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A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces

By A. BRANCIARI (Macerata)

Abstract. We give a fixed point theorem related to the contraction mapping principle of Banach and Caccioppoli; here we have considered generalized metric spaces, that is metric spaces with the triangular inequality replaced by similar ones which involve four or more points instead of three. At the end of the paper an example is provided to show the improvement of our result with respect to the classical one.

1. Introduction

Since the appearing of the contraction mapping principle (see [B] and [C]), a lot of papers were dedicated to the improvement of that result.

Most of these deal with the generalization of the contractive condition, (see the survey-article of Rhoades [R] for a formal discussion, or the work of Meszaros [M] for more recent developments). On the other hand some authors have studied how to generalize fixed point theorems for contractive-type mappings to more general settings (see for example the d-complete topological spaces in [H]).

Our intent is to give a generalization of the Banach–Caccioppoli theorem for a class of spaces containing as proper subset the class of complete metric spaces.

From now on we will denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{Z}^+$ the set of all non-negative integers and by $\mathbb{R}^+$ the set of all non-negative real numbers.

We begin with some preliminary definitions.

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Definition 1.1. Let X be a set and $d : X^2 \to \mathbb{R}^+$ a mapping such that for all $x, y \in X$ and for all distinct point $\xi, \eta \in X$, each of them different from x and y, one has

$$d(x,y) = 0 \iff x = y$$

$$d(x,y) = d(y,x)$$

$$d(x,y) \leq d(x,\xi) + d(\xi,\eta) + d(\eta,y)$$

then we will say that (X,d) is a generalized metric space, (shortly a g.m.s.).

As in the metric setting, such spaces X became topological spaces with neighbourhood basis given by

$$\mathbb{B} = \big\{B(x,r) \mid x \in X,\ r \in \mathbb{R}^+ - \{0\}\big\}$$

where $B(x,r) := \{y \in X; d(x,y) < r\}$ is the *ball of center x and radius r.*

Definition 1.2. Let (X,d) be a g.m.s. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be a Cauchy sequence if for all $\varepsilon > 0$ there exists a natural number $n_\varepsilon \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n \geq n_\varepsilon$, one has $d(x_n, x_{n+m}) < \varepsilon$.

Further a g.m.s. (X,d) will be called complete if every Cauchy sequence in X is convergent.

We observe now that the function d is continuous in each coordinates, in fact if x_n, a, b are distinct points in X $(n \in \mathbb{N})$ and if $\lim_{n\to\infty} x_n = a$ then we have

$$d(x_n,b) \leq d(x_n,x_m) + d(x_m,a) + d(a,b)$$

$$d(a,b) \leq d(a,x_m) + d(x_m,x_n) + d(x_n,b)$$

these two together give

$$|d(x_n,b) - d(a,b)| \leq d(x_n,x_m) + d(x_m,a) \xrightarrow{n,\,m\to\infty} 0$$

which proves the continuity of d with respect to the first coordinate, and thus, by symmetry, in both of them.

2. Main results

Theorem 2.1. *Let (X, d) be a complete g.m.s., $c \in [0, 1[$ and $f : X \to X$ a mapping such that for each $x, y \in X$ one has*

$$d(fx, fy) \leq c\, d(x, y) \tag{2.1}$$

then

(i) *there exists a point $a \in X$ such that for each $x \in X$ one has $\lim_{n\to\infty} f^n x = a$;*

(ii) *$fa = a$ and for each $e \in X$ such that $fe = e$ one has $e = a$;*

(iii) *for all $n \in \mathbb{N}$ one has*

$$d(f^n x, a) \leq \frac{c^n}{1-c} \max\left\{d(x, fx), d(x, f^2 x)\right\}.$$

Proof. Let us start with a generic point $x \in X$ and consider the sequence $(f^n x)_{n\in\mathbb{N}}$; we can suppose that x is not a periodic point, in fact if $f^\nu x = x$ for some $\nu \in \mathbb{N}$ then

$$d(x, fx) = d\left(f^\nu x, f^{\nu+1} x\right) \leq c^\nu d(x, fx)$$

and being $c < 1$ one has $x = fx$ that is a fixed point. Thus in the sequel of the proof we can suppose that $f^n x \neq f^m x$ for all distinct $n, m \in \mathbb{N}$.

Let us now prove that for all $y \in X$ one has

$$\text{(2.2)}\quad \begin{aligned}
&\text{(a)} \quad d\left(y, f^{2k} y\right) \leq \sum_{i=0}^{2k-3} c^i d(y, fy) + c^{2k-2} d(y, f^2 y) && \text{for } k = 2, 3, 4, \dots\\
&\text{(b)} \quad d\left(y, f^{2k+1} y\right) \leq \sum_{i=0}^{2k} c^i d(y, fy) && \text{for } k = 0, 1, 2, \dots
\end{aligned}$$

We first prove part (a) of (2.2) by mathematical induction: for $k = 2$ one has

$$\begin{aligned}
d(y, f^4 y) &\leq d(y, fy) + d(fy, f^2 y) + d(f^2 y, f^4 y)\\
&\leq d(y, fy) + c\, d(y, fy) + c^2 d(y, f^2 y)
\end{aligned}$$

as we wanted. Let us now suppose that for a certain $k_0 \in \mathbb{N}$ (a) of (2.2) is true for all $k \in \mathbb{N}$ such that $2 \leq k \leq k_0$, then

$$\begin{aligned} d\left(y, f^{2k_0+2}y\right) &\leq d(y, fy) + d(fy, f^2y) + d\left(f^2y, f^{2k_0+2}y\right) \\ &\leq d(y, fy) + cd(y, fy) + c^2 d\left(y, f^{2k_0}y\right) \\ &\leq d(y, fy) + c\, d(y, fy) + c^2 \left[\sum_{i=0}^{2k_0-3} c^i d(y, fy) + c^{2k_0-2} d(y, f^2y)\right] \\ &= \sum_{i=0}^{2k_0-1} c^i d(y, fy) + c^{2k_0} d(y, f^2y). \end{aligned}$$

We prove by mathematical induction part (b) of (2.2) too: for $k = 0$ one has $d(y, fy) = d(y, fy)$, while if we suppose that (2.2) part (b) is true for all $k \in \mathbb{Z}^+$ with $0 \leq k \leq k_0$ for a certain $k_0 \in \mathbb{Z}^+$ then for $k_0 + 1$ we still have

$$\begin{aligned} d\left(y, f^{2k_0+3}y\right) &\leq d(y, fy) + d(fy, f^2y) + d\left(f^2y, f^{2k_0+3}y\right) \\ &\leq d(y, fy) + c\, d(y, fy) + c^2 d\left(y, f^{2k_0+1}y\right) \\ &\leq d(y, fy) + c\, d(y, fy) + c^2 \sum_{i=0}^{2k_0} c^i d(y, fy) \\ &= \sum_{i=0}^{2k_0+2} c^i d(y, fy). \end{aligned}$$

Finally by (2.2) for all $n, k \in \mathbb{N}$ (k is allowed to be in $\mathbb{Z}^+$ in the second formula below) one has

$$\begin{aligned} d(f^n x, f^{n+2k}x) &\leq c^n d(x, f^{2k}x) \\ &\leq c^n \sum_{i=0}^{2k-2} c^i \max\{d(x, fx), d(x, f^2x)\} \\ &\leq \frac{c^n}{1-c} \max\{d(x, fx), d(x, f^2x)\} \end{aligned}$$

$$\begin{aligned} d(f^n x, f^{n+2k+1}x) &\le c^n d(x, f^{2k+1}x) \\ &\le c^n \sum_{i=0}^{2k} c^i \max\{d(x,fx), d(x,f^2x)\} \\ &\le \frac{c^n}{1-c} \max\{d(x,fx), d(x,f^2x)\} \end{aligned}$$

that is for all $n, m \in \mathbb{N}$ one has

$$d(f^n x, f^{n+m}x) \le \frac{c^n}{1-c} \max\{d(x,fx), d(x,f^2x)\} \tag{2.3}$$

thus $(f^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d) which is a complete g.m.s., so that there exists a point $a \in X$ such that $a = \lim_{n\to\infty} f^n x$.

Also a is a fixed point for f, in fact one has

$$d(f^{n+1}x, fa) \le c\ d(f^n x, a) \xrightarrow{n\to\infty} 0$$

thus

$$a = \lim_{n\to\infty} f^n x = fa$$

(this is a consequence of the fact that a g.m.s. is a Hausdorff space); if further one has $fe = e$, then

$$d(a,e) = d(fa, fe) \le c\ d(a,e)$$

which implies that $a = e$ and thus (i) and (ii) are satisfied.

To see (iii) it is sufficient to let $m \to \infty$ in (2.3) so that

$$d(f^n x, a) \le \frac{c^n}{1-c}\ \max\left\{d(x,fx), d(x,f^2x)\right\}$$

which ends the proof. $\square$

Now with Definition 1.1 in hand, we can generalize the concept of g.m.s. in the following way:

Definition 2.1. Let X be a set, $\nu \in \mathbb{N}$ and $d : X^2 \to \mathbb{R}^+$ a mapping such that for all $x, y \in X$ and for all distinct point $\xi_i \in X$ $i \in \{1, \dots, \nu\}$, each of them different from x and y, one has ($\xi_0 := x$, $\xi_{\nu+1} := y$)

$$d(x,y) = 0 \iff x = y$$
$$d(x,y) = d(y,x)$$
$$d(x,y) \leq \sum_{i=0}^{\nu} d(\xi_i, \xi_{i+1})$$

then we will say that (X,d) is a generalized metric space of order ν, (shortly a ν-g.m.s.).

Remark 1. According to this a g.m.s. as introduced in Definition 1.1 and a standard metric space are respectively generalized metric spaces of order 2 and 1.

Remark 2. The related definition of balls, Cauchy sequence and completeness of a ν-g.m.s. are the same with "g.m.s." replaced by "ν-g.m.s." ($\nu \in \mathbb{N}$).

Using Definition 2.1 we are now ready to generalize Theorem 2.1:

Theorem 2.2. *Let (X,d) be a complete ν-g.m.s. for some $\nu \in \mathbb{N}$, $c \in [0,1[$ and $f : X \to X$ a mapping such that for each $x, y \in X$ one has* (2.1)*; then*

(i) *there exists a point $a \in X$ such that for each $x \in X$ one has $\lim_{n\to\infty} f^n x = a$;*

(ii) *$fa = a$ and for each $e \in X$ such that $fe = e$ one has $e = a$;*

(iii) *for all $n \in \mathbb{N}$ one has*

$$d(f^n x, a) \leq \frac{c^n}{1-c} \max\left\{d(x, f^i x) \mid i = 1, \ldots, \nu\right\}$$

The proof of Theorem 2.2 is a simple generalization of that one of Theorem 2.1 and we left it to the interested reader.

3. An example

We give here an easy example of a contraction mapping in a 2-g.m.s., that is not a metric space in the usual sense.

Let $X := \{a, b, c, e\}$ and let $d : X^2 \to \mathbb{R}^+$ be the following mapping: $d(a, b) = 3$, $d(a, c) = d(b, c) = 1$, $d(a, e) = d(b, e) = d(c, e) = 2$ and $d(x, x) = 0$ for every $x \in X$; further let $f : X \to X$ be the function:

$$fx \stackrel{\text{def}}{=} \begin{cases} c & \text{if } x \in \{a, b, c\} \\ a & \text{if } x = e \end{cases}$$

then it is easy to see that (X, d) is a generalized metric space of order 2 (i.e. a g.m.s. according to Definition 1.1) and that f is a contraction, in fact for every $x, y \in X$ one has

$$d(fx, fy) \leq \frac{1}{2}\, d(x, y)$$

but (X, d) is not a standard metric space because it lacks the triangular property: if we take a, b and c we have

$$d(a, b) = 3 > 1 + 1 = d(a, c) + d(c, b)$$

We note finally that, as Theorem 2.1 states, f has a unique fixed point, namely c.

References

[B] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* **3** (1922), 133–181.

[C] R. Caccioppoli, Un teorema generale sull' esistenza di elementi uniti in una trasformazione funzionale, *Rend. Accad. dei Lincei* **11** (1930), 794–799.

[H] T. L. Hicks, Fixed point theorems for d-complete topological spaces I, *Internat. J. Math. & Math. Sci.* **15** (1992), 435–440.

[M] J. Meszaros, A comparison of various definitions of contractive type mappings, *Bull. Cal. Math. Soc.* **84** (1992), 167–194.

[R] B. E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.* **226** (1977), 257–290.

A. BRANCIARI
VIALE MARTIRI DELLA LIBERTÀ N. 20
62100 MACERATA
ITALY

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