$ such that (1) holds for every $a \in \mathbb{R}$, is denoted by $\mathcal{N}_t(\mathbb{R}^n)$. It is an ideal of $\mathcal{E}_{t,M}(\mathbb{R}^n)$. The quotient space $\mathcal{G}_t(\mathbb{R}^n) = \mathcal{E}_{t,M}(\mathbb{R}^n)/\mathcal{N}_t(\mathbb{R}^n)$ is called Colombeau space of tempered generalized functions. Note that $\mathcal{G}_t$ is not a subspace of G because

$$\mathcal{N}(\mathbb{R}^n) \cap \mathcal{E}_{t,M}(\mathbb{R}^n) \ne \mathcal{N}_t(\mathbb{R}^n),$$

but there is a canonical mapping: $\mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$, and $[G_\varepsilon] = [G_\varepsilon + \mathcal{N}(\mathbb{R}^n)]$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathcal{F}(\hat\phi) = \hat\phi \in \mathcal{D}(\mathbb{R}^n)$ and $\hat\phi \equiv 1$ on be neighbourhood of zero. Put $\phi_\epsilon(x) = \frac{1}{\epsilon^{2n}}\phi(\frac{x}{\epsilon^2})$, $x \in \mathbb{R}^n$, $\epsilon \in (0,1)$. We call ϕ the “vision” function.

If $g \in \mathcal{D}'$, then by

$$\left\langle g(\xi),\ \epsilon^{-2n}\phi\left(\frac{x-\xi}{\epsilon^2}\right)\right\rangle, \quad x \in \mathbb{R}^n$$

is denoted the representative of the corresponding element in $\mathcal{G}$. Its class is called Colombeau regularization of g and it is denoted by $Cd\ g$.

Quasiasymptotic behaviour

Let $\mathcal{K}$ be a set of positive measurable functions c defined on $(0,1)$ with the following property:

$$A^{-1}\epsilon^r < c(\epsilon) < A\epsilon^{-r}, \quad \epsilon \in (0,1)$$

for some $A > 0$ and $r > 0$.

The notion of quasiasymptotic behaviour at zero in $\mathcal{G}(\Omega)$ is introduced in [8].

Let $F \in \mathcal{G}(\Omega)$. If for every $\psi \in \mathcal{D}(\Omega)$ there is $C_\psi \in \mathbb{C}$, $C_\psi \neq 0$, such that

$$\lim_{\epsilon \to 0^+} \left\langle \frac{F_\epsilon(\epsilon x)}{c(\epsilon)},\ \ \psi(x)\right\rangle = C_\psi,$$

then it is said that F has quasiasymptotics at zero with respect to $c(\epsilon) \in \mathcal{K}$.

The consequences of this definition are contained in Proposition 2 and Proposition 3 of [8].

3. The quasiasymptotic expansion at zero of Colombeau generalized functions

We denote by Λ the set $\mathcal{N}$ or a finite set of the form $\{1,2,\dots,N\}$, $N \in \mathbb{N}$.

Let $c_k \in K$, $k \in \Lambda$, such that

$$(1) \qquad \lim_{\varepsilon \to 0} \frac{c_{k+1}(\varepsilon)}{c_k(\varepsilon)} \to 0, \quad k = 1,\dots,N-1 \text{ (or } k \in \mathbb{N},\ \text{if } \Lambda = \mathbb{N})$$

and

$$P_k = [P_\varepsilon] \in \mathcal{G}(\Omega), \quad k \in \Lambda.$$

Then $G = [G_\varepsilon] \in \mathcal{G}(\Omega)$ has quasiasymptotic expansion (strong asymptotic expansion) at zero and this equals $\sum_{k\in\Lambda} P_k$ with respect to $\{c_k(\varepsilon);\ k \in \Lambda\}$ if

$$\frac{\left(G_\varepsilon - \sum_{k=1}^{m} P_{k\varepsilon}\right)(\varepsilon x)}{c_m(\varepsilon)} \to 0,\ \varepsilon \to 0^+,\ \text{in } \mathcal{D}'(\Omega) \text{ for every } m \in \Lambda$$

$$\left(\frac{\left(G_\varepsilon - \sum_{k=1}^{m} P_{k\varepsilon}\right)(\varepsilon x)}{c_m(\varepsilon)} \to 0,\ \varepsilon \to 0^+ \text{ for every } x \in \Omega, \text{ and every } m \in \Lambda\right).$$

In this case we write

$$G(\varepsilon x) \overset{\text{q.e.c.}}{\sim} \sum_{k\in\Lambda} P_k(\varepsilon x) \text{ with respect to } \{c_k(\varepsilon);\ k \in \Lambda\} \tag{2}$$

$$\left(G(\varepsilon x) \overset{s.e.c.}{\sim} \sum_{k\in\Lambda} P_k(\varepsilon x) \text{ with respect to } \{c_k(\varepsilon);\ k \in \Lambda\}\right)$$

and say that G has quasiasymptotic (strong asymptotic) expansion in the Colombeau sense.

If $G \in \mathcal{G}_t(\mathbb{R})$ and $P_k \in \mathcal{G}_t(\mathbb{R})$, $k \in \Lambda$, then with the same definitions we obtain the quasiasymptotic expansion in $\mathcal{G}_t(\mathbb{R})$.

One can simply prove that this definition does not depend on representatives.

In the sequel (if no additional condition is given), we will allways assume that $\{c_k(\varepsilon); k \in \Lambda\}$ is a subset of $\mathcal{K}$ satisfying (1).

Remark 1. Let $G \in \mathcal{G}(\mathbb{R})$ ($G \in \mathcal{G}_t(\mathbb{R})$) and $P_k \in \mathcal{G}(\mathbb{R})$ ($P_k \in \mathcal{G}_t(\mathbb{R})$), $k \in \Lambda$. We define the strong asymptotic expansion at ∞ as follows:

G has strong asymptotic expansion at ∞ with respect to $\{c_k(\varepsilon);\ k \in \Lambda\}$ if

$$\frac{\left(G_\varepsilon - \sum_{k=1}^{m} P_{k\varepsilon}\right)\left(\frac{x}{\varepsilon}\right)}{c_m\left(\frac{1}{\varepsilon}\right)} \to 0, \quad \varepsilon \to 0^+,$$

for every $x > 0$ and every $m \in \Lambda$.

In this case we write

$$G\left(\frac{x}{\varepsilon}\right) \overset{\text{s.e.c.}}{\sim} \sum_{k\in\Lambda} P_k\left(\frac{x}{\varepsilon}\right) \quad \text{at } \infty \text{ with respect to } \{c_k;\ k \in \Lambda\}.$$

We will use this definition in Proposition 3 below.

Remark 2. We give a definition related to a special choice of P_k, $k \in \Lambda$. Let α_k, $k \in \Lambda$ be an increasing sequence of real numbers, and $G = [G_\varepsilon] \in \mathcal{G}_t(\Omega)$.

Then G has quasiasymptotic expansion at zero as $\sum_{k\in\Lambda} A_k F_{\alpha_k+1}$ with respect to $\{c_k(\varepsilon);\ k \in \Lambda\}$ if there are complex numbers $A_k \neq 0$, $k \in \Lambda$, such that for any $m \in \Lambda$

$$\frac{\left(G_\varepsilon - \sum_{k=1}^{m} A_k F_{\alpha_k+1,\varepsilon}\right)(\varepsilon x)}{c_m(\varepsilon)} \to 0, \quad \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\Omega),$$

where $F_{\alpha+1,\varepsilon} = f_{\alpha+1} * \phi_{\varepsilon^2}$. We write

$$G(\varepsilon x) \overset{\text{q.e.c.}}{\sim} \sum_{k\in\Lambda} A_k F_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to } \{c_k(\varepsilon);\ k \in \Lambda\}.$$

In an adequate way one defines the strong asymptotic expansion at zero as $\sum_{k\in\Lambda} A_k F_{\alpha_k+1}$.

Proposition 1. *If* (2) *holds, then*

a) $$G'(\varepsilon x) \overset{\text{q.e.c.}}{=} \sum_{k\in\Lambda} P'_k(\varepsilon x) \qquad \textit{with respect to } \{c_k(\varepsilon);\ k \in \Lambda\}.$$

b) $$\varphi G(\varepsilon x) \overset{\text{q.e.c.}}{\sim} \sum_{k\in\Lambda} \varphi(0) P_k(\varepsilon x) \quad \textit{with respect to } \{c_k(\varepsilon);\ k \in \Lambda\}$$

where $\varphi \in C^\infty(\mathbb{R})$.

c) *The strong asymptotic expansion at zero with respect to* $\{c_k(\varepsilon);\ k \in \Lambda\}$ *implies the quasiasymptotic expansion at zero in the Colombeau sense if for every compact set* $K \subset\subset \Omega$ *and* $m \in \Lambda$ *there exists* $\varepsilon_m > 0$ *such that*

$$\sup\left\{\frac{|G_\varepsilon(\varepsilon x) - \sum_{k=1}^{m} P_{k\varepsilon}(\varepsilon x)|}{c_m(\varepsilon)};\ x \in \Omega,\ \varepsilon \in (0, \varepsilon_m)\right\} < \infty.$$

Proof. Assertion a) is obvious, c) follows by Lebesgue's theorem on dominated convergence and b) follows from the equivalence of weak and strong convergence in $\mathcal{D}'(\Omega)$ since for every $\psi \in \mathcal{D}(\mathbb{R})$, $\{\psi(x)\varphi(\varepsilon x);\ \varepsilon \in (0,1)\}$ is a bounded set in $\mathcal{D}(\mathbb{R})$. $\square$

Recall [5] that if $f \in \mathcal{E}'$ and

$$\frac{\left(f - \sum_{k=1}^{N} A_k f_{\alpha_k+1}\right)(\varepsilon x)}{c_N(\varepsilon)} \xrightarrow{\mathcal{D}'} 0, \ \varepsilon \to 0^+, \quad \text{for every } N \in \Lambda,$$

then we say that

$$f(\varepsilon x) \overset{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x) \quad \text{at zero in } \mathcal{D}'$$

with respect to the scale $\{c_k(\varepsilon);\ k \in \Lambda\}$.

The next assertion shows the coherence of quasiasymptotic expansions in Colombeau and Schwartz spaces.

A measurable and positive function L defined on $(0, M)$, $M > 0$, is called slowly varying at 0 if

$$\lim_{\varepsilon \to 0} \frac{L(\varepsilon t)}{L(\varepsilon)} = 1$$

uniformly for $t \in [a, b] \subset (0, M)$. A function of the form $\rho(x) = x^{\alpha} L(x)$, $x \in (0, M)$ is called a regularly varying function. $\square$

Proposition 2. *Let $f(x) \in \mathcal{E}'(\Omega)$. Then $f(\varepsilon x) \overset{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x)$ with respect to $\{\varepsilon^{\alpha_k} L_k(\varepsilon); k \in \Lambda\}$ if and only if*

$$Cd\, f(\varepsilon x) \overset{\text{q.e.c.}}{\sim} \sum_{k \in \Lambda} A_k F_{\alpha_k+1}(\varepsilon x) \quad \textit{with respect to } \{\varepsilon^{\alpha_k} L_k(\varepsilon);\ k \in \Lambda\}.$$

Proof. It is proved in [7] that an $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behaviour at zero in the sense of convergence in $\mathcal{S}'$ if and only if it has quasiasymptotic behaviour at zero in the sense of convergence in $\mathcal{D}'$.

The same result is true for the quasiasymptotic expansion at zero. If $f \in \mathcal{S}'(\mathbb{R})$ then $f(\varepsilon x) \overset{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x)$ with respect to $\{c_k(\varepsilon);\ k \in \Lambda\}$ in the sense of convergence in $\mathcal{S}'$ iff $f(\varepsilon x) \overset{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x)$ with respect to $\{c_k(\varepsilon);\ k \in \Lambda\}$ in the sense of convergence in $\mathcal{D}'$.

Let $\alpha \in \mathcal{S}$, $\eta > 0$, $\varepsilon > 0$. We have

$$\left\langle \frac{(f * \phi_\eta)(\varepsilon x) - \sum_{k=1}^m A_m(f_{\alpha_k+1} * \phi_\eta)(\varepsilon x)}{c_m(\varepsilon)},\ \alpha(x) \right\rangle$$

$$= \left\langle \frac{f(x) - \sum_{k=1}^m A_k f_{\alpha_k+1}(x)}{\varepsilon c_m(\varepsilon)},\ (\check{\phi}_\eta(t) * \alpha(t/\varepsilon))(x) \right\rangle$$

and for $\eta = \varepsilon^2$

$$= \left\langle \frac{(f * \phi_{\varepsilon^2})(\varepsilon x) - \sum_{k=1}^m A_k(f_{\alpha_k+1} * \phi_{\varepsilon^2})(\varepsilon x)}{c_m(\varepsilon)},\ \alpha(x) \right\rangle$$

$$= \left\langle \frac{f(\varepsilon x) - \sum_{k=1}^m A_k f_{\alpha_k+1}(\varepsilon x)}{c_m(\varepsilon)},\ \int_{-\infty}^{\infty} \check{\phi}_{\varepsilon^2}(t)\alpha(x - t/\varepsilon)dt \right\rangle$$

$$= \left\langle \frac{f(\varepsilon x) - \sum_{k=1}^m A_k f_{\alpha_k+1}(\varepsilon x)}{c_m(\varepsilon)},\ \psi_\varepsilon(x) \right\rangle,$$

where

$$\psi_\varepsilon(x) = \int_{-\infty}^{\infty} \check{\phi}_{\varepsilon^2}(t)\alpha(x - t/\varepsilon)dt = \int_{-\infty}^{\infty} \check{\phi}(t)\alpha(x - \varepsilon t)dt.$$

Since $\{\psi_\varepsilon; \varepsilon \in (0,1)\}$ is a bounded set in $\mathcal{S}$ and $\psi_\varepsilon \to \alpha$ in $\mathcal{S}$, it follows

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{(f * \phi_{\varepsilon^2})(\varepsilon x) - \sum_{k=0}^m A_k(f_{\alpha_k+1} * \phi_{\varepsilon^2})(\varepsilon x)}{c_m(\varepsilon)},\ \alpha(x) \right\rangle$$

$$= \left\langle g(x) - \sum_{k=0}^m A_k f_{\alpha_k+1}(x),\ \alpha(x) \right\rangle.$$

This implies the assertion in both directions.

4. Generalized Watson lemma

The Laplace transformation $\mathcal{L}_g$ for an element $G \in \mathcal{G}_t(\mathbb{R})$ supported by $[0,\infty)$ is defined in [4] by

$$\mathcal{L}_g(G)(p) = \left[\int\limits_{\mathbb{R}} e^{-px} G_\varepsilon(x)\eta(x)dx\right], \quad \operatorname{Re} p > 0,$$

where G_ε is a representative of G, and $\eta \in C_0^\infty(\mathbb{R})$ has the properties $|\eta| \le 1$; $\eta = 1$ on $[-a/2,\infty)$ and $\eta = 0$ on $(-\infty,-a)$ for some $a > 0$.

We will use the notation $\mathcal{L}$ for the usual distributional or classical Laplace transformation (if it exists).

It is well known that

$$\begin{aligned}(3) \qquad \mathcal{L}_g(F_{\alpha+1,\varepsilon})(\lambda) &= \mathcal{L}(f_{\alpha+1} * \phi_{\varepsilon^2})(\lambda) = \mathcal{L}(f_{\alpha+1})\mathcal{L}(\phi_{\varepsilon^2})(\lambda)\\ &= (-i\lambda)^{-\alpha-1}\mathcal{L}(\phi)(\varepsilon^2\lambda), \quad \operatorname{Re}\lambda > 0.\end{aligned}$$

We will give an Abelian-type result for the Laplace transformation in $\mathcal{G}_t$ by using the quasiasymptotic expansion given in Remark 2.

Proposition 3. *Let α_k, $k \in \mathbb{N}$, be an increasing sequence of complex numbers, A_k, $k \in \mathbb{N}$, be a sequence of real numbers and let $c_k(\varepsilon)$, $k \in \mathbb{N}$, be a sequence in K which satisfies* (1).

Assume that $G \in \mathcal{G}_t(\mathbb{R})$, $\operatorname{supp} G \subset [0,\infty)$ and G has a representative G_ε with the property $\operatorname{supp} G_\varepsilon \subset [-b\varepsilon,\infty)$, $\varepsilon \le \varepsilon_0$, for some $b > 0$. Let

$$G(\varepsilon x) \overset{\text{q.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k F_{\alpha_k+1}(\varepsilon x) \quad \textit{with respect to } \{c_k(\varepsilon),\ k \in \mathbb{N}\}.$$

Then

$$\mathcal{L}_g(G)\left(\frac{x}{\varepsilon}\right) \overset{\text{s.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k i^{\alpha_k+1}\left[t^{-\alpha_k-1}\mathcal{L}(\phi)(\varepsilon^3 t)\right]\left(\frac{x}{\varepsilon}\right) \quad \textit{at } \infty,$$

with respect to $\{c_k(\varepsilon);\ k \in \mathbb{N}\}$.

Proof. We again use the fact that

$$f(\varepsilon x) \overset{\text{q.e.}}{\sim} \sum_{k=1}^{m} A_k f_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to } \{c_k(\varepsilon); k \in \Lambda\}$$

in the sense of convergence in $\mathcal{S}'$ if and only if this holds in the sense of convergence in $\mathcal{D}'$.

Let $a = 2b$ and η be defined as above. We have

$$G_\varepsilon(x) = \eta\left(\frac{x}{\varepsilon}\right) G_\varepsilon(x), \quad x \in \mathbb{R},\ \varepsilon \leq \varepsilon_0.$$

This implies

$$\frac{1}{\varepsilon c(\varepsilon)} \mathcal{L}_g(G_\varepsilon)\left(\frac{p}{\varepsilon}\right) = \frac{1}{\varepsilon c(\varepsilon)} \int_{\mathbb{R}} e^{-pu} G_\varepsilon(u) \eta\left(\frac{u}{\varepsilon}\right) \eta(u) du$$
$$= \frac{1}{c(\varepsilon)} \int_{\mathbb{R}} e^{-pu} G_\varepsilon(u\varepsilon) \eta(u) \eta(u\varepsilon) du, \quad p > 0.$$

Thus, if $\frac{G_\varepsilon(u\varepsilon)}{c(\varepsilon)}$ converges in $\mathcal{S}'$ as $\varepsilon \to 0^+$, then by using the fact that $\{e^{-pu}\eta(u\varepsilon)\eta(u);\ \varepsilon \in (0,1)\}$ is bounded in $\mathcal{S}$ it follows that $\frac{\mathcal{L}_g(G_\varepsilon)\left(\frac{p}{\varepsilon}\right)}{\varepsilon c(\varepsilon)}$ converges as $\varepsilon \to 0^+$ for every $p > 0$, $\operatorname{Re} p > 0$.

By applying (3) and previous arguments to

$$\frac{1}{c_N(\varepsilon)} \left[\mathcal{L}_g(G_\varepsilon)\left(\frac{x}{\varepsilon}\right) - \sum_{k=1}^{N} A_k \mathcal{L}_g(F_{\alpha_k+1})\left(\frac{x}{\varepsilon}\right) \right], \quad \forall N \in \mathbb{N},$$

we obtain

$$\mathcal{L}_g(G)\left(\frac{x}{\varepsilon}\right) \overset{\text{s.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k \left(-i\frac{x}{\varepsilon}\right)^{-\alpha_k - 1} \left[(\mathcal{L}\phi)(\varepsilon^3 x)\right]\left(\frac{x}{\varepsilon}\right) \quad \text{at } \infty,$$

with respect to $\{c_k;\ k \in \mathbb{N}\}$. $\square$

In the next proposition we define an element of $\mathcal{G} \backslash \mathcal{D}'$ which we call δ^2. Note that "various squares of δ" can be defined in this way.

Proposition 4. *Let* $\delta^2 = [\frac{1}{\varepsilon^4}\phi^2(\frac{t}{\varepsilon^2})]$, *where* $\phi \in C_0^\infty$, $\int \phi = 1$, $\int x^m \phi(x) dx = 0$, $m = 1, \ldots, N$, $N \in \mathbb{N}$. *Then*

a) $$\delta^2(\varepsilon x) \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \left[\frac{1}{\varepsilon^4}\phi^{(k)}\left(\frac{t}{\varepsilon^2}\right)\right](\varepsilon x)$$

with respect to the scale $\{\varepsilon^{-3+m},\ m = 0, 1, \ldots, N\}$.

b) $$\big(\mathcal{L}_g(\delta^2)\big)\Big(\frac{x}{\varepsilon}\Big) \overset{\text{s.e.c.}}{\sim} \sum_{k=0}^{m} (-1)^k \frac{\mu_k}{k!} \Big[\varepsilon^{2k} t^k \mathcal{L}\Big(\phi\Big(\frac{u}{\varepsilon^2}\Big)\Big)(t)\Big]\Big(\frac{x}{\varepsilon}\Big), \ \ \textit{at } \infty,$$

with respect to the scale $\{\varepsilon^{-3+m};\ m=0,1,\dots,N\}$.

PROOF. a) Let

$$\mu_k=\int x^k\phi^2(x)dx,\quad k\le N.$$

We have (as $\varepsilon\to 0^+$)

$$\frac{1}{\varepsilon^{-3}}\int\left(\frac{1}{\varepsilon^4}\phi^2\left(\frac{\varepsilon x}{\varepsilon^2}\right)-\frac{\mu_0}{\varepsilon^4}\phi\left(\frac{\varepsilon x}{\varepsilon^2}\right)\right)\psi(x)dx\to 0,$$

$$\frac{1}{\varepsilon^{-2}}\int\left(\frac{1}{\varepsilon^4}\phi^2\left(\frac{\varepsilon x}{\varepsilon^2}\right)-\frac{\mu_0}{\varepsilon^4}\phi\left(\frac{\varepsilon x}{\varepsilon^2}\right)+\frac{\mu_1}{\varepsilon^4}\phi'\left(\frac{\varepsilon x}{\varepsilon^2}\right)\right)\psi(x)dx\to 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\frac{1}{\varepsilon^{s}}\int\Bigg(\frac{1}{\varepsilon^4}\phi^2\left(\frac{\varepsilon x}{\varepsilon^2}\right)-\frac{\mu_0}{\varepsilon^4}\phi\left(\frac{\varepsilon x}{\varepsilon^2}\right)+\cdots+\frac{(-1)^{s+2}}{\varepsilon^4(s+3)!}\phi^{(s+3)}\left(\frac{\varepsilon x}{\varepsilon^2}\right)\Bigg)&\\ \times\psi(x)dx\to 0,\quad s\in\mathbb{N}.&\end{aligned}$$

Thus,

$$\left[\frac{1}{\varepsilon^4}\phi^2\left(\frac{t}{\varepsilon^2}\right)\right](\varepsilon x)\overset{\text{q.e.c.}}{\sim}\sum_{k=0}^{m}(-1)^k\frac{\mu_k}{k!}\left[\frac{1}{\varepsilon^4}\phi^{(k)}\left(\frac{t}{\varepsilon^2}\right)\right](\varepsilon x)$$

with respect to the scale $\{\varepsilon^{-3+m};\ m=0,1,\dots,N\}$.

b) Let $m\le N$. We have

$$\begin{aligned}\left(\mathcal{L}_g(\delta^2)\right)\left(\frac{x}{\varepsilon}\right)&=\left[\mathcal{L}\left(\frac{1}{\varepsilon^4}\phi^2\left(\frac{t}{\varepsilon^2}\right)\right)\left(\frac{x}{\varepsilon}\right)\right]\\&=\left[\frac{1}{\varepsilon^2}\left(\mathcal{L}(\phi^2)\right)(\varepsilon x)\right],\qquad x>0,\end{aligned}$$

$$\begin{aligned}\mathcal{L}\left(\phi^{(k)}\left(\frac{t}{\varepsilon^2}\right)\right)(x)&=\int\exp(-xt)\phi^{(k)}\left(\frac{t}{\varepsilon^2}\right)dt\\&=\varepsilon^{2k}x^k\left(\mathcal{L}\phi\left(\frac{t}{\varepsilon^2}\right)\right)(x),\qquad x>0,\end{aligned}$$

and

$$\frac{1}{\varepsilon^{-3+m}}\mathcal{L}\left(\frac{1}{\varepsilon^4}\phi^2\left(\frac{\varepsilon t}{\varepsilon^2}\right)-\frac{1}{\varepsilon^4}\sum_{k=0}^{m}(-1)^k\frac{\mu_k}{k!}\phi^{(k)}\left(\frac{\varepsilon t}{\varepsilon^2}\right)\right)(x)\to 0,$$
$$x>0,\ \text{as}\ \varepsilon\to 0^+.$$

This implies

$$\frac{1}{\varepsilon^{-2+m}}\mathcal{L}\left(\frac{1}{\varepsilon^4}\phi^2\left(\frac{t}{\varepsilon^2}\right)\right)\left(\frac{x}{\varepsilon}\right)$$
$$-\frac{1}{\varepsilon^{-2+m}}\sum_{k=0}^{m}\frac{(-1)^k\mu_k}{k!}\mathcal{L}\left(\phi^{(k)}\left(\frac{t}{\varepsilon^2}\right)\right)\left(\frac{x}{\varepsilon}\right)\to 0,$$

$x>0$, $\varepsilon\to 0^+$, and

$$\mathcal{L}_g(\delta^2)\left(\frac{x}{\varepsilon}\right)\overset{\text{s.e.c.}}{\sim}\sum_{k=0}^{m}(-1)^k\frac{\mu_k}{k!}\left[\varepsilon^{2k}t^k\mathcal{L}\left(\phi\left(\frac{t}{\varepsilon^2}\right)\right)(t)\right]\left(\frac{x}{\varepsilon}\right)\quad\text{at }\infty,$$

with respect to the scale $\{\varepsilon^{-3+m};\ m=0,1,\dots,N\}$.

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