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Hardy classes and Briot–Bouquet differential subordinations

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Abstract. In this paper we determined Hardy classes for the operator of Singh and his logarithm when it is applied to a function satisfying an Briot–Bouquet differential subordination.

1. Introduction

In this paper we obtain a result for Hardy classes of some integral operators, applying some functions satisfying Briot–Bonquet differential subordination.

Let β and γ be complex numbers, let h be univalent in the unit disk U, and let $p(z) = h(0) + p_1 z + \dots$ be analytic in U and satisfy

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z). \tag{1}$$

This first-order differential subordination is said to be of Briot–Bouquet type.

This particular differential subordination has some special properties and has many applications in the theory of univalent functions.

In 1973 R. Singh has shown that if

$$I[f](z) = \left[\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma - 1} dt\right]^{\frac{1}{\beta}} \tag{2}$$

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for $\beta, \gamma = 1, 2, \ldots$ then $I(S^*) \subset S^*$, where S^* is the class of starlike functions.

In [2] we determined Hardy classes of Singh's operator. Here we obtain Hardy classes for the operator of Singh and its logarithm when it is applied to a function satisfying a Briot–Bouquet differential subordination.

2. Preliminaries

For $f \in \mathcal{H}(U)$ and $z = re^{i\theta}$ we denote

$$M(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \quad \text{for } 0 < p < \infty$$

$$M(r,f) = \sup_{0\leq\theta\leq 2\pi} \lfloor f(re^{i\theta})\rfloor, \quad \text{for } p = \infty.$$

A function is said to be of Hardy class H^p, $0 < p < \infty$, if $M(r,f)$ remains bounded as $r \to 1^-$. H^∞ is the class of bounded analytic functions in the unit disk.

First we introduce a special mapping from U onto a slit domain [4].

Let c be a complex number such that $\operatorname{Re} c > 0$ and

$$N = N(c) = \frac{1}{\operatorname{Re} c}\left[|c|(1 + 2\operatorname{Re} c)^{\frac{1}{2}} + \operatorname{Im} c\right].$$

If h is the univalent function $h(z) = \frac{2Nz}{1-z^2}$ and $b = h^{-1}(c)$ then we define the "open door" function Q_c as

$$Q_c(z) = h\left(\frac{z+b}{1+\bar{b}z}\right), \quad z \in U.$$

From its definition we see that Q_c is univalent, $Q_c(0) = c$, and $Q_c(U) = h(U)$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geq N$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leq -N$.

Let β and γ be complex numbers with $\beta \neq 0$, let Q_c be the univalent function given, and define the following subclasses of analytic functions:

$$K_{\beta,\gamma} = \{f \in A \mid \operatorname{Re}(\beta z f'(z)/f(z) + \gamma) > 0\}$$

$$H_{\beta,\gamma} = \{f \in A \mid \beta z f'(z)/f(z) z\gamma \prec Q_{\beta+\gamma}\},$$

where A denotes the class of functions f that are analytic in the unit disk U, normalized by $f(0)=0$ and $f'(0)=1$.

From the properties of the open door function $Q_{\beta+\gamma}$ we have $K_{\beta,\gamma}\subset H_{\beta,\gamma}$.

Lemma 1 ([4]). *Let h be convex in U with $\mathrm{Re}(\beta h(z)+\gamma)>0$. If p is analytic in U with $p(0)=h(0)$ and if p satisfies (1) then $p(z)\prec h(z)$.*

Lemma 2 ([4]). *Let h be convex in U with $\mathrm{Re}(\beta h(z)+\gamma)>0$ and $h(0)=1$. If $f\in A$ and $\frac{zf'(z)}{f(z)}\prec h(z)$, then $F=I[f]$, and I given by (2) satisfies $\frac{zF'(z)}{F(z)}\prec h(z)$.*

Lemma 3 ([4]). *Let $f\in A$ and let $g\in K_{\beta,\gamma}$ be such that $\frac{zg'(z)}{g(z)}$ is convex. Let $F=I[f]$ and $G=I[g]$, where I is given by (2) and suppose $\frac{zG'(z)}{G(z)}$ is univalent in U.*

If $\frac{zf'(z)}{f(z)}\prec\frac{zg'(z)}{g(z)}$ then $\frac{zF'f(z)}{F(z)}\prec\frac{zG'(z)}{G(z)}$, and this result is sharp.

Lemma 4 ([4]). *Let h be analytic in U, with $h(0)=a$. If $\mathrm{Re}(\beta a+\gamma)>0$ and $\beta h(z)+\gamma\prec Q_{\beta a+\gamma}(z)$, where Q_c is the open door function, then the solution q of the differential equation*

$$q(z)+\frac{zq(z)}{\beta q(z)+\gamma}=h(z),\quad q(0)=h(0) \tag{3}$$

is analytic and $\mathrm{Re}(\beta q(z)+\gamma)>0$.

3. Main results

Theorem 1. *If h is a convex function with $\mathrm{Re}(\beta h(z)+\gamma)>0$ and p is analytic in U with $p(0)=h(0)$ and satisfies (1), then*

(i) *if $\beta>n\lambda$, $0<\lambda<1$ then $I^n(p)\in H^{\frac{\beta\lambda}{\beta-n\lambda}}$.*

(ii) *if $\beta\le n\lambda$, $0<\lambda<1$ then $I^n(p)\in H^{\infty}$, $\Big(I^n=\underbrace{I\circ I\circ\cdots\circ I}_{n}\Big)$.*

Proof. Because h is a convex function, from Lemma 1 we have $p\prec h$. Applying the subordination theorem of Littlewood we obtain $p\in H^{\lambda}$, $\lambda<1$. From Theorem 5 [2] we obtain for the operator of Singh

$$I^n(f)\in H^{\frac{\beta\lambda}{\beta-n\lambda}}\ \text{for}\ \beta-n\lambda>0\ \text{and}\ I^n(f)\in H^{\infty}\ \text{for}\ \beta-n\lambda\le 0.$$

Theorem 2. *Let h be convex in U with $\mathrm{Re}(\beta h(z)+\gamma)>0$ and $h(0)=1$. If $f\in A$ and $\frac{zf'(z)}{f(z)}\prec h(z)$ and I is the operator of Singh, then*

(i) $\log\frac{I[f](z)}{z}\in H^{\infty}$,

(ii) $\left[\log\frac{I[f](z)}{z}\right]'\in H^{\lambda}$, *for all* λ, $\lambda<1$.

Proof. (ii) We have $\left[\log\frac{I[f](z)}{z}\right]'=\frac{[I[f](z)]'}{I[f](z)}-\frac{1}{z}=\frac{1}{z}\left(\frac{z[I[f](z)]'}{I[f](z)}-1\right)$. Hence $\left[\log\frac{I[f](z)}{z}\right]'$ has the same Hardy class as $\frac{z[I[f](z)]'}{I[f]}$ and from Lemma 2 $\frac{z[I[f](z)]'}{I[f]}\prec h(z)$. From the subordination theorem $\frac{z[I[f](z)]'}{I[f]}\in H^{\lambda}$. Hence $\left[\log\frac{I[f](z)}{z}\right)'\in H^{\lambda}$, $\lambda<1$.

(i) From the theorem of Hardy–Littlewood [1] we obtain (i). □

Theorem 3. *Let $f\in A$ and $g\in K_{\beta,\gamma}$ and let $\frac{zg'(z)}{g(z)}$ be a convex function. Also, let $F=I[f]$ and $G=I[g]$ where I is the operator of Singh and suppose $\frac{zG'(z)}{G(z)}$ is univalent in U. If $\frac{zf'(z)}{f(z)}<\frac{zg'(z)}{g(z)}$ then*

(i) $\log\frac{f(z)}{z}\in H^{\infty}$; $\left[\log\frac{f(z)}{z}\right]'\in H^{\lambda}$, $\lambda<1$,

(ii) $\log\frac{F(z)}{z}\in H^{\frac{\lambda}{1-\lambda}}$; $\left[\log\frac{F(z)}{z}\right]\in H^{\lambda}$, $\lambda<\frac{1}{2}$,

(iii) $\log\frac{G(z)}{z}\in H^{\frac{\lambda}{1-\lambda}}$; $\left[\log\frac{G(z)}{z}\right]'\in H^{\lambda}$, $\lambda<\frac{1}{2}$.

Proof. (i) $\left[\log\frac{f(z)}{z}\right]'=\frac{1}{2}\left(\frac{zf'(z)}{f(z)}-1\right)$.

Because $\frac{zg'(z)}{g(z)}$ is convex we have $\frac{zf'(z)}{z}\in H^{\lambda}$, for all $\lambda<1$. Hence $\left[\log\frac{f(z)}{z}\right]'\in H^{\lambda}$, $\lambda<1$ and from the theorem of Hardy–Littlewood [1] $\log\frac{f(z)}{z}\in H^{\infty}$.

(ii) From Lemma 3 we have $\frac{zF'(z)}{F(z)}\prec\frac{zG'(z)}{G(z)}$.

Because $\frac{zG'(z)}{G(z)}$ is univalent we obtain $\frac{zF'(z)}{F(z)}\in H^{\lambda}$, $\lambda<\frac{1}{2}$ and $\left[\log\frac{F(z)}{z}\right]'\in H^{\lambda}$, $\lambda<\frac{1}{2}$. Hence we obtain (ii).

(iii) $\frac{zG'(z)}{G(z)}$ is univalent. Hence $\frac{zG'(z)}{G(z)}\in H^{\lambda}$, $\lambda<\frac{1}{2}$ and analogously to (i) $\left[\log\frac{G(z)}{z}\right]'$ has the same Hardy class as $\frac{zG'(z)}{G(z)}$. Hence $\left[\log\frac{G(z)}{z}\right]'\in H^{\lambda}$, $\lambda<\frac{1}{2}$.

Applying the theorem of Hardy–Littlewood [1] we obtain $\left[\log\frac{G(z)}{z}\right]' \in H^{\lambda}$, $\lambda < \frac{1}{2}$. □

Theorem 4. *Let h be analytic in U with $h(0) = a$. If $\mathrm{Re}(\beta a+\gamma) > 0$ and $\beta h(z) + \gamma \prec Q_{\beta a+\gamma}(z)$ where Q_c is the open door function, then for the solution q of* (3) *and I* (2) *we have*

(i) *if $\beta > n\lambda$, $\lambda < 1$ then $I^n(q) \in H^{\frac{\beta\lambda}{\beta-n\lambda}}$,*

(ii) *if $\beta \le n\lambda$, $\lambda < 1$ then $I^n(q) \in H^{\infty}$.*

Proof. From Lemma 4 we have $\mathrm{Re}(\beta q(z) + \gamma) > 0$.

Let $\beta q(z) + \gamma = f(z)$. Hence $q(z) = \frac{1}{\beta} f(z) - \frac{\gamma}{\beta}$, and q and f have the same Hardy class.

Because $\mathrm{Re}\, f(z) > 0$, from [1] we have $f \in H^{\lambda}$, $\lambda < 1$.

From Theorem 5 [2] we obtain for the operator of Singh

$$I^n[f] \in H^{\frac{\beta\lambda}{\beta-n\lambda}} \text{ for } \beta - \lambda n \ge 0 \text{ and } I^n[f] \in H^{\infty} \text{ for } \beta - \lambda n \le 0. \qquad \square$$

References

[1] P. L. Duren, Theory of H^p spaces, *Academic Press, New York, London*, 1970.

[2] Gh. Miclăuş, Integral operators of Singh and Hardy classes, *Studia Univ. Babeş-Bolyai, Mathematica* **42**, 2 (1997), 71–77.

[3] Gh. Miclăuş, Some integral operators and Hardy classes, *Studia Univ. Babeş-Bolyai, Mathematica* **41**, 3 (1996), 57–64.

[4] S. S. Miller and P. T. Mocanu, The theory and applications of second-order differntial subordinations, *Studia Univ. Babeş-Bolyai, Math.* **34**, 4 (1989), 3–33.

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