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Paraopen spaces – a class of peculiar spaces

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Abstract. If a Hausdorff space has a base which consists of clopen sets and each union of $< \kappa$ basic sets is closed, we call it a κ-paraopen space. Properties of such spaces are investigated and this class of spaces is compared with some classes of peculiar spaces. Using these results some results concerning paracompactness in products and topological groups are obtained.

0. Introduction

The definition and some basic properties of paraopen spaces are given in the first section. Although paraopen spaces are very pathological (never k-spaces and always totally non-compact) they are often paracompact and this property is preserved in box products.

In Section 2, this class of spaces is compared with some classes of peculiar spaces. For example, a κ-paraopen space X satisfying $\chi(X) \leq |X| = \kappa$ is extremally disconnected iff it is discrete. Also, X is basically disconnected iff it is an F-space, iff it is a P-space.

A construction of nontrivial paraopen spaces is given in Section 3: a reduced ideal-product (r.i.p.) of a countable family of regular spaces is always ω_1-paraopen.

Paracompactness in products is considered in Section 4. So, Theorem 11 is an extension of results of K. Kunen and M.E. Rudin and the result of P. Bankston. Paracompactness of the special r.i.p. investigated by B. Lawrence is considered in Theorem 14. By the theorems of Section 3 we can construct paraopen spaces which are not κ-metrizable. So,

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Theorem 12 extends the results about box products of κ-metrizable spaces obtained by M.E. Rudin and J.E. Vaughan.

A.V. Arhangel'skii asked whether there is a non-discrete extremally disconnected (n.d.e.d.) topological group. Such groups were constructed: under CH by S.M. Sirota, under MA by A. Louveau and under $P(c)$ by V.I. Malyhin. Sirota's group is countable and it can be embedded into the ideal-product $Y = \prod^{\Lambda} D$ given in Section 5. It is natural to ask whether the group Y or the corresponding r.i.p. $X = \prod_{\Psi}^{\Lambda} D$ is a n.d.e.d. group. This problem is solved (under some set-theoretic assumptions) at the end of the paper.

1. Paraopen spaces (basic facts)

Let κ be an infinite cardinal. We will say that a Hausdorff space $(X, \mathcal{O})$ is *κ-paraopen* iff there is a base $\mathcal{B}$ for the topology $\mathcal{O}$ consisting of clopen sets such that each union of $< \kappa$ elements of $\mathcal{B}$ is a closed set. Such a base will be called *canonical*. By [6] Theorem 1.1.15 it always may be chosen such that $|\mathcal{B}| = w(X)$.

Clearly, κ-paraopen spaces are zero-dimensional and completely regular (i.e. $T_3\frac{1}{2}$). A κ-paraopen space is λ-paraopen for all $\lambda < \kappa$. Discrete spaces are κ-paraopen for each cardinal κ. Moreover, there holds

Theorem 1. *Let $(X, \mathcal{O})$ be a Hausdorff space and $\kappa = \min\{|X|, w(X)\}$. Then the space $(X, \mathcal{O})$ is κ^+-paraopen $\Leftrightarrow$ $(X, \mathcal{O})$ is a discrete space.*

Proof. $(\Rightarrow)$ Let $(X, \mathcal{O})$ be a κ^+-paraopen T_2-space and $x \in X$. Since the set $X \setminus \{x\}$ is open, it is the union of $\leq \kappa$ members of a canonical base, thus it is closed and $\{x\}$ is an open set. $(\Leftarrow)$ is trivial. $\square$

Theorem 2. *κ-paraopenness is a property which is hereditary, additive, finitely multiplicative and preserved in box products. It is preserved by clopen continuous mappings.*

Proof. We prove preservation in box-products only. Let $(X_i, \mathcal{O}_i)$, $i \in I$ be κ-paraopen spaces and $\mathcal{B}_i, i \in I$ the corresponding canonical bases. Then the base $\mathcal{B} = \{\prod_{i\in I} B_i : B_i \in \mathcal{B}_i, i \in I\}$ for the box product $X = \Box_{i\in I} X_i$ consists of clopen sets. Let $\mu < \kappa$ be a cardinal and $B^\alpha \in \mathcal{B}$, $\alpha < \mu$, where $B^\alpha = \prod_{i\in I} \mathcal{B}_i^\alpha$. We prove that $\bigcup_{\alpha<\mu} B^\alpha$ is a closed set. If $f \notin \bigcup_{\alpha<\mu} B^\alpha$ and $S_i = \{\alpha < \mu : f_i \notin B_i^\alpha\}$, then for each $\alpha < \mu$ there is $i \in I$ such that $\alpha \in S_i$, thus $\mu = \bigcup_{i \in I} S_i$. Since $f_i \notin \bigcup_{\alpha \in S_i} B_i^\alpha$ and $|S_i| < \kappa$, the set $\bigcup_{\alpha \in S_i} B_i^\alpha$ is closed and we can choose $V_i \in \mathcal{O}_i$ such that $f_i \in V_i$ and $V_i \cap \bigcup_{\alpha \in S_i} B_i^\alpha = \emptyset$. Now, $V = \prod_{i \in I} V_i$ is a neighbourhood of f which does not intersect $\bigcup_{\alpha < \mu} B^\alpha$. □

The Cantor cube 2^ω is a (Tychonov) product of ω_1-paraopen spaces which is not ω_1-paraopen.

Theorem 3. *Let $(X, \mathcal{O})$ be a κ-paraopen space and $\mathcal{B}$ a canonical base for $\mathcal{O}$. Then*

(a) *Each $Y \in [X]^{<\kappa}$ is a closed discrete subspace of X.*

(b) *If $\kappa > \omega$, then each compact subspace of X is finite (i.e. the space X is totally non-compact).*

(c) *Each open set $O \subset X$ of cardinality $\leq \kappa$ is the union of $\leq \kappa$ disjoint sets from $\mathcal{B}$.*

(d) *For each open $O \subset X$ there are disjoint sets $B_j \in \mathcal{B}$, $j \in J$ such that $B_j \subset O$ and $\overline{\bigcup_{j \in J} B_j} = \overline{O}$.*

(e) *If $\kappa \geq L(X)$, then $(X, \mathcal{O})$ is a strongly paracompact space.*

(f) *If $\kappa \geq \min\{|X|, w(X)\}$, then the space $(X, \mathcal{O})$ is hereditarily strongly paracompact.*

Proof. (a) If $|Y| = \mu < \kappa$, $x \in X$ and $Y \setminus \{x\} = \{y_\alpha : \alpha < \mu\}$, we choose $B_\alpha \in \mathcal{B}$ satisfying $x \notin B_\alpha \ni y_\alpha$ and define $O = X \setminus \bigcup_{\alpha < \mu} B_\alpha$. Then $x \in O \in \mathcal{O}$ and $O \cap Y \subset \{x\}$.

(b) Suppose K is an infinite compact subset of X. Let $K_0 \subset K$, $|K_0| = \omega$. By (a), the set K_0 is closed in X (and clearly in K) thus K_0 is a compact subspace of X. But by (a), K_0 is a discrete subspace of X. A contradiction.

(c) Let $O = \{x_\alpha : \alpha < \mu\}$, where $\mu \leq \kappa$. We define sequences α_ξ, $\xi < \mu$ and B_ξ, $\xi < \mu$ as follows. Let α_ξ and B_ξ be defined for all $\xi < \eta (< \mu)$. If $\bigcup_{\xi < \eta} B_\xi = O$, we put $\alpha_\eta = \mu$ and $B_\eta = \emptyset$. If $\bigcup_{\xi < \eta} B_\xi \neq O$ we define $\alpha_\eta = \min\{\alpha \in \mu : x_\alpha \in O \setminus \bigcup_{\xi < \eta} B_\xi\}$ and choose $B_\eta \in \mathcal{B}$ satisfying $x_{\alpha_\eta} \in B_\eta \subset O \setminus \bigcup_{\xi < \eta} B_\xi$ (such a choice is possible since $\bigcup_{\xi < \eta} B_\xi$ is a closed set). Now, $\{B_\xi : \xi < \mu\} \setminus \{\emptyset\}$ is the desired family.

(d) The set $\mathcal{P} = \{\mathcal{D} : \mathcal{D} \subset \mathcal{B} \wedge \bigcup \mathcal{D} \subset O \wedge \mathcal{D} \text{ is a disjoint family}\}$ partially ordered by the set inclusion satisfies the conditions of the Zorn's lemma. Let $\mathcal{D} = \{B_j : j \in J\}$ be a maximal element of $\mathcal{P}$. Clearly $\mathcal{D} \subset \mathcal{B}$ and $\overline{O} \supset \overline{\bigcup_{j\in J} B_j}$. If $x \in \overline{O}$ and $V \in \mathcal{U}(x)$, then $\emptyset \neq V \cap O \in \mathcal{O}$. Suppose $(V \cap O) \cap \bigcup_{j\in J} B_j = \emptyset$. Then there is $B \in \mathcal{B}$ satisfying $B \subset V \cap O$, a contradiction to the maximality of $\mathcal{D}$. Thus $V \cap \bigcup_{j\in J} B_j \neq \emptyset$ for each $V \in \mathcal{U}(x)$, so $x \in \overline{\bigcup_{j\in J} B_j}$. Hence $\overline{O} \subset \overline{\bigcup_{j\in J} B_j}$.

(e) Let $\kappa \geq L(X)$ and let $\{O_i : i \in I\}$ be an open cover of X. If for $x \in X$ we pick $i_x \in I$ and $B_x \in \mathcal{B}$ such that $x \in B_x \subset O_{i_x}$, then $\{B_x : x \in X\}$ is an open cover of X. It contains a subcover $\{B_\alpha : \alpha < \mu\}$ where $\mu \leq L(X)$, which is an open refinement of $\{O_i : i \in I\}$. Now, we define $V_\alpha = B_\alpha \setminus \bigcup_{\xi<\alpha} B_\xi$, $\alpha < \mu$. Since $|\alpha| < \mu \leq \kappa$, the sets V_α are open. Clearly, $\{V_\alpha : \alpha < \mu\}$ is a disjoint (hence star-finite) open refinement of $\{O_i : i \in I\}$.

(f) Follows from $hL(X) \leq \min\{|X|, w(X)\}$ and (e). □

2. Paraopen vs peculiar spaces

We compare the class of paraopen spaces with some related classes described in [7]. A space $(X, \mathcal{O})$ is *κ-open* iff each intersection of $< \kappa$ open sets is open. P-spaces are completely regular, ω_1-open spaces. For $\kappa > \omega$, all κ-open $T_{3\frac{1}{2}}$ spaces are zero-dimensional. Clearly it holds

Theorem 4. *For each $\kappa > \omega$, κ-open $T_{3\frac{1}{2}}$ spaces are κ-paraopen. Specially, P-spaces are ω_1-paraopen.*

According to [22], Definition 2.7, a completely regular space $(X, \mathcal{O})$ is κ-metrizable iff $uw(X) \leq \kappa$ and $(X, \mathcal{O})$ is a κ-open space. Thus we have

Theorem 5. *For each $\kappa > \omega$, κ-metrizable spaces are κ-paraopen.*

According to [7], a completely regular space $(X, \mathcal{O})$ is *extremally disconnected* iff the closure of each open set is open, it is *basically disconnected* iff the closure of each cozero-set is open, and it is an *F-space* provided any two disjoint cozero-sets are completely separated. The well-known facts are: e.d. $\Rightarrow$ b.d. $\Rightarrow$ F-sp. and P-sp. $\Rightarrow$ b.d. $\Rightarrow$ zero-dim.

Theorem 6. *Let $(X, \mathcal{O})$ be a κ-paraopen, extremally disconnected space. Then*

(a) *Each point $x \in X$ of character $\leq \kappa$ is an isolated point.*

(b) *If $|X| = \kappa$ and κ is a (Ulam) nonmeasurable cardinal, then each singleton is a G_δ-set.*

Proof. If all the points of X are isolated, then (a) and (b) hold trivially.

(a) Let $x \in X$ be a nonisolated point. By Theorem 3(d) there are disjoint $B_\alpha \in \mathcal{B}$, $\alpha < \mu$ satisfying $B_\alpha \subset X \setminus \{x\}$ and $\overline{\bigcup_{\alpha<\mu} B_\alpha} = \overline{X \setminus \{x\}} = X$. Clearly, $\mu \geq \kappa$. If $\mathcal{B}(x) = \{V_\xi : \xi < \lambda\}$ is a local base at x such that $\lambda = \chi(x)$, for each $\xi < \lambda$ we define $S_\xi = \{\alpha \in \mu : V_\xi \cap B_\alpha \neq \emptyset\}$. Now, $|S_\xi| < \kappa$ would imply that $V_\xi \setminus \bigcup_{\alpha \in S_\xi} B_\alpha$ is a neighbourhood of x disjoint from $\bigcup_{\alpha<\mu} B_\alpha$ which is impossible so, $|S_\xi| \geq \kappa$ for all $\xi < \lambda$. Obviously, $\{S_\xi : \xi < \lambda\}$ is a base for some filter Ψ on μ.

Suppose Ψ is not an ultrafilter. Choose $A \subset \mu$ such that $A, \mu \setminus A \notin \Psi$. Define $G = \bigcup_{\alpha \in A} B_\alpha$ and $H = \bigcup_{\alpha \in \mu \setminus A} B_\alpha$. For each $\xi < \lambda$ we have $S_\xi \cap A \neq \emptyset$ and $V_\xi \cap G \neq \emptyset$, so $x \in \overline{G}$. Similarly $x \in \overline{H}$. But G and H are disjoint open sets in an extremally disconnected space so they have disjoint closures. A contradiction.

Thus Ψ is an (clearly non-principal) ultrafilter. By [4] Corollary 7.8, we have $\chi(x) = \lambda > \mu \geq \kappa$.

(b) By Theorem 3(c) there are disjoint $B_\alpha \in \mathcal{B}$, $\alpha < \mu$ such that $X \setminus \{x\} = \bigcup_{\alpha<\mu} B_\alpha$. Now $\mu = \kappa$. Like in (a) we define S_ξ's and Ψ and we conclude that Ψ is non-principal ultrafilter on κ. Since κ is nonmeasurable, Ψ is countably incomplete i.e. there are $\xi_n \in \lambda$, $n \in \omega$ such that $\bigcap_{n\in\omega} S_{\xi_n} = \emptyset$.

Suppose $y \in \bigcap_{n\in\omega} V_{\xi_n} \setminus \{x\}$. Then $y \in B_{\alpha_0}$ for some $\alpha_0 \in \kappa$, hence $B_{\alpha_0} \cap V_{\xi_n} \neq \emptyset$ for all $n \in \omega$ and $\alpha_0 \in \bigcap_{n\in\omega} S_{\xi_n}$. A contradiction. Thus $\bigcap_{n\in\omega} V_{\xi_n} = \{x\}$, that is $\{x\}$ is a G_δ-set. □

Theorem 7. *Let $(X, \mathcal{O})$ be a κ-paraopen F-space, where $|X| = \kappa$. Then each point $x \in X$ of character $\leq \kappa$ is a P-point.*

Proof. Suppose that for some $x \in X$, $\chi(x) = \mu \leq \kappa$ and x is not a P-point. Let $\mathcal{B}$ be a canonical base for $\mathcal{O}$ and $U_n \in \mathcal{B}$, $n \in \omega$, such that $x \in \bigcap_{n\in\omega} U_n \notin \mathcal{O}$. Then $O = X \setminus \bigcap_{n\in\omega} U_n$ is an open F_σ-set and $V \cap O \neq \emptyset$ for each $V \in \mathcal{U}(x)$, that is $x \in \overline{O}$. By Theorem 3(c), there are disjoint $B_\alpha \in \mathcal{B}$, $\alpha < \lambda$ such that $O = \bigcup_{\alpha<\lambda} B_\alpha$. Since O is not closed, $\lambda = \kappa$. So, $O = \bigcup_{\alpha<\kappa} B_\alpha$.

Let $\mathcal{B}(x) = \{V_\xi : \xi < \mu\}$ be a local base at x. Again define $S_\xi = \{\alpha < \kappa : V_\xi \cap B_\alpha \neq \emptyset\}$, $\xi < \mu$. Like in the preceding theorem, $x \in \overline{O}$ gives $|S_\xi| = \kappa$ for each $\xi < \mu$ and $\{S_\xi : \xi < \mu\}$ is a base for the filter Ψ on κ. Since $\mu \leq \kappa, \Psi$ is not an ultrafilter. Thus there is $A \subset \kappa$ such that $A, \kappa \setminus A \notin \Psi$. Let $G = \bigcup_{\alpha\in A} B_\alpha$ and $H = \bigcup_{\alpha\in\kappa\setminus A} B_\alpha$.

By Theorem 3(f), X is a T_4-space so, by [6] Corollary 1.5.12, G and H are disjoint cozero-sets. But $\overline{G} \cap \overline{H} \neq \emptyset$, thus G and H are not completely separated. A contradiction because X is an F-space. □

Corollary 1. *Let $(X, \mathcal{O})$ be a $|X|$-paraopen space, where $|X| > \omega$ and $\chi(X) \leq |X|$. Then there holds*

(a) *X is extremally disconnected $\Leftrightarrow$ X is discrete.*

(b) *X is basically disconnected $\Leftrightarrow$ X is a F-space $\Leftrightarrow$ X is a P-space.*

Example 1. Let κ be an infinite cardinal, Ψ an uniform ultrafilter on κ and $p \notin \kappa$. The family $\mathcal{B} = \{\{\alpha\} : \alpha \in \kappa\} \cup \{\{p\} \cup F : F \in \Psi\}$ is a base for some topology $\mathcal{O}$ on $X = \kappa \cup \{p\}$. It is easy to verify that $(X, \mathcal{O})$ is a $|X|$-paraopen, extremally disconnected, non-discrete space. If κ is nonmeasurable, then, by a theorem of Isbell (see 12H of [7]), $(X, \mathcal{O})$ is not a P-space.

A Hausdorff space $(X, \mathcal{O})$ is a *k-space* iff each $A \subset X$ such that $K \cap A$ is closed for all compact $K \subset X$, is closed in X. By Theorem 3(b) we have

Theorem 8. *Let $(X, \mathcal{O})$ be a κ-paraopen space where $\kappa > \omega$. Then, X is a k-space $\Leftrightarrow$ X is discrete.*

The space described in Section 5 is (under CH or MA or some weaker assumptions) an example of a paraopen space which is not an F-space.

According to M. Henriksen (see [11]) a base $\mathcal{B}$ for a topological space is called *pretty* if $\mathcal{B}$ consists of clopen sets and the closure of each countable union of elements of $\mathcal{B}$ is in $\mathcal{B}$. Clearly, each ω_1-paraopen space has a pretty base (this is the family of all countable unions of elements of the canonical base). The two classes are not equal because the spaces with a cocompact pretty base are not ω_1-paraopen. (These spaces are, by [11], ω-bounded, but Theorem 3(b) holds.)

3. A construction of paraopen spaces

Completely regular κ-open spaces and specially, κ-metrizable spaces are the trivial examples of κ-paraopen spaces. A κ-paraopen space which is not κ-open was constructed in Example 1. Now, the construction given in [9] will be used for obtaining more such spaces.

Given a family of spaces $(X_i, \mathcal{O}_i)$, $i \in I$ and an ideal $\Lambda \subset P(I)$ we observe the topology $\mathcal{O}^\Lambda$ on $\prod X_i$ defined by the base $\mathcal{B}^\Lambda = \{\bigcap_{i\in L} \pi_i^{-1}(O_i) : L \in \Lambda, O_i \in \mathcal{O}_i\}$. The space $(\prod_{i\in I} X_i, \mathcal{O}^\Lambda)$ will be called the ideal-product and denoted by $\prod^\Lambda X_i$. If Ψ is a filter on $P(I)$ it determines the equivalence relation $\sim$ on $\prod X_i$ given by: $f \sim g$ iff $\{i \in I : f_i = g_i\} \in \Psi$. The quotient space $(\prod X_i, \mathcal{O}^\Lambda)/\sim$ is the reduced ideal-product (r.i.p.) in notation $\prod_\Psi^\Lambda X_i$. The quotient mapping $q : \prod X_i \to \prod X_i/\sim$ assigns to each $f \in \prod X_i$ its equivalence class $[f]$ that is $q(f) = [f]$.

"Nice" r.i.p.'s preserve separation axioms T_k, for $k \le 3\frac{1}{2}$. By [9], this holds iff the condition

$$(\Lambda\Psi) \qquad \forall B \notin \Psi \quad \exists L \in \Lambda \quad (L \subset B^c \wedge L^c \notin \Psi)$$

is satisfied. Some special "nice" r.i.p.'s are Tychonov products, box products, reduced products and ultraproducts.

If I is an infinite set and $\Phi \subset P(I)$ an ultrafilter, then $\Lambda = \{I\backslash F : F \in \Phi\}$ is a maximal ideal on $P(I)$ (i.e. for each $A \subset I$, $A \in \Lambda$ or $I\backslash A \in \Lambda$). In the sequel we will observe r.i.p.'s $\prod_\Psi^\Lambda X_i$ where Λ is as above and $\Psi = \{A \subset I : |I \setminus A| < |I|\}$. Firstly we check that such r.i.p.'s are "nice" and prove one combinatorial lemma.

Lemma 1. *Let κ be an infinite cardinal. Then each maximal ideal $\Lambda \subset P(\kappa)$ and the filter $\Psi = \{F \subset \kappa : |\kappa\backslash F| < \kappa\}$ satisfy the condition $(\Lambda\Psi)$.*

Proof. If $B \notin \Psi$, then $|\kappa\backslash B| = \kappa$. Let $\kappa\backslash B = L_1 \cup L_2$ where $L_1 \cap L_2 = \emptyset$ and $|L_1| = |L_2| = \kappa$. Then $L_1 \in \Lambda$ or $L_2 \in \Lambda$. Suppose $L_1 \in \Lambda$. Clearly $L_1 \subset \kappa\backslash B$ and since $|L_1| = \kappa$ we have $L_1^c \notin \Psi$. □

Lemma 2. *Let $\kappa \ge \omega$ be a cardinal satisfying $\kappa^{\underline{\kappa}} = \kappa$ and $\Lambda \subset P(\kappa)$ a maximal ideal. If $\Delta_\alpha \in \Lambda$, $\alpha \in \kappa$, and $|\Delta_\alpha| = \kappa$ for each $\alpha \in \kappa$, then*

$$\exists L \in \Lambda \quad \forall \alpha \in \kappa \quad |L \cap \Delta_\alpha| = \kappa.$$

Proof. Suppose that $\forall L \in \Lambda \quad \exists \alpha \in \kappa(|L \cap \Delta_\alpha| < \kappa)$. Let $\mathcal{D} = \{\Delta_\alpha\backslash K : \alpha < \kappa,\ K \in [\kappa]^{<\kappa}\}$. Since $|[\kappa]^{<\kappa}| = \kappa^{\underline{\kappa}} = \kappa$ we have $|\mathcal{D}| = \kappa$, thus there is an enumeration $\mathcal{D} = \{D_\beta : \beta \in \kappa\}$. By the assumption $\forall L \in \Lambda\ \exists \beta \in \kappa (L \cap D_\beta) = \emptyset$ so, if Φ is the ultrafilter corresponding to Λ, then $\forall F \in \Phi\ \exists \beta \in \kappa (D_\beta \subset F)$. Since $|D_\beta| = \kappa$, $\beta < \kappa$, by the Disjoint refinement lemma (see [4], pp. 146) there are disjoint $S_\beta, \beta \in \kappa$ such that $S_\beta \subset D_\beta$ and $|S_\beta| = \kappa$, for all $\beta < \kappa$. Now $\forall F \in \Phi\ \exists \beta \in \kappa (S_\beta \subset F)$. We pick $s_\beta \in S_\beta$, $\beta < \kappa$ and define $S = \{s_\beta : \beta < \kappa\}$. So, $F \cap S \neq \emptyset$ for each $F \in \Phi$ hence $S \in \Phi$ since Φ is an ultrafilter. But there is $\beta_0 \in \kappa$ satisfying $S_{\beta_0} \subset S$ which is impossible, because $S_{\beta_0} \cap S = \{s_{\beta_0}\}$. □

Theorem 9. *Let $\kappa \geq \omega$ be a cardinal satisfying $\kappa^{\underline{\kappa}} = \kappa$, $\Lambda \subset P(\kappa)$ a non-principal maximal ideal and $\Psi = \{F \subset \kappa : |\kappa \backslash F| < \kappa\}$. If $(X_i, \mathcal{O}_i)$, $i \in \kappa$ are zero-dimensional, κ-open spaces, then $X = \prod_{\Psi}^{\Lambda} X_i$ is a κ^+-paraopen space.*

Proof. Let $\mathcal{B}_i$, $i \in \kappa$ be the bases for the topologies $\mathcal{O}_i$ consisting of clopen sets. The family $\mathcal{B}$ of sets of the form $q(\bigcap_{i \in L} \pi_i^{-1}(B_i))$, where $L \in \Lambda$ and $B_i \in \mathcal{B}_i$ is a clopen base for the topology $\mathcal{O}$ on X. Thus, X is a zero-dimensional space. We show that any union of κ elements of $\mathcal{B}$ is closed in X.

Suppose $B^\alpha \in \mathcal{B}$, $\alpha < \kappa$, where $B^\alpha = q(\bigcap_{i \in L_\alpha} \pi_i^{-1}(B_i^\alpha))$ and $[f] \notin \bigcup_{\alpha < \kappa} B^\alpha$. Then for $\Delta_\alpha = \{i \in L_\alpha : f_i \notin B_i^\alpha\}$, $\alpha < \kappa$, we have $|\Delta_\alpha| = \kappa$. By the previous lemma there is $L \in \Lambda$ such that

$$|L \cap \Delta_\alpha| = \kappa, \quad \text{for all } \alpha < \kappa. \tag{1}$$

For each $\alpha < \kappa$ we define open sets $U_i^\alpha, i \in \kappa$ as follows : if $i \in L \cap \Delta_\alpha$, we choose $U_i^\alpha \in \mathcal{O}_i$ satisfying $f_i \in U_i^\alpha \subset X_i \backslash B_i^\alpha$; if $i \notin L \cap \Delta_\alpha$, we put $U_i^\alpha = X_i$. We also define $V_i = \bigcap_{\alpha=0}^{i} U_i^\alpha$ for all $i \in \kappa$. Clearly

$$\forall \alpha < \kappa \quad \forall i \geq \alpha \quad V_i \subset U_i^\alpha \tag{2}$$

and $V_i \in \mathcal{O}_i$ (X_i are κ-open). Moreover $f_i \in V_i$ for all $i \in \kappa$, so for $B = q(\bigcap_{i \in L} \pi_i^{-1}(V_i))$ we have $[f] \in B \in \mathcal{O}$. It remains to show that $B \cap \bigcup_{\alpha < \kappa} B^\alpha = \emptyset$. Suppose that there is $[g] \in B \cap \bigcup_{\alpha < \kappa} B^\alpha$. Then

$$|\{i \in L : g_i \notin V_i\}| < \kappa \tag{3}$$

and $[g] \in B^{\alpha_0}$ for some $\alpha_0 \in \kappa$, that is $|\{i \in L_{\alpha_0} : g_i \notin B_i^{\alpha_0}\}| < \kappa$. Since $\Delta_{\alpha_0} \subset L_{\alpha_0}$ it follows that

$$|\{i \in \Delta_{\alpha_0} : g_i \notin B_i^{\alpha_0}\}| < \kappa. \tag{4}$$

By (3) and (4), $|\{i \in L \cap \Delta_{\alpha_0} : g_i \notin V_i \vee g_i \notin B_i^{\alpha_0}\}| < \kappa$ and since (1) gives $|L \cap \Delta_{\alpha_0}| = \kappa$ we have

$$(5) \qquad |\{i \in L \cap \Delta_{\alpha_0} : g_i \in V_i \cap B_i^{\alpha_0}\}| = \kappa.$$

The last set is cofinal in κ, so there is $i_0 \geq \alpha_0$ such that $i_0 \in L \cap \Delta_{\alpha_0}$ and $g_{i_0} \in V_{i_0} \cap B_{i_0}^{\alpha_0}$. By (2), $V_{i_0} \subset U_{i_0}^{\alpha_0}$ and we have $U_{i_0}^{\alpha_0} \cap B_{i_0}^{\alpha_0} \neq \emptyset$. A contradiction with the choice of U_i^{α}. □

If $\kappa = \omega$ Then the assumption that the spaces X_i are zero-dimensional can be weakened.

Theorem 10. *Let $\Lambda \subset P(\omega)$ be a non-principal maximal ideal and $\Psi \subset P(\omega)$ the Frechét filter. If the spaces $(X_i, \mathcal{O}_i)$, $i \in \omega$ are T_3, then $X = \prod_{\Psi}^{\Lambda} X_i$ is an ω_1-paraopen space.*

Proof. If $(X_i, \mathcal{O}_i)$, $i \in \omega$ are T_3-spaces, then, by Theorem 3.2 of [14], the space $\prod_{\Psi}^{\Lambda} X_i$ is zero-dimensional. If we modify the proof of the mentioned theorem for the case when $\Lambda \subset P(\omega)$ is a maximal ideal and Ψ is the Frechét filter, we easily conclude that the sets of the form $\bigcap_{m\in\omega} q(\bigcap_{i\in L} \pi_i^{-1}(B_{i,m}))$, where $L \in \Lambda$, $B_{i,m} \in \mathcal{O}_i$ for all $i \in L$ and $m \in \omega$ and $B_{i,0} \supset \overline{B_{i,1}} \supset B_{i,1} \supset \overline{B_{i,2}} \supset B_{i,2} \supset \dots$ make a clopen base $\mathcal{B}$ for the topology on X. The ω_1-paraopenness remains to be shown.

Let $U^k \in \mathcal{B}$, $k \in \omega$, where $U^k = \bigcap_{m\in\omega} q(\bigcap_{i\in L_k} \pi_i^{-1}(B_{i,m}^k))$ and let $[f] \notin \bigcup_{k\in\omega} U_k$. For each $k \in \omega$ we choose $m_k \in \omega$ such that $[f] \notin \overline{q(\bigcap_{i\in L_k} \pi_i^{-1}(B_{i,m_k}^k)}$. If we define $\Delta^k = \{i \in L_k : f_i \notin \overline{B_{i,m_k}^k}\}$ then $|\Delta^k| = \omega$ for every $k \in \omega$ and applying Lemma 2 there is $L \in \Lambda$ satisfying

$$(1) \qquad \forall k \in \omega \quad |L \cap \Delta^k| = \omega.$$

For each $k \in \omega$ we define open sets U_i^k, $i \in \omega$ as follows: if $i \in L \cap \Delta^k$ we choose $U_i^k \in \mathcal{O}_i$ satisfying $f_i \in U_i^k \subset X_i \setminus \overline{B_{i,m_k}^k}$, and if $i \notin L \cap \Delta^k$ we put $U_i^k = X_i$. The sets $V_i, i \in \omega$ defined by $V_i = \bigcup_{k=0}^{i} U_i^k$ obviously satisfy

$$(2) \qquad \forall k \in \omega \quad \forall i \geq k \quad V_i \subset U_i^k.$$

Now $B = q(\bigcap_{i\in L} \pi_i^{-1}(V_i))$ is open in X and $[f] \in B$.

Suppose that there exists a $[g] \in B \cap \bigcup_{k\in\omega} U^k$. Then

$$(3) \qquad |\{i \in L : g_i \notin V_i\}| < \omega$$

and $[g] \in U^{k_0}$, for some $k_0 \in \omega$, that is for each $m \in \omega$, $|\{i \in L_{k_0} : g_i \notin B^{k_0}_{i,m}\}| < \omega$. Thus, for $m = m_{k_0}$ we have $|\{i \in L_{k_0} : g_i \notin B^{k_0}_{i,m_{k_0}}\}| < \omega$ and since $\Delta^{k_0} \subset L_{k_0}$, it holds

$$(4) \qquad |\{i \in \Delta^{k_0} : g_i \notin B^{k_0}_{i,m_{k_0}}\}| < \omega.$$

Now (3) and (4) give $|\{i \in L \cap \Delta^{k_0} : g_i \notin V_i \vee g_i \notin B^{k_0}_{i,m_{k_0}}\}| < \omega$. By (1), $L \cap \Delta^{k_0}$ is an infinite set, thus there is $i_0 \in L \cap \Delta^{k_0}$ such that $i_0 \geq k_0$ and $g_{i_0} \in V_{i_0} \cap B^{k_0}_{i_0,m_{k_0}}$. Because of (2), from $i_0 \geq k_0$ follows $V_{i_0} \subset U^{k_0}_{i_0}$ so, $U^{k_0}_{i_0} \cap B^{k_0}_{i_0,m_{k_0}} \neq \emptyset$, which is false regarding the construction of $U^{k_0}_{i_0}$. Finally, $B \cap \bigcup_{k \in \omega} U^k = \emptyset$, thus $\bigcup_{k \in \omega} U^k$ is a closed set. □

4. Paraopenness and paracompactness in products

Paracompactness in topological products is widely considered. Specially, there are many results when the box products of countably many factors are in question. The results are mainly obtained for compact, locally compact paracompact, metrizable, κ-metrizable and countable factors (see the survey of S.W. Williams [22]). By Theorem 3, paraopen spaces having some additional properties are paracompact. This fact will be used in the sequel.

Theorem 11. *Let $(X_i, \mathcal{O}_i)$, $i \in I$ be regular spaces, $\Psi \subset P(I)$ an $|I|$-regular filter and $X = \square_\Psi X_i$ (reduced box product). Then*

(a) *If $L(X) \leq |I|^+, X$ is strongly paracompact.*

(b) (GCH) *If $\sup |X_i| \leq |I|^+$ or $\sup w(X_i) \leq |I|^+$, then X is a hereditarily strongly paracompact space.*

Proof. Here $\Lambda = P(I)$, so Λ is a $|I|^+$-complete ideal. Since Ψ is a $|I|$-regular filter, by [15] Corollary 3.1 the r.i.p. $\prod^\Lambda_\Psi X_i = X$ is a $|I|^+$-open space. Now, (a) follows from Theorem 3(e) or from [13], Lemma 1.3 and (b) follows from Theorem 3(f) and $|X| \leq \prod_{i \in I} 2^{|I|} = 2^{|I|} = |I|^+$ in the first case or $w(X) \leq |I|^+$ in the second. □

A consequence of the previous theorem is the result about paracompactness of the nabla-product (see Rudin [19] or Kunen[13]) and Theorem 6.2 of Bankston (in [2]) concerning paracompactness of ultraproducts.

According to Theorem 5, the following statement is an extension of the results concerning box products of κ-metrizable spaces obtained by M. E. Rudin and J. E. Vaughan (see [22]).

Theorem 12. *If $(X_i, \mathcal{O}_i)$, $i \in I$ are κ-paraopen spaces, then for the box product $X = \Box X_i$ we have*

(a) *If $L(X) \leq \kappa$, then X is strongly paracompact.*

(b) *If $\kappa^{|I|} = \kappa$ and $\sup |X_i| \leq \kappa$ or $\sup w(X_i) \leq \kappa$, then X is a hereditarily, strongly paracompact space. (The condition $\kappa^{|I|} = \kappa$ holds if, for example, the GCH holds and $\mathrm{cf}(\kappa) > |I|$.)*

Proof. By Theorem 2, X is a κ-paraopen space. Now, (a) follows from Theorem 3(e) and (b) follows from Theorem 3(f) and $|X| \leq \prod_{i \in I} \kappa = \kappa^{|I|} = \kappa$ in the first case or $w(X) \leq \kappa$ in the second. □

Example 2. If in Example 1 we put $\kappa = 2^\omega$, then, under the CH, $(X, \mathcal{O})$ is a c-paraopen space of cardinality c. Since $c^\omega = c$, the box product $\Box_{i \in \omega} X$ is a paracompact space (which is not a P-space).

Example 3. Let $\Lambda \subset P(\omega)$ be a maximal ideal, $\Psi \subset P(\omega)$ the Frechét filter and R the usual real line. If the CH holds, then by Theorem 10 the space $Y = \prod_{\Psi}^{\Lambda} R$ is a c-paraopen space of cardinality c. Using the previous theorem we conclude that the box product $\Box_{i \in \omega} Y$ is a paracompact space. (Clearly, instead of R we may take arbitrary regular spaces $X_i, i \in \omega$ of cardinality c.)

Using Theorem 3(f), Theorem 9 and Theorem 10 in the similar way we obtain two results concerning products considered by B. Lawrence in [16].

Theorem 13 (GCH). *Let κ be an infinite cardinal and let Λ and Ψ be as in Theorem 9. If $(X_i, \mathcal{O}_i)$, $i \in \kappa$, are zero-dimensional, κ-open spaces and $\sup |X_i| \leq 2^\kappa$ or $\sup w(X_i) \leq 2^\kappa$, then $\prod_{\Psi}^{\Lambda} X_i$ is a hereditarily strongly paracompact space.*

Theorem 14 (CH). *Let $\Lambda \subset P(\omega)$ be a non-principal maximal ideal and Ψ the Frechét filter on ω. If the spaces $(X_i, \mathcal{O}_i)$, $i \in \omega$ are T_3 and $\sup |X_i| \leq 2^\omega$ or $\sup w(X_i) \leq 2^\omega$, then $\prod_{\Psi}^{\Lambda} X_i$ is a hereditarily strongly paracompact space.*

5. An example

In the sequel Φ will be a non-principal ultrafilter on ω (i.e. $\Phi \in \beta\omega\backslash\omega$) and Λ the corresponding maximal ideal: $\Lambda = \{\omega\backslash F : F \in \Phi\}$. The Fréchet filter on $P(\omega)$ will be denoted by Ψ and by D we will denote the two-element discrete space $\{0,1\}$. We consider the r.i.p. $X = \prod_{\Psi}^{\Lambda} D$ and introduce a convenient notation: for $L \in \Lambda$ and $\varphi \in {}^{L}2$ we define $\langle L, \varphi\rangle = \{f \in \prod_{i\in\omega} D : f \mid L = \varphi\}$. Obviously, $\langle L, \varphi\rangle = \bigcap_{i\in L} \pi_i^{-1}(\{\varphi_i\})$, so $\mathcal{B} = \{q(\langle L, \varphi\rangle) : L \in \Lambda,\ \varphi \in {}^{L}2\}$ is a base for the topology on X. If φ is the zero-function, we will simply write $\langle L, 0\rangle$.

$\langle D, \cdot\,\rangle$ is a (discrete) topological group, if the operation is defined by $0 \cdot 0 = 1 \cdot 1 = 0$ and $0 \cdot 1 = 1 \cdot 0 = 1$. The corresponding operation on X is defined (as usual in reduced products) by: $[\langle f_i : i \in I\rangle] \cdot [\langle g_i : i \in I\rangle] = [\langle f_i g_i : i \in I\rangle]$. The element $[\langle 0 : i \in I\rangle]$ of X will be denoted by $\mathbf{0}$. By [10], X is a non-discrete Hausdorff topological group. By Theorem 10, it is an ω_1-paraopen space and by Theorem 8, it is not a k-space.

Theorem 15.

(a) *X is a P-space iff Φ is a P-point of $\beta\omega\backslash\omega$.*

(b) *If Φ is a P-point of $\beta\omega \setminus \omega$, then X does not contain a non-discrete extremally disconnected subspace. Specially, X is not an extremally disconnected space.*

Proof. (a) follows from Theorem 1 of [16].

(b) If Φ is a P-point, then by (a), X is a P-space. Moreover, each subspace of X is a P-space. But extremally disconnected P-spaces of non-measurable cardinality are discrete (see [7]). □

Theorem 16 (CH)**.**

(a) *X is a hereditarily strongly paracompact space.*

(b) *X does not contain a nondiscrete extremally disconnected subspace.*

(c) *X is an F-space iff Φ is a P-point of $\beta\omega \setminus \omega$.*

Proof. (a) follows from Theorem 14. (b) is true since each $Y \subset X$ is $\omega_1 = 2^\omega$-paraopen and $\chi(Y) \leq w(X) = 2^\omega$, so, we can apply Theorem 6(a). (c) follows from Corollary 1(b) and Theorem 15. □

Let $\Phi \in \beta\omega \setminus \omega$. A family $\mathcal{A} \subset \Phi$ is a base for Φ iff $\forall F \in \Phi\ \exists A \in \mathcal{A}$ $(A \subset F)$. A family $\mathcal{P} \subset [\omega]^\omega$ is a π-base for Φ iff $\forall F \in \Phi\ \exists P \in \mathcal{P}(P \subset F)$.

So, let $u(\Phi) = \min\{|\mathcal{A}| : \mathcal{A}$ is a base for $\Phi\}$ and $\pi u(\Phi) = \min\{|\mathcal{P}| : \mathcal{P} \subset [\omega]^\omega$ is a π-base for $\Phi\}$.

Also, we remind that the small cardinals u and πu are defined by: $u = \min\{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega$ and $\mathcal{A}$ is a base for some $\Phi \in \beta\omega \setminus \omega\}$ and $\pi u = \min\{|\mathcal{P}| : \mathcal{P} \subset [\omega]^\omega$ and $\mathcal{P}$ is a π-base for some $\Phi \in \beta\omega \setminus \omega\}$. Basic facts concerning these cardinals are available in [5] and [21]. The following lemma will be used in the sequel.

Lemma 3. *If $\Phi \in \beta\omega \setminus \omega$ and $\kappa < \pi u(\Phi)$, then for each family $\{\Delta_\alpha : \alpha < \kappa\} \subset \Lambda \cap [\omega]^\omega$ there holds*

$$(*) \qquad \exists L \in \Lambda \quad \forall \alpha < \kappa \quad |L \cap \Delta_\alpha| = \omega.$$

Proof. Suppose that $\forall L \in \Lambda\ \exists \alpha < \kappa(|L \cap \Delta_\alpha| < \omega)$. Then $\forall F \in \Phi$ $\exists \alpha < \kappa(|\Delta_\alpha \setminus F| < \omega)$. The family $\mathcal{D} = \{\Delta_\alpha \setminus K : \alpha \in \kappa, K \in [\omega]^{<\omega}\}$ is of cardinality $\kappa\omega = \kappa$ and, by the assumption there holds $\forall F \in \Phi$ $\exists D \in \mathcal{D}\ (D \subset F)$. So, $\mathcal{D}$ is a π-base for Φ which is impossible since $|\mathcal{D}| = \kappa < \pi u(\Phi)$. □

Theorem 17.

(a) $\chi(X) = u(\Phi)$.

(b) *The space X is $\pi u(\Phi)$-paraopen.*

(c) *If $\pi u(\Phi) = u(\Phi)$ or specially, if $\pi u = c$, then X does not contain a non-discrete extremally disconnected subspace.*

Proof. (a) Let $\{q(\langle L_\alpha, 0\rangle) : \alpha < \kappa\}$ be a local base at $\mathbf{0}$. Then for each $L \in \Lambda$ there is $\alpha < \kappa$ satisfying $q(\langle L_\alpha, 0\rangle) \subset q(\langle L, 0\rangle)$ that is $|L \setminus L_\alpha| < \omega$. If we put $L^c = F$ and $L^c_\alpha = F_\alpha$ we have

$$(1) \qquad \forall F \in \Phi \quad \exists \alpha < \kappa \quad |F_\alpha \setminus F| < \omega.$$

Define $\mathcal{A} = \{F_\alpha \setminus K : \alpha \in \kappa,\ K \in [\omega]^{<\omega}\}$. Then $\mathcal{A} \subset \Phi$ and $|\mathcal{A}| = \kappa$. From (1) we have $\forall F \in \Phi\ \exists A \in \mathcal{A}(A \subset F)$ so, $\mathcal{A}$ is a base for Φ and there holds $\kappa \geq u(\Phi)$. Thus $\chi(X) \geq u(\Phi)$.

On the other hand, let $\mathcal{A} = \{F_\alpha : \alpha < \kappa\}$ be a base for Φ and $\kappa = u(\Phi)$. Then for $L_\alpha = \omega \setminus F_\alpha$ we have $\forall L \in \Lambda\ \exists \alpha < \kappa(L \subset L_\alpha)$. Hence for each neighbourhood $q(\langle L, 0\rangle)$ of $\mathbf{0}$, there exists $\alpha < \kappa$ such that $L \subset L_\alpha$, that is $q(\langle L_\alpha, 0\rangle) \subset q(\langle L, 0\rangle)$. Now, $\{q(\langle L_\alpha, 0\rangle) : \alpha < \kappa\}$ is a local base at $\mathbf{0}$, thus $\chi(X) \leq u(\Phi)$.

(b) Let $\kappa < \pi u(\Phi)$ and $[f] \notin \bigcup_{\alpha<\kappa} q(\langle L_\alpha, \varphi_\alpha\rangle)$. Then for $\Delta_\alpha = \{i \in L_\alpha : f_i \neq \varphi_\alpha(i)\}$ we have $|\Delta_\alpha| = \omega$, $\alpha < \kappa$. By the previous lemma, there is $L \in \Lambda$ satisfying

$$\forall \alpha < \kappa \quad |L \cap \Delta_\alpha| = \omega. \tag{2}$$

Define $B = q(\langle L, f \mid L\rangle)$ and suppose $[g] \in B \cap q(\langle L_\alpha, \varphi_\alpha\rangle)$ for some $\alpha < \kappa$. Then $f_i \neq \varphi_\alpha(i)$ for at most finitely many $i \in L \cap L_\alpha$, and since $\Delta_\alpha \subset L_\alpha$ we have $|\{i \in L \cap \Delta_\alpha : f_i \neq \varphi_\alpha(i)\}| < \omega$. Now, by (2) there is $i \in L \cap \Delta_\alpha$ such that $f_i = \varphi_\alpha(i)$ which is not possible because of the definition of Δ_α. Thus, $[f] \in B \subset X \setminus \bigcup_{\alpha<\kappa} q(\langle L_\alpha, \varphi_\alpha\rangle)$ and $\bigcup_{\alpha<\kappa} q(\langle L_\alpha, \varphi_\alpha\rangle)$ is a closed set.

(c) If $\pi u(\Phi) = u(\Phi) = \kappa$, then each subspace of X is a κ-paraopen space of character $\leq \kappa$ and we can apply Theorem 6(a). □

A.V. Arhangel'skii asked whether there is a non-discrete extremally disconnected group (see [1]). The answer “Yes” is consistent. Namely, $\langle G, \Delta, \mathcal{O}_\Phi\rangle$ is a topological group, where $G = [\omega]^{<\omega}$, Δ is the symmetric difference operation on G, $\Phi \in \beta\omega \setminus \omega$ and the topology $\mathcal{O}_\Phi$ on G is generated by sets of the form $U_F(K) = \{K \Delta K_1 : K_1 \in [F]^{<\omega}\}$ where $K \in G$ and $F \in \Phi$. Each of the following conditions implies that G is an extremally disconnected space: (1) Φ is a k-ultrafilter (Sirota [20], CH is used); (2) Φ is a selective ultrafilter which exists if MA holds (Louveau [17]); (3) Φ is a $P(c)$-point of $\beta\omega \setminus \omega$ (Malyhin [18]). Such an ultrafilter exists if there holds $p = c$.

If Λ is the maximal ideal corresponding to Φ and if we identify the elements of G with their characteristic functions, it is easy to prove that $U_F(K) = \langle L, \chi_{K\cap L} \mid L\rangle$, where $L = \omega \setminus F$. Thus, the space G is homeomorphic to the subgroup $Z = \{\chi_K : K \in G\}$ of the ideal-product $Y = \prod^\Lambda D$. If $X = \prod_\Psi^\Lambda = q(Y)$ is the space from the previous paragraph, it is natural to ask whether X (or Y) is an extremally disconnected space if there holds some of the conditions given above.

According to the results of the previous paragraph we know that if Φ is a P-point or if $\pi u(\Phi) = u(\Phi)$ (obviously, CH $\rightarrow p = c \rightarrow \pi u = c \rightarrow \pi u(\Phi) = u(\Phi)$) then X is not extremally disconnected and it does not contain a non-discrete e.d. subgroup. Since e.d. is preserved by continuous open surjections, the space Y is not e.d. too. Also, Y does not contain a non-discrete open e.d. subgroup.

Question. Is it consistent that there is $\Phi \in \beta\omega \setminus \omega$ such that the group X (or some non-discrete subgroup of X) is extremally disconnected?

Clearly, such Φ cannot be a P-point and it must satisfy $\pi u(\Phi) < u(\Phi)$. It is possible to satisfy the second condition. Namely, M. Goldstern and S. Shelah in [8] constructed a model of ZFC such that $\pi u < u$, so one can take $\Phi \in \beta\omega \setminus \omega$ satisfying $\pi u(\Phi) = \pi u$.

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